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Extremum Problems on an Infinite Network

Maretsugu YAMASAKI

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Introduction

Network problems are discussed usually on a finite graph. Duffin [5] investigated the extremal length of a network on a finite graph and suggested a relation between potential theory and network theory. Derrick [4] and Ohtsuka [6] generalized Duffin's results to the continuous case without using network theory.

We shall study in this paper the extremal length of a network on an infinite graph which has a countably infinite number of nodes and arcs. We use some techniques which are standard in potential theory (for instance [1], [2] and [3]) and go along Duffin's arguments.

Some definitions and notations related to network theory are given in § 1. The extremal length of a network is studied in § 4 with the aid of the functional spaces defined in § 2 and the fact in § 3 that max-potential equals min-work. The duality relation between the max-flow problem and the min-cut problem, which is investigated in § 6, does not hold in general for infinite linear programming problems. We shall treat three kinds of the extremal widths of a network in § 7 by using some results in § 5 and § 6. The reciprocal relation between the extremal length and one of the extremal widths is also studied in § 7. We shall be concerned with Duffin's path-cut inequality in § 8.

§ 1. Notations and network definitions

A graph is intuitively a geometric figure consisting of points (which we shall call nodes) and line segments (which we shall call arcs) connecting a node to another. To each arc we assign a direction. Denote by X the set of nodes and by Y the set of arcs. Since we always consider the case where X and Y consist of a countably infinite number of elements, we put for simplicity

$$X = \{0, 1, 2,\dots, n,\dots\},$$

$$Y = \{1, 2,\dots, n,\dots\}.$$

Define the node-arc incidence matrix $K=(K_{vj})$ by $K_{vj}=1$ if arc j is directed toward node v, $K_{vj}=-1$ if arc j is directed away from node v and $K_{vj}=0$ if arc j and node

v do not meet.

In the formal terminology, we define an infinite graph as follows.

DEFINITION 1. An infinite graph G is the triple $<X, Y, K>$ which satisfies the following conditions:

(1.1) $\{j \in Y; K_{vj} \neq 0\}$ is a finite set for each $v \in X$.

(1.2) $\{v \in X; K_{vj} \neq 0\}$ consists of two nodes for each $j \in Y$.

Let $L(X)$ and $L(Y)$ be the sets of all real-valued functions on X and Y respectively. For $u \in L(X)$ and $w \in L(Y)$, we put

$$u_v = u(v) \qquad \text{and} \quad w_j = w(j).$$

A path P from node α to node β is the triple $(C_X(P), C_Y(P), p)$ of a finite ordered set $C_X(P)=\{v_0, v_1,..., v_n\}$ of nodes, a finite ordered set $C_Y(P)=\{j_1, j_2,..., j_n\}$ of arcs and a function $p=p(P)$ on Y called the index of P such that

$$v_0 = \alpha, \ v_n = \beta, \ v_i \neq v_k \qquad (i \neq k),$$

$$\{v \in X; K_{vj_i} \neq 0\} = \{v_{i-1}, v_i\},$$

$$p_j = 0 \qquad \text{if} \quad j \notin C_Y(P),$$

$$p_j = -K_{vj} \quad \text{with} \quad v = v_{i-1} \quad \text{if} \quad j = j_i.$$

Intuitively a path from node α to node β is a finite set of nodes and arcs which forms a simple curve. Denote by $\boldsymbol{P}_{\alpha\beta}$ the set of all paths from node α to node β. For simplicity, we set $\boldsymbol{P}_{\alpha\alpha}=\{\alpha\}$. Let A and B be mutually disjoint nonempty subsets of X. A path P from A to B is a path from some node $\alpha \in A$ to some node $\beta \in B$ such that

$$C_X(P) \cap A = \{\alpha\} \qquad \text{and} \quad C_X(P) \cap B = \{\beta\}.$$

Denote by $\boldsymbol{P}_{AB}$ the set of all paths from A to B.

DEFINITION 2. We say that the pair (G, r) of an infinite graph $G=<X, Y, K>$ and a function $r \in L(Y)$ is an infinite network if the following conditions are fulfilled:

(1.3) G is connected, i.e., $\boldsymbol{P}_{\alpha\beta} \neq \phi$ for any $\alpha, \beta \in X$.

(1.4) $r_j > 0 \quad$ for all $\ j \in Y$.

The infinite network (G, r) is denoted by $<X, Y, K, r>$ or $<X, Y, r>$ or $<X, Y>$ if there is no confusion from the context. In this paper we always consider extremum problems on an infinite network $<X, Y, K, r>$.

DEFINITION 3. We say that a subset F of X is connected if for any α, $\beta \in F$ there is $P \in \boldsymbol{P}_{\alpha\beta}$ such that $C_X(P) \subset F$.

We say that $<X', Y', K, r'> = <X', Y'>$ is a finite subnetwork of $<X, Y, K, r>$ if X' and Y' are finite subsets of X and Y respectively, if conditions (1.2) and (1.3) are fulfilled replacing X and Y by X' and Y' respectively and $r'_j = r_j$ for each $j \in Y'$.

DEFINITION 4. We say that a sequence $\{<X^{(n)}, Y^{(n)}>\}$ of finite subnetworks of $<X, Y>$ is an exhaustion of $<X, Y>$ if

$$(1.5) \qquad X = \bigcup_{n=1}^{\infty} X^{(n)} \quad \text{and} \quad Y = \bigcup_{n=1}^{\infty} Y^{(n)},$$

$$(1.6) \qquad \{j \in Y; K_{vj} \neq 0\} \subset Y^{(n+1)} \quad \text{for each} \quad v \in X^{(n)}.$$

We have by definition

$$X^{(n)} = \cup \{\{v \in X; K_{vj} \neq 0\}; j \in Y^{(n)}\},$$

$$X^{(n)} \subset X^{(n+1)} \quad \text{and} \quad Y^{(n)} \subset Y^{(n+1)}.$$

A sequence $\{<X^{(n)}, Y^{(n)}>\}$ of finite subnetworks of $<X, Y>$ is said to be the elementary exhaustion of $<X, Y>$ starting from a finite connected subset A of X if

$$(1.7) \qquad Y^{(1)} = \{j \in Y; K_{vj} \neq 0 \text{ for some } v \in A\},$$

$$(1.8) \qquad X^{(n)} = \{v \in X; K_{vj} \neq 0 \text{ for some } j \in Y^{(n)}\} \qquad (n = 1, 2, \ldots),$$

$$(1.9) \qquad Y^{(n)} = \{j \in Y; K_{vj} \neq 0 \text{ for some } v \in X^{(n-1)}\} \qquad (n = 2, 3, \ldots).$$

Let A and B be mutually disjoint nonempty subsets of X. We say that a subset Q of Y is a cut between A and B if there exist mutually disjoint subsets $Q(A)$ and $Q(B)$ of X such that $A \subset Q(A)$, $B \subset Q(B)$, $X = Q(A) \cup Q(B)$ and the set

$$Q(A) \ominus Q(B) = \{j \in Y; K_{vj} K_{\mu j} = -1 \text{ for some } v \in Q(A) \text{ and } \mu \in Q(B)\}$$

is equal to Q. The pair of $Q(A)$ and $Q(B)$ is called a dissection of X. Denote by $\boldsymbol{Q}_{AB}$ the set of all cuts between A and B and put

$$\boldsymbol{Q}_{AB}^{(f)} = \{Q \in \boldsymbol{Q}_{AB}; Q \text{ is a finite set}\}.$$

A circuit is a finite set of nodes and arcs forming a simple closed curve. To each circuit we assign a direction. Let C_{kj} be the circuit-arc incidence matrix. Namely $C_{kj} = 1$ if arc j lies on circuit C_k in the same direction, $C_{kj} = -1$ if arc j lies on circuit C_k in the opposite direction, and $C_{kj} = 0$ if arc j does not lie on circuit C_k.

For any $u \in L(X)$ and $w \in L(Y)$, let us put

$$Su = \{i \in X; u_i \neq 0\}, \qquad Sw = \{j \in Y; w_j \neq 0\},$$

$$D(u) = \sum_{j=1}^{\infty} r_j^{-1}\Big(\sum_{\nu=0}^{\infty} K_{\nu j}u_\nu\Big)^2, \tag{1.10}$$

$$H(w) = \sum_{j=1}^{\infty} r_j w_j^2. \tag{1.11}$$

The Laplacian $\Delta u \in L(X)$ of $u \in L(X)$ is defined by

$$(\Delta u)_i = -\sum_{j=1}^{\infty} r_j^{-1} K_{ij}\Big(\sum_{\nu=0}^{\infty} K_{\nu j}u_\nu\Big). \tag{1.12}$$

We shall use the following classes of functions on X and Y.

$$L_0(X) = \{u \in L(X);\ Su \text{ is a finite set}\},$$

$$L_0(Y) = \{w \in L(Y);\ Sw \text{ is a finite set}\},$$

$$L^+(Y) = \{w \in L(Y);\ w_j \geqq 0 \text{ for all } j \in Y\},$$

$$L_0^+(Y) = L_0(Y) \cap L^+(Y),$$

$$L_2(Y; r) = \{w \in L(Y); H(w) < \infty\},$$

$$L_2^+(Y; r) = L_2(Y; r) \cap L^+(Y).$$

$L_2(Y; r)$ is a Hilbert space with the norm $[H(w)]^{1/2}$ and the inner product $<w, w'>$ defined by

$$<w, w'> = \sum_{j=1}^{\infty} r_j w_j w'_j. \tag{1.13}$$

If $H(w-w^{(n)}) \to 0$ as $n \to \infty$, then $w_j^{(n)} \to w_j$ as $n \to \infty$ for each j.

Let A and B be mutually disjoint nonempty finite subsets of X. We say that $w \in L(Y)$ is a flow from A to B of strength $I(w)$ if

$$I(w) = -\sum_{\nu \in A}\sum_{j=1}^{\infty} K_{\nu j}w_j = \sum_{\nu \in B}\sum_{j=1}^{\infty} K_{\nu j}w_j, \tag{1.14}$$

$$\sum_{j=1}^{\infty} K_{\nu j}w_j = 0 \qquad (\nu \notin A \cup B). \tag{1.15}$$

Denote by $F(A, B)$ the set of all flows from A to B and by $F_0(A, B)$ the closure of $F(A, B) \cap L_0(Y)$ in $L_2(Y; r)$. Thus for any $w \in F_0(A, B)$, there exists a sequence $\{w^{(n)}\} \subset F(A, B) \cap L_0(Y)$ such that $H(w-w^{(n)}) \to 0$ as $n \to \infty$. It follows that $F_0(A, B) \subset F(A, B)$ and $I(w^{(n)}) \to I(w)$ as $n \to \infty$.

We often use the following well-known theorem to assure the existence of an optimal (or extremal) solution of an extremum problem.

THEOREM A. *Let Z be a Hilbert space with the norm $\|z\|$ and the inner product (z, z') and C be a nonempty closed convex set in Z. Then there exists a unique $\hat{z} \in C$ such that*

$$\|\hat{z}\| = \min\{\|z\|; z \in C\}. \tag{1.16}$$

The element $\hat{z} \in C$ is characterized by the relation

$$\|\hat{z}\|^2 \leqq (\hat{z}, z) \tag{1.17}$$

for all $z \in C$. If $\hat{z} \pm z \in C$, then

$$(\hat{z}, z) = 0. \tag{1.18}$$

§2. Functional spaces D and $\mathbf{D}_0$

Beurling and Deny constructed a Dirichlet space on a finite set of points and arcs which can be interpreted as a finite network ([1], p. 223). Analogously to their method, we shall introduce a functional space on an infinite network $<X, Y, r>$. Let A be a nonempty subset of X and put

$$\boldsymbol{B}_0(X) = \boldsymbol{B}_0^A(X) = \{u \in L_0(X); u = 0 \text{ on } A\},$$

$$\boldsymbol{D} = \boldsymbol{D}^A = \{u \in L(X); D(u) < \infty \text{ and } u = 0 \text{ on } A\},$$

where $D(u)$ is defined by (1.10). It is easily seen that $\boldsymbol{D}$ is a real linear space and contains $\boldsymbol{B}_0(X)$. For $u, v \in \boldsymbol{D}$, we define $\|u\|$ and (u, v) by

$$\|u\| = [D(u)]^{1/2},$$

$$(u, v) = \sum_{j=1}^{\infty} r_j^{-1} \left(\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu\right)\left(\sum_{\nu=0}^{\infty} K_{\nu j} v_\nu\right).$$

Then we have

$$\|u+v\| \leqq \|u\| + \|v\| \qquad \text{and} \qquad \|tu\| = |t|\,\|u\|$$

for any $u, v \in \boldsymbol{D}$ and any real number t.

First we shall prove

LEMMA 1. *There exists a constant M_n such that*

$$\sum_{i=0}^{n} |u_i| \leqq M_n \|u\|$$

for all $u \in \boldsymbol{D}$.

PROOF. Let $\alpha \in A$, let $\{<X^{(n)}, Y^{(n)}>\}$ be the elementary exhaustion of $<X, Y>$ starting from $\{\alpha\}$ and let $u \in \boldsymbol{D}$. From the relation

$$\|u\|^2 \geqq \sum_{j \in Y^{(1)}} r_j^{-1} (\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu)^2 \geqq r_j^{-1} u_\nu^2$$

for all $\nu \in X^{(1)}$ and $j \in Y^{(1)}$ such that $K_{\nu j} \neq 0$, we derive

$$|u_\nu| \leqq a_1 \|u\| \quad \text{with} \quad a_1 = \max\{r_j^{1/2}; j \in Y^{(1)}\}$$

for all $\nu \in X^{(1)}$. We have by induction

$$|u_\nu| \leqq (\sum_{k=1}^{n} a_k) \|u\| \quad \text{with} \quad a_k = \max\{r_j^{1/2}; j \in Y^{(k)} - Y^{(k-1)}\}$$

for all $\nu \in X^{(n)}$. Given $\{0, 1, \dots, n\} \subset X$, we can find m such that $\{0, 1, \dots, n\} \subset X^{(m)}$. Then we have

$$\sum_{\nu=0}^{n} |u_\nu| \leqq M_n \|u\| \quad \text{with} \quad M_n = n \sum_{k=1}^{m} a_k.$$

It is clear by our construction that M_n is independent of u.

COROLLARY 1. *If* $\|u\| = 0$, *then* $u = 0$.

COROLLARY 2. *Assume that* $u^{(k)}, u \in \boldsymbol{D}$ *and* $\|u - u^{(k)}\| \to 0$ *as* $k \to \infty$. *Then* $u_i^{(k)} \to u_i$ *as* $k \to \infty$ *for each* $i \in X$.

From these facts we obtain

THEOREM 1. $\boldsymbol{D}$ *is a Hilbert space with the norm* $\|u\|$ *and the inner product* (u, v).

DEFINITION 5. We call a function T on the real line R into itself a normal contraction of R is $T0 = 0$ and

$$|Tx_1 - Tx_2| \leqq |x_1 - x_2|$$

for any $x_1, x_2 \in R$. Define $Tu \in L(X)$ for $u \in L(X)$ by

$$(Tu)_i = Tu_i.$$

We have

LEMMA 2. *Let* T *be a normal contraction of* R *and* $u \in \boldsymbol{D}$. *Then* $Tu \in \boldsymbol{D}$ *and* $\|Tu\| \leqq \|u\|$.

PROOF. For $\nu \in A$, we have $(Tu)_\nu = Tu_\nu = T0 = 0$. If $\{\nu \in X; K_{\nu j} \neq 0\} =$

$\{a, b\}$, then

$$|\sum_{\nu=0}^{\infty} K_{\nu j} Tu_\nu| = |Tu_a - Tu_b| \leqq |u_a - u_b| = |\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu|,$$

so that

$$\|Tu\|^2 = \sum_{j=1}^{\infty} r_j^{-1} (\sum_{\nu=0}^{\infty} K_{\nu j} Tu_\nu)^2 \leqq \sum_{j=1}^{\infty} r_j^{-1} (\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu)^2 = \|u\|^2.$$

COROLLARY 1. *Let $u \in \boldsymbol{D}$ and c be a non-negative real number and define $\min(u, c) \in L(X)$ by $(\min(u, c))_i = \min(u_i, c)$. Then $\min(u, c) \in \boldsymbol{D}$ and $\|\min(u, c)\| \leqq \|u\|$.*

COROLLARY 2. *If $u \in \boldsymbol{D}$, then $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$ belong to $\boldsymbol{D}$ and $\|u^+\| \leqq \|u\|$, $\|u^-\| \leqq \|u\|$.*

DEFINITION 6. Denote by $\boldsymbol{D}_0 = \boldsymbol{D}_0^A$ the closure of $\boldsymbol{B}_0^A(X)$ in $\boldsymbol{D}^A$ and set

$$\boldsymbol{H} = \boldsymbol{H}^A = \{u \in \boldsymbol{D}^A; (\Delta u)_i = 0 \text{ for } i \notin A\}.$$

We have

LEMMA 3. *Let $u \in \boldsymbol{D}$ and $f \in \boldsymbol{B}_0(X)$. Then*

$$(u, f) = -\sum_{i=0}^{\infty} (\Delta u)_i f_i = -\sum_{i=0}^{\infty} u_i (\Delta f)_i.$$

PROOF. Since Sf is a finite set,

$$(u, f) = \sum_{j=1}^{\infty} r_j^{-1} (\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu)(\sum_{i=0}^{\infty} K_{ij} f_i)$$

$$= \sum_{i=0}^{\infty} f_i (\sum_{j=1}^{\infty} r_j^{-1} K_{ij} (\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu)) = -\sum_{i=0}^{\infty} f_i (\Delta u)_i.$$

COROLLARY 1. *Let $u \in \boldsymbol{D}$ and $v \in \boldsymbol{D}_0$. If $\{i \in X; (\Delta u)_i \neq 0\}$ is a finite set, then*

$$(u, v) = -\sum_{i=0}^{\infty} (\Delta u)_i v_i.$$

PROOF. There exists a sequence $\{f^{(n)}\}$ in $\boldsymbol{B}_0(X)$ such that $\|v - f^{(n)}\| \to 0$ as $n \to \infty$. We have by Lemma 3 and Corollary 2 of Lemma 1

$$(u, v) = \lim_{n \to \infty} (u, f^{(n)}) = -\lim_{n \to \infty} \sum_{i=0}^{\infty} (\Delta u)_i f_i^{(n)} = -\sum_{i=0}^{\infty} (\Delta u)_i v_i.$$

COROLLARY 2. *$\boldsymbol{H}$ is the orthogonal complement of $\boldsymbol{D}_0$ in $\boldsymbol{D}$, i.e.,*

$$H = \{h \in D; (h, v) = 0 \text{ for all } v \in D_0\}.$$

COROLLARY 3. *Every $u \in D$ can be decomposed uniquely in the form*

$$u = v + h, \quad \text{where} \quad v \in D_0 \quad \text{and} \quad h \in H.$$

PROOF. Since D_0 is a closed linear subspace of a Hilbert space D and H is the orthogonal complement of D_0 in D, our assertion follows from the orthogonal decomposition theorem (cf. [7], p. 82, Theorem 1).

THEOREM 2. (Dirichlet principle) *Let $u \in D$ and let B be a subset of X such that $A \cap B = \phi$. Then $S_u^B = \{v \in D; v = u \text{ on } B\}$ is a closed subset of D and there exists a unique $\hat{u}^B \in S_u^B$ such that*

$$\|\hat{u}^B\| = \min\{\|v\|; v \in S_u^B\}.$$

It is valid that

$$(2.1) \qquad (\Delta \hat{u}^B)_i = 0 \quad \text{if} \quad i \notin A \quad \text{and} \quad i \notin B,$$

$$(2.2) \qquad \|\hat{u}^B\|^2 = (\hat{u}^B, v) \quad \text{for all} \quad v \in S_u^B.$$

The function $\hat{u}^B$ is characterized by (2.2).

PROOF. It is clear that S_u^B is a nonempty closed convex set. Then the existence of $\hat{u}^B$ follows from Theorems 1 and A. For any $f \in B_0(X)$ such that $f_i = 0$ for $i \in B$, we have $\hat{u}^B \pm f \in S_u^B$, so that $(\hat{u}^B, f) = 0$ by (1.18) of Theorem A. Thus (2.1) follows from Lemma 3. If $v \in S_u^B$, then

$$\hat{u}^B \pm (\hat{u}^B - v) \in S_u^B$$

and $(\hat{u}^B, \hat{u}^B - v) = 0$ by (1.18). This shows (2.2). If $\bar{u} \in S_u^B$ satisfies $\|\bar{u}\|^2 = (\bar{u}, v)$ for all $v \in S_u^B$, then

$$\|\bar{u}\|^2 = (\bar{u}, v) \leqq \|\bar{u}\| \, \|v\|$$

and hence $\|\bar{u}\| \leqq \|v\|$. Thus $\bar{u} = \hat{u}^B$.

§3. Max-potential equals min-work

We shall generalize a Duffin's theorem which assures that max-potential equals min-work.

Let $c \in L^+(Y)$ and A and B be mutually disjoint nonempty subsets of X. We shall be concerned with the following two extremum problems on an infinite network $<X, Y>$.

(3.1) (Min-work problem) Find

$$N(A, B; c) = \inf\{\sum_P c_j; P \in \boldsymbol{P}_{AB}\},$$

where $\sum_P c_j$ is an abbreviation of $\sum_{C_Y(P)} c_j$.

(3.2) (Max-potential problem) Find

$$N^*(A, B; c) = \sup\{\inf\{u_\nu; \nu \in B\} - \sup\{u_\nu; \nu \in A\}; u \in S^*\},$$

where

$$S^* = \{u \in L(X); |\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu| \leqq c_j \text{ on } Y\}.$$

We have

THEOREM 3. *$N(A, B; c) = N^*(A, B; c)$ holds and there exists an optimal solution u of (3.2) such that $u = 0$ on A.*

PROOF. Let us put

$$\hat{N}(A, B; c) = \inf\{\sum_P c_j; P \in \hat{\boldsymbol{P}}_{AB}\},$$

where $\hat{\boldsymbol{P}}_{AB} = \cup\{\boldsymbol{P}_{\alpha\beta}; \alpha \in A, \beta \in B\}$. Then $\boldsymbol{P}_{AB} \subset \hat{\boldsymbol{P}}_{AB}$ and $N(A, B; c) = \hat{N}(A, B; c)$, since every $P \in \hat{\boldsymbol{P}}_{AB}$ contains some $P' \in \boldsymbol{P}_{AB}$ and $c_j \geqq 0$. Let us show that $\hat{N}(A, B; c) = N^*(A, B; c)$. Let $P \in \hat{\boldsymbol{P}}_{\alpha\beta}$ with $\alpha \in A$ and $\beta \in B$ and put $C_X(P) = \{\nu_0, \nu_1, \ldots, \nu_n\}$ and $C_Y(P) = \{j_1, \ldots, j_n\}$. If $u \in S^*$, then

$$\sum_P c_j = \sum_{i=1}^{n} c_{j_i} \geqq \sum_{i=1}^{n} |u_{\nu_i} - u_{\nu_{i-1}}| \geqq u_\beta - u_\alpha$$
$$\geqq \inf\{u_\nu; \nu \in B\} - \sup\{u_\nu; \nu \in A\}.$$

Therefore $\hat{N}(A, B; c) \geqq N^*(A, B; c)$. On the other hand, let us define $\hat{u} \in L(X)$ by $\hat{u}_\nu = 0$ if $\nu \in A$ and

$$\hat{u}_\nu = \inf\{\sum_P c_j; P \in \hat{\boldsymbol{P}}_{A\{\nu\}}\} \quad \text{if} \quad \nu \notin A.$$

We show that $\hat{u} \in S^*$, i.e.,

$$|\sum_{\nu=0}^{\infty} K_{\nu k} \hat{u}_\nu| \leqq c_k \tag{3.3}$$

for each $k \in Y$. Let $k \in Y$ and $\{\nu \in X; K_{\nu k} \neq 0\} = \{a, b\}$. In case $a \in A$ and $b \in A$, we have $\hat{u}_a = \hat{u}_b = 0$. Then (3.3) is clear. In case $a \in A$ and $b \notin A$, let us consider the path $\tilde{P} \in \boldsymbol{P}_{ab} \subset \hat{\boldsymbol{P}}_{A\{b\}}$ defined by

$$C_X(\tilde{P}) = \{a, b\}, \quad C_Y(\tilde{P}) = \{k\},$$

$$p_j(\tilde{P}) = 0 \ (j \neq k) \quad \text{and} \quad p_k(\tilde{P}) = -K_{ak}.$$

Then we have

$$0 \leqq \hat{u}_b \leqq \sum_{\tilde{P}} c_j = c_k,$$

and hence $|\hat{u}_a - \hat{u}_b| = \hat{u}_b \leqq c_k$, which shows (3.3). In case $a \notin A$ and $b \in A$, we have (3.3) similarly. Finally we consider the case where $a \notin A$ and $b \notin A$. For any $P \in \hat{\boldsymbol{P}}_{A\{a\}}$, let us define $\bar{P} \in \hat{\boldsymbol{P}}_{A\{b\}}$ by $C_X(\bar{P}) = C_X(P) \cup \{b\}$, $C_Y(\bar{P}) = C_Y(P) \cup \{k\}$, $p_j(\bar{P}) = p_j(P)$ if $j \neq k$ and $p_k(\bar{P}) = -K_{ak}$. Then

$$\hat{u}_b \leqq \sum_{\bar{P}} c_j = \sum_{P} c_j + c_k,$$

so that $\hat{u}_b \leqq \hat{u}_a + c_k$. Interchanging the roles of node a and node b in the above discussion, we obtain $\hat{u}_a \leqq \hat{u}_b + c_k$, and hence $|\hat{u}_a - \hat{u}_b| \leqq c_k$. Therefore $\hat{u} \in S^*$ and

$$N^*(A, B; c) \geqq \inf\{\hat{u}_v; v \in B\} - \sup\{\hat{u}_v; v \in A\}$$
$$= \inf\{\hat{u}_v; v \in B\} = \hat{N}(A, B; c).$$

Thus $N(A, B; c) = N^*(A, B; c)$ and $\hat{u}$ is an optimal solution of max-potential problem.

There is no optimal solution of min-work problem in general. This is shown by

EXAMPLE 1. Let us consider an infinite graph such as shown in Fig. 1, where we number nodes and arcs. To each arc of the graph a direction is assigned.

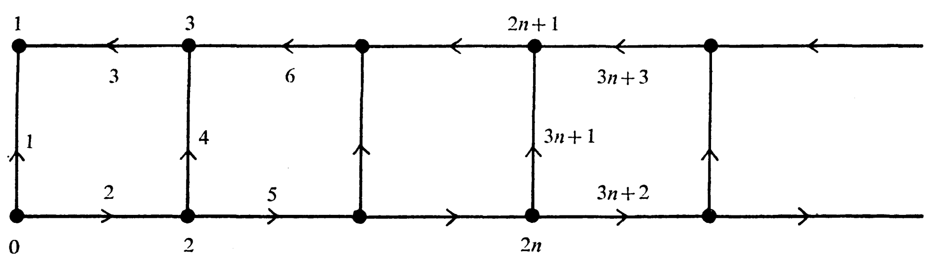

Fig. 1. An infinite graph.

Denote this infinite graph by $G = <X, Y, K>$. If $r \in L(Y)$ is defined by $r_j = 1$ on Y, then (G, r) is an infinite network.

Let $A = \{0\}$ and $B = \{1\}$ and define $c \in L^+(Y)$ by

$$c_1 = 1,\ c_{3n+1} = 4^{-n}\quad (n \neq 0),\qquad c_{3n+2} = c_{3n+3} = 4^{-n-1}\quad (n \geqq 0).$$

Then we have $N(A, B; c) = 2/3$ and $\sum_P c_j > 2/3$ for any $P \in \boldsymbol{P}_{AB}$. Namely there is no optimal solution of min-work problem.

§4. The extremal length of a network

Let A and B be mutually disjoint nonempty subsets of X and define the extremal length $EL = EL(A, B)$ of an infinite network $<X, Y, r>$ relative to two sets A and B as the value of the following extremum problem.

(4.1) Find $$EL^{-1} = \inf\{H(W);\ W \in E_{AB}\},$$

where

$$E_{AB} = \{W \in L_2^+(Y; r);\ \sum_P r_j W_j \geqq 1 \text{ for all } P \in \boldsymbol{P}_{AB}\}.$$

We use the convention in this paper that the infimum of a real-valued function on the empty set ϕ is equal to ∞. If $E_{AB} \neq \phi$, then there is a unique $\hat{W} \in E_{AB}$ such that $EL^{-1} = H(\hat{W})$ by Theorem A, since E_{AB} is a nonempty closed convex subset of $L_2(Y; r)$. Note that $\inf\{\sum_P r_j \hat{W}_j;\ P \in \boldsymbol{P}_{AB}\} = 1$.

In connection with (4.1), we consider the following extremum problem.

(4.2) Find

$$d = \inf\{D(v);\ v \in L(X),\ v = 0 \text{ on } A \text{ and } v = 1 \text{ on } B\}.$$

We have

THEOREM 4. $EL^{-1} = d$.

PROOF. First we shall prove $EL^{-1} \leqq d$. We may suppose that d is finite, i.e., there is $q \in \boldsymbol{D}^A$ such that $q = 1$ on B. There exists $\hat{q} = \hat{q}^B \in S_q^B$ such that $d = D(\hat{q}) = \|\hat{q}\|^2$ by Theorem 2. Define $W \in L^+(Y)$ by

$$W_j = r_j^{-1} \left| \sum_{\nu=0}^{\infty} K_{\nu j} \hat{q}_\nu \right|.$$

Let $P \in \boldsymbol{P}_{AB}$ and $C_X(P) = \{\nu_0, \nu_1, \ldots, \nu_n\}$ with $\nu_0 = \alpha \in A$ and $\nu_n = \beta \in B$. Then

$$\sum_P r_j W_j = \sum_{i=1}^{n} |\hat{q}_{\nu_i} - \hat{q}_{\nu_{i-1}}| \geqq |\hat{q}_\beta - \hat{q}_\alpha| = 1$$

and

$$H(W) = D(\hat{q}) = d < \infty.$$

Therefore $W \in E_{AB}$ and $EL^{-1} \leqq H(W) = d$. Next we shall show $d \leqq EL^{-1}$. We may suppose that $EL^{-1} < \infty$, i.e., $E_{AB} \neq \phi$. Then there is $\hat{W} \in E_{AB}$ such that $EL^{-1} = H(\hat{W})$. Define $c \in L^+(Y)$ by $c_j = r_j \hat{W}_j$. Then $N(A, B; c) = 1$. By means of Theorem 3, we have the existence of $u \in L(X)$ such that $u = 0$ on A, $u \geqq 1$ on B and

$$|\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu| \leqq c_j = r_j \hat{W}_j.$$

Observing that

$$D(u) = \sum_{j=1}^{\infty} r_j^{-1} (\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu)^2 \leqq H(\hat{W}) = EL^{-1} < \infty,$$

we have by Corollary 1 of Lemma 2

$$v = \min(u, 1) \in \boldsymbol{D} \qquad \text{and} \quad D(v) \leqq D(u).$$

Since $v = 0$ on A and $v = 1$ on B, we have $d \leqq D(v) \leqq EL^{-1}$. Therefore $EL^{-1} = d$.

It is easily seen that $d < \infty$ if either A or B is a finite set.

§5. A fundamental equality for a double series

First we have

LEMMA 4. $$\{\sum_{j=1}^{\infty} |w_j| \, |\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu|\}^2 \leqq H(w) D(u)$$

for every $u \in L(X)$ and $w \in L(Y)$.

PROOF. We have

$$\{\sum_{j=1}^{\infty} |w_j| \, |\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu|\}^2 = \{\sum_{j=1}^{\infty} (r_j^{1/2} |w_j|)(r_j^{-1/2} |\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu|)\}^2$$

$$\leqq (\sum_{j=1}^{\infty} r_j w_j^2)(\sum_{j=1}^{\infty} r_j^{-1} (\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu)^2) = H(w) D(u).$$

THEOREM 5. *If A and B are finite sets, then*

$$\sum_{j=1}^{\infty} w_j (\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu) = \sum_{\nu=0}^{\infty} u_\nu (\sum_{j=1}^{\infty} K_{\nu j} w_j) \tag{5.1}$$

for every $w \in F_0(A, B)$ and $u \in L(X)$ such that $D(u) < \infty$.

PROOF. Let $w \in F_0(A, B)$ and $u \in L(X)$ such that $D(u) < \infty$. There exists a sequence $\{w^{(n)}\}$ in $L_0(Y)$ such that $w^{(n)} \in F(A, B)$ and $H(w - w^{(n)}) \to 0$ as $n \to \infty$. We have

$$\begin{aligned}\sum_{j=1}^{\infty} w_j^{(n)}\Big(\sum_{\nu=0}^{\infty} K_{\nu j}u_\nu\Big) &= \sum_{\nu=0}^{\infty} u_\nu\Big(\sum_{j=1}^{\infty} K_{\nu j}w_j^{(n)}\Big)\\ &= \sum_{\nu\in A} u_\nu\Big(\sum_{j=1}^{\infty} K_{\nu j}w_j^{(n)}\Big)+\sum_{\nu\in B} u_\nu\Big(\sum_{j=1}^{\infty} K_{\nu j}w_j^{(n)}\Big)\\ &\to \sum_{\nu\in A} u_\nu\Big(\sum_{j=1}^{\infty} K_{\nu j}w_j\Big)+\sum_{\nu\in B} u_\nu\Big(\sum_{j=1}^{\infty} K_{\nu j}w_j\Big)\\ &= \sum_{\nu=0}^{\infty} u_\nu\Big(\sum_{j=1}^{\infty} K_{\nu j}w_j\Big)\end{aligned}$$

as $n\to\infty$, since $w_j^{(n)}\to w_j$ as $n\to\infty$ for each j and A and B are finite sets. On the other hand, we have

$$\Big\{\sum_{j=1}^{\infty} |w_j|\,\Big|\sum_{\nu=0}^{\infty} K_{\nu j}u_\nu\Big|\Big\}^2 \leqq H(w)D(u) < \infty\,,$$

$$\Big\{\sum_{j=1}^{\infty} |w_j-w_j^{(n)}|\,\Big|\sum_{\nu=0}^{\infty} K_{\nu j}u_\nu\Big|\Big\}^2 \leqq H(w-w^{(n)})D(u)$$

by Lemma 4. It follows that

$$\begin{aligned}\sum_{j=1}^{\infty} w_j\Big(\sum_{\nu=0}^{\infty} K_{\nu j}u_\nu\Big) &= \lim_{n\to\infty}\sum_{j=1}^{\infty} w_j^{(n)}\Big(\sum_{\nu=0}^{\infty} K_{\nu j}u_\nu\Big)\\ &= \sum_{\nu=0}^{\infty} u_\nu\Big(\sum_{j=1}^{\infty} K_{\nu j}w_j\Big).\end{aligned}$$

This completes the proof.

REMARK 1. We also have (5.1) if any one of the following conditions is fulfilled:

(i) $w\in L_0(Y)$ or $u\in L_0(X)$.

(ii) $\sum_{j=1}^{\infty}|w_j|<\infty$ and $\{u_\nu\}$ is bounded.

(iii) $u\in \boldsymbol{D}_0$ and $w\in L_2(Y;r)$ such that $\{\nu\in X;\ \sum_{j=1}^{\infty} K_{\nu j}w_j\neq 0\}$ is a finite set.

Let $Q\in \boldsymbol{Q}_{AB}$ and $Q=Q(A)\ominus Q(B)$, where the pair of $Q(A)$ and $Q(B)$ is a dissection of X such that $A\subset Q(A)$ and $B\subset Q(B)$. We define the characteristic function $u=u(Q)\in L(X)$ of Q and the index $s=s(Q)\in L(Y)$ by

$$\begin{aligned}&u_\nu = 0 \quad\text{if}\quad \nu\in Q(A) \quad\text{and}\quad u_\nu = 1 \quad\text{if}\quad \nu\in Q(B),\\ &s_j = \sum_{\nu=0}^{\infty} K_{\nu j}u_\nu\,.\end{aligned} \tag{5.2}$$

We have $s_j=0$ if $j \notin Q$ and $|s_j|=1$ if $j \in Q$. If $j \in Q$ and $\{v \in X; K_{vj} \neq 0\}=\{a, b\}$ with $a \in Q(A)$ and $b \in Q(B)$, then

$$s_j = K_{bj} = -K_{aj}. \tag{5.3}$$

We have

COROLLARY 1. *Let A and B be finite sets, $w \in F_0(A, B)$, $Q \in \boldsymbol{Q}_{AB}^{(f)}$ and $s=s(Q)$ be the index of Q. Then*

$$\sum_Q s_j w_j = I(w).$$

PROOF. Since Q is a finite set, $u=u(Q) \in \boldsymbol{D}$. It follows from Theorem 5 that

$$\sum_Q s_j w_j = \sum_{j=1}^{\infty} w_j(\sum_{v=0}^{\infty} K_{vj} u_v) = \sum_{v=0}^{\infty} u_v(\sum_{j=1}^{\infty} K_{vj} w_j)$$
$$= \sum_{v \in B} \sum_{j=1}^{\infty} K_{vj} w_j = I(w).$$

COROLLARY 2. *If A and B are finite sets and $w \in F_0(A, B)$, then*

$$\sum_Q |w_j| \geqq I(w)$$

for all $Q \in \boldsymbol{Q}_{AB}^{(f)}$.

Theorem 5 and its corollaries do not hold in general if we replace $F_0(A, B)$ by $F(A, B)$. This is shown by

EXAMPLE 2. Let us consider an infinite graph such as shown in Fig. 2, where we number nodes and arcs. Denote this infinite graph by $G=<X, Y, K>$.

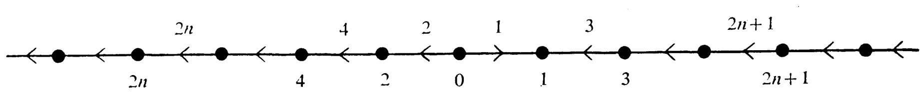

Fig. 2. An infinite graph.

Define $r \in L(Y)$ by $r_j=1$ on Y. Then (G, r) is an infinite network. Let $A=\{0\}$ and $B=\{1\}$ and define $u \in L(X)$ and $w \in L(Y)$ by

$$u_v = 1 \quad \text{if} \quad v = 2n+1 \quad \text{and} \quad u_v = 0 \quad \text{if} \quad v = 2n$$

for $n=0, 1, 2,\ldots$ and

$$w_j = 1 \qquad \text{on} \quad Y.$$

It is clear that $w \in F(A, B)$ and u is the characteristic function of $Q=\{1\} \in \boldsymbol{Q}_{AB}^{(f)}$.

We have

$$\sum_{j=1}^{\infty} w_j(\sum_{\nu=0}^{\infty} K_{\nu j}u_\nu) = w_1 = 1,$$

$$\sum_{\nu=0}^{\infty} u_\nu(\sum_{j=1}^{\infty} K_{\nu j}w_j) = \sum_{j=1}^{\infty} K_{1j}w_j = 2,$$

$$I(w) = 2 > 1 = \sum_Q |w_j| = \sum_Q s_j w_j .$$

§ 6. Max-flows and min-cuts

Let A and B be mutually disjoint nonempty finite subsets of X and $W \in L^+(Y)$. We consider the following extremum problems on an infinite network $<X, Y, r>$.

(6.1) (Max-flow problem) Find

$$M(W; F_0(A, B)) = \sup\{I(w); w \in F_0(A, B) \text{ and } |w_j| \leqq W_j \text{ on } Y\}.$$

(6.2) (Min-cut problem) Find

$$M^*(W; \boldsymbol{Q}_{AB}^{(f)}) = \inf\{\sum_Q W_j; Q \in \boldsymbol{Q}_{AB}^{(f)}\}.$$

We can define $M^*(W; \boldsymbol{Q}_{AB})$ similarly. Let us put

$$G(A, B) = F_0(A, B) \cap L_0(Y) = F(A, B) \cap L_0(Y)$$

and consider one more extremum problem.

(6.3) (Weak max-flow problem) Find

$$M(W; G(A, B)) = \sup\{I(w); w \in G(A, B) \text{ and } |w_j| \leqq W_j \text{ on } Y\}.$$

We have

Lemma 5. $M(W; G(A, B)) \leqq M(W; F_0(A, B)) \leqq M^*(W; \boldsymbol{Q}_{AB}^{(f)})$.

Proof. Since $G(A, B) \subset F_0(A, B)$, we have $M(W; G(A, B)) \leqq M(W; F_0(A, B))$. Let $Q \in \boldsymbol{Q}_{AB}^{(f)}$ and $w \in F_0(A, B)$ such that $|w_j| \leqq W_j$ on Y. Then we have by Corollary 2 of Theorem 5

$$I(w) \leqq \sum_Q |w_j| \leqq \sum_Q W_j,$$

which leads to the desired inequality $M(W; F_0(A, B)) \leqq M^*(W; \boldsymbol{Q}_{AB}^{(f)})$.

Lemma 6. *If* $W \in L_2^+(Y; r)$, *then there exists* $\hat{w} \in F_0(A, B)$ *such that* $|\hat{w}_j| \leqq W_j$

on Y *and* $M(W; F_0(A, B)) = I(\hat{w})$, *i.e.,* $\hat{w}$ *is an optimal solution of* (6.1).

PROOF. There exists a sequence $\{w^{(n)}\}$ in $F_0(A, B)$ such that $|w_j^{(n)}| \leqq W_j$ on Y and $I(w^{(n)})$ converges to $M(W; F_0(A, B))$. Since $\{w \in F_0(A, B); |w_j| \leqq W_j$ on $Y\}$ is weakly compact in $L_2(Y; r)$, we may assume that $\{w^{(n)}\}$ converges weakly to $\hat{w} \in F_0(A, B)$ with $|\hat{w}_j| \leqq W_j$ on Y. Then $w_j^{(n)} \to \hat{w}_j$ as $n \to \infty$ for each j. Since $\{j \in Y; K_{vj} \neq 0\}$ is a finite set for each v, we have

$$I(\hat{w}) = \sum_{v \in B} \sum_{j=1}^{\infty} K_{vj} \hat{w}_j = \lim_{n \to \infty} \sum_{v \in B} \sum_{j=1}^{\infty} K_{vj} w_j^{(n)}$$
$$= \lim_{n \to \infty} I(w^{(n)}) = M(W; F_0(A, B)).$$

This completes the proof.

Problem (6.3) has no optimal solution in general. This is shown by Example 5 below. But by the same method as in the above proof, we can prove

LEMMA 7. *If* $W \in L_2^+(Y; r)$, *then there exists* $\hat{w} \in F_0(A, B)$ *such that* $|\hat{w}_j| \leqq W_j$ *on* Y *and* $M(W; G(A, B)) = I(\hat{w})$. *This* $\hat{w}$ *is called a weak optimal solution of* (6.3).

Let $w \in F_0(A, B)$ with $|w_j| \leqq W_j$ on Y. Define a subset $Q(A; w)$ of X as follows. Node v belongs to $Q(A; w)$ if and only if either $v \in A$ or there exist $P \in \boldsymbol{P}_{A\{v\}}$ and a positive number t such that

$$(6.4) \qquad \begin{aligned} &w_j + t \leqq W_j && \text{if} \quad p_j = 1, \\ &w_j - t \geqq -W_j && \text{if} \quad p_j = -1, \end{aligned}$$

where p is the index of P. Let $Q(B; w) = X - Q(A; w)$.

We have

LEMMA 8. *Let* $\hat{w}$ *be a weak optimal solution of* (6.3) *and set* $\hat{Q}(A) = Q(A; \hat{w})$ *and* $\hat{Q}(B) = Q(B; \hat{w})$. *Then* $\hat{Q} = \hat{Q}(A) \ominus \hat{Q}(B) \in \boldsymbol{Q}_{AB}$. *We say that* $\hat{Q}$ *is the cut determined by* $\hat{w}$.

PROOF. It suffices to show that $B \subset \hat{Q}(B)$. If we suppose the contrary, there is $\beta \in B$ such that $\beta \in \hat{Q}(A)$. We can find $P \in \boldsymbol{P}_{A\{\beta\}}$ and a positive number t which satisfy (6.4). There exists a sequence $\{w^{(n)}\}$ in $G(A, B)$ such that $|w_j^{(n)}| \leqq W_j$ on Y and $w_j^{(n)} \to \hat{w}_j$ as $n \to \infty$ for each j. For any ε with $0 < \varepsilon < t/2$, there is n_0 such that

$$|w_j^{(n)} - \hat{w}_j| < \varepsilon \qquad \text{for all} \quad j \in C_Y(P)$$

and

$$|I(w^{(n)}) - I(\hat{w})| < \varepsilon$$

whenever $n \geqq n_0$. Taking $w' = w^{(n_0)}$, we have

$$w'_j + t/2 \leqq W_j \qquad \text{if} \quad p_j = 1,$$
$$w'_j - t/2 \geqq -W_j \qquad \text{if} \quad p_j = -1.$$

Thus $w' + (t/2)p \in G(A, B)$ and $|w'_j + (t/2)p_j| \leqq W_j$ on Y, so that

$$M(W; G(A, B)) \geqq I(w' + (t/2)p) = I(w') + (t/2)I(p)$$
$$= I(w') + t/2 > I(\hat{w}) - \varepsilon + t/2 > I(\hat{w}).$$

This is a contradiction. Therefore $B \subset \hat{Q}(B)$.

REMARK 2. If $\hat{w}$ is an optimal solution of (6.1), then we have $Q(A; \hat{w}) \ominus Q(B; \hat{w}) \in \boldsymbol{Q}_{AB}$.

LEMMA 9. *Let $\hat{w}$ be a weak optimal solution of* (6.3), *$\hat{Q}$ be the cut determined by $\hat{w}$ and $\hat{s} = s(\hat{Q})$ be the index of $\hat{Q}$. Then $\hat{s}_j \hat{w}_j = W_j$ for each $j \in \hat{Q}$.*

PROOF. Let $k \in \hat{Q}$ and $\{v \in X; K_{vk} \neq 0\} = \{a, b\}$ with $a \in \hat{Q}(A)$ and $b \in \hat{Q}(B)$. Suppose that $\hat{s}_k \hat{w}_k \neq W_k$. In the case where $\hat{s}_k = -K_{ak} = 1$, we have $\hat{w}_k < W_k$. If $a \in A$, then there exist $\bar{P} \in \boldsymbol{P}_{A\{b\}}$ and a positive number t such that $\hat{w}_k + t \leqq W_k$, $p_k(\bar{P}) = 1$ and $p_j(\bar{P}) = 0$ if $j \neq k$. In fact, we may take $t = W_k - \hat{w}_k$, $C_X(\bar{P}) = \{a, b\}$, $C_Y(\bar{P}) = \{k\}$ and $p_k(\bar{P}) = 1$. This implies $b \in \hat{Q}(A)$, which is a contradiction. If $a \in \hat{Q}(A) - A$, then there exist $P \in \boldsymbol{P}_{A\{a\}}$ and a positive number t which satisfy (6.4). Let $t_0 = \min(t, W_k - \hat{w}_k)$ and $\bar{P}$ be the path from A to $\{b\}$ which is generated by P and $\{k\}$, i.e.,

$$C_X(\bar{P}) = C_X(P) \cup \{b\}, \qquad C_Y(\bar{P}) = C_Y(P) \cup \{k\},$$
$$p_j(\bar{P}) = p_j \qquad \text{if} \quad j \neq k,$$
$$p_k(\bar{P}) = -K_{ak} = 1.$$

Then we have

$$\hat{w}_j + t_0 \leqq W_j \qquad \text{if} \quad p_j(\bar{P}) = 1,$$
$$\hat{w}_j - t_0 \geqq -W_j \qquad \text{if} \quad p_j(\bar{P}) = -1.$$

This implies $b \in \hat{Q}(A)$, which is a contradiction. In the case where $\hat{s}_k = -1$, we can arrive at a contradiction similarly. Therefore $\hat{s}_k \hat{w}_k = W_k$.

COROLLARY 1. *If $j \in \hat{Q}$ and $\{v \in X; K_{vj} \neq 0\} = \{a, b\}$ with $a \in \hat{Q}(A)$ and $b \in \hat{Q}(B)$, then*

$$-K_{aj}\hat{w}_j = K_{bj}\hat{w}_j = W_j.$$

COROLLARY 2. *If* $\hat{Q} \in \boldsymbol{Q}_{AB}^{(f)}$, *then* $M(W; G(A, B)) = M(W; F_0(A, B)) = M^*(W; \boldsymbol{Q}_{AB}^{(f)})$.

PROOF. In view of Lemma 5, it suffices to show that $M^*(W; \boldsymbol{Q}_{AB}^{(f)}) \leqq M(W; G(A, B))$. Since $\hat{Q} \in \boldsymbol{Q}_{AB}^{(f)}$ by our assumption, we have by Corollary 1 of Theorem 5 and Lemma 9

$$M(W; G(A, B)) = I(\hat{w}) = \sum_{Q} \hat{s}_j \hat{w}_j = \sum_{Q} W_j \geqq M^*(W; \boldsymbol{Q}_{AB}^{(f)}).$$

COROLLARY 3. *If* $W \in L_0^+(Y)$, *then* $M(W; G(A, B)) = M(W; F_0(A, B)) = M^*(W; \boldsymbol{Q}_{AB}^{(f)})$.

PROOF. Since $W \in L_0^+(Y) \subset L_2^+(Y; r)$, there exists a weak optimal solution $\hat{w}$ of (6.3) by Lemma 7. We have $\hat{w}_j = 0$ whenever $W_j = 0$. It follows from (6.4) that $\hat{Q}(A)$ is a finite set, so that $\hat{Q} \in \boldsymbol{Q}_{AB}^{(f)}$. Our assertion is now an immediate consequence of the above corollary.

COROLLARY 4. *If* $W \in L_2^+(Y; r)$ *and* $\boldsymbol{Q}_{AB} = \boldsymbol{Q}_{AB}^{(f)}$, *then* $M(W; G(A, B)) = M(W; F_0(A, B)) = M^*(W; \boldsymbol{Q}_{AB}^{(f)})$.

We shall prove

LEMMA 10. *There exists* $Q \in \boldsymbol{Q}_{AB}$ *such that* $M^*(W; \boldsymbol{Q}_{AB}) = \sum_{Q} W_j$.

PROOF. There exists a sequence $\{Q_n\}$ in $\boldsymbol{Q}_{AB}$ such that

$$\sum_{Q_n} W_j - 1/n < M^*(W; \boldsymbol{Q}_{AB}).$$

Denote by $u^{(n)}$ the characteristic function of Q_n. Since $u_\nu^{(n)} = 0$ or 1, we may assume that $\{u_\nu^{(n)}\}$ converges to u_ν for each ν, by using the diagonal method. Then $u_\nu = 0$ or 1. Let us put

$$Q(A) = \{\nu \in X; u_\nu = 0\} \quad \text{and} \quad Q(B) = \{\nu \in X; u_\nu = 1\}.$$

Then the pair of $Q(A)$ and $Q(B)$ is a dissection of X. Since $u^{(n)} = 0$ on A and $u^{(n)} = 1$ on B for all n, we have $A \subset Q(A)$ and $B \subset Q(B)$. Thus $Q = Q(A) \ominus Q(B) \in \boldsymbol{Q}_{AB}$ and

$$M^*(W; \boldsymbol{Q}_{AB}) \geqq \varliminf_{n \to \infty} \sum_{Q_n} W_j = \varliminf_{n \to \infty} \sum_{j=1}^{\infty} W_j \left| \sum_{\nu=0}^{\infty} K_{\nu j} u_\nu^{(n)} \right|$$

$$\geqq \sum_{j=1}^{\infty} W_j \left| \sum_{\nu=0}^{\infty} K_{\nu j} u_\nu \right| = \sum_{Q} W_j \geqq M^*(W; \boldsymbol{Q}_{AB}).$$

Namely we have $M^*(W; \boldsymbol{Q}_{AB}) = \sum_{Q} W_j$.

THEOREM 6. $M(W; G(A, B)) = M^*(W; \boldsymbol{Q}_{AB})$.

PROOF. We have (5.1) for any $w \in G(A, B)$ with $|w_j| \leqq W_j$ on Y and any characteristic function $u(Q)$ of $Q \in \boldsymbol{Q}_{AB}$ by Remark 1 (i). It follows that $M(W; G(A, B)) \leqq M^*(W; \boldsymbol{Q}_{AB})$ (cf. the proof of Lemma 5). On the other hand, let $\{<X^{(n)}, Y^{(n)}>\}$ be an exhaustion of $<X, Y>$ such that $A \cup B \subset X^{(n)}$ for all n. Define $W^{(n)} \in L_0^+(Y)$ by

$$W_j^{(n)} = W_j \quad \text{if} \quad j \in Y^{(n)} \quad \text{and} \quad W_j^{(n)} = 0 \quad \text{if} \quad j \in Y - Y^{(n)}.$$

We have by Corollary 3 of Lemma 9

$$\begin{aligned} (6.5) \qquad M(W; G(A, B)) &\geqq M(W^{(n)}; G(A, B)) \\ &= M^*(W^{(n)}; \boldsymbol{Q}_{AB}^{(f)}) \geqq M^*(W^{(n)}; \boldsymbol{Q}_{AB}). \end{aligned}$$

By Lemma 10, we can find $Q_n \in \boldsymbol{Q}_{AB}$ such that

$$(6.6) \qquad M^*(W^{(n)}; \boldsymbol{Q}_{AB}) = \sum_{Q_n} W_j^{(n)}.$$

Let $u^{(n)}$ be the characteristic function of Q_n. We may assume that $\{u_\nu^{(n)}\}$ converges to u_ν for each ν by using the diagonal method. There exists $Q \in \boldsymbol{Q}_{AB}$ such that u is the characteristic function of Q (cf. the proof of Lemma 10). We have by (6.5) and (6.6)

$$\begin{aligned} M(W; G(A, B)) &\geqq \varliminf_{n\to\infty} \sum_{Q_n} W_j^{(n)} = \varliminf_{n\to\infty} \sum_{j=1}^{\infty} W_j^{(n)} \Big| \sum_{\nu=0}^{\infty} K_{\nu j} u_\nu^{(n)} \Big| \\ &\geqq \sum_{j=1}^{\infty} W_j \Big| \sum_{\nu=0}^{\infty} K_{\nu j} u_\nu \Big| = \sum_{Q} W_j \geqq M^*(W; \boldsymbol{Q}_{AB}). \end{aligned}$$

This completes the proof.

COROLLARY. $M^*(W; \boldsymbol{Q}_{AB}) \leqq M(W; F_0(A, B)) \leqq M^*(W; \boldsymbol{Q}_{AB}^{(f)})$.
We have

THEOREM 7. *Let* $W \in L_2^+(Y; r)$ *and* $\{<X^{(n)}, Y^{(n)}>\}$ *be an exhaustion of* $<X, Y>$, *and put* $Z_n = Y^{(n)} - Y^{(n-1)}$ $(Y^{(0)} = \phi)$ *and* $\mu_n = \sum_{Z_n} r_j^{-1}$ $(n = 1, 2, \ldots)$. *If* $\sum_{n=1}^{\infty} \mu_n^{-1} = \infty$, *then* $M(W; G(A, B)) = M^*(W; \boldsymbol{Q}_{AB}^{(f)})$.

PROOF. Let us put for simplicity

$$a = M(W; G(A, B)) \qquad \text{and} \qquad b = M^*(W; \boldsymbol{Q}_{AB}^{(f)}).$$

In view of Lemma 5, it suffices to show that $a \geqq b$. There is a weak optimal solution $\hat{w}$ of (6.3) by Lemma 7. Let $\hat{Q}$ be the cut determined by $\hat{w}$. If $\hat{Q} \in \boldsymbol{Q}_{AB}^{(f)}$, then $a=b$ by Corollary 2 of Lemma 9. We consider the case where $\hat{Q}\notin \boldsymbol{Q}_{AB}^{(f)}$. There exist $Q_n \in \boldsymbol{Q}_{AB}^{(f)}$ such that

$$\hat{Q} \cap Y^{(n-1)} \subset Q_n \subset Y^{(n)} \tag{6.7}$$

for large n. In fact, let $\hat{Q}(A)$ and $\hat{Q}(B)$ be as in Lemma 8. For large n such that $A \cup B \subset X^{(n-1)}$, set

$$Q_n(A) = \hat{Q}(A) \cap X^{(n-1)},$$

$$Q_n(B) = (\hat{Q}(B) \cap X^{(n-1)}) \cup (X - X^{(n-1)}).$$

Then $Q_n = Q_n(A) \ominus Q_n(B)$ satisfies (6.7). We have $\sum_{Q_n} W_j \geqq b$. Let $s^{(n)} = s(Q_n)$ be the index of Q_n. Then $s_j^{(n)} = \hat{s}_j$ if j belongs to $\hat{Q} \cap Y^{(n-1)}$, and hence $W_j = s_j^{(n)} \hat{w}_j$ for $j \in \hat{Q} \cap Y^{(n-1)} = Q_n \cap Y^{(n-1)}$ by Lemma 9. On the other hand,

$$a = I(\hat{w}) = \sum_{Q_n} s_j^{(n)} \hat{w}_j$$

by Corollary 1 of Theorem 5. Hence

$$0 \leqq b - a \leqq \sum_{Q_n} W_j - \sum_{Q_n} s_j^{(n)} \hat{w}_j$$

$$= \sum_{Z_n \cap Q_n} (W_j - s_j^{(n)} \hat{w}_j) \leqq \sum_{Z_n} (W_j + |\hat{w}_j|) \leqq 2 \sum_{Z_n} W_j .$$

It follows that

$$(b-a)^2 \leqq 4\Big(\sum_{Z_n} W_j\Big)^2 \leqq 4\Big(\sum_{Z_n} r_j^{-1}\Big)\Big(\sum_{Z_n} r_j W_j^2\Big) = 4\mu_n\Big(\sum_{Z_n} r_j W_j^2\Big),$$

so that for any $m \geqq 0$

$$(b-a)^2 \sum_{k=n}^{n+m} \mu_k^{-1} \leqq 4 \sum_{k=1}^{\infty} \sum_{Z_k} r_j W_j^2 = 4H(W) < \infty .$$

Since $\sum_{n=1}^{\infty} \mu_n^{-1} = \infty$, we conclude $b=a$. This completes the proof.

COROLLARY. *If $W \in L_2^+(Y; r)$ and if there exists an exhaustion $\{<X^{(n)}, Y^{(n)}>\}$ of $<X, Y>$ such that $\sum_{n=1}^{\infty} \mu_n^{-1} = \infty$, then $M^*(W; \boldsymbol{Q}_{AB}) = M(W; F_0(A, B)) = M^*(W; \boldsymbol{Q}_{AB}^{(f)})$.*

REMARK 3. If $\mu = \sum_{j=1}^{\infty} r_j^{-1} < \infty$, then $\sum_{n=1}^{\infty} \mu_n^{-1} = \infty$ for any exhaustion $\{<X^{(n)}, Y^{(n)}>\}$ of $<X, Y>$. In fact, $\mu_n^{-1} \geqq \mu^{-1} > 0$.

We show by examples that there is a duality gap between the max-flow problem and the min-cut problem in general. First we show that Theorem 7 does not always hold if $W \notin L_2^+(Y; r)$.

EXAMPLE 3. Let us consider the infinite network (G, r) defined in Example 1 and let $A=\{0\}$ and $B=\{1\}$. Define $W \in L^+(Y)$ by

$$(6.8) \qquad W_{3n+1} = 0 \quad \text{and} \quad W_{3n+2} = W_{3n+3} = 1 \qquad (n \geqq 0).$$

Then we have $W \notin L_2^+(Y; r)$ and

$$(6.9) \qquad M(W; F_0(A, B)) = M^*(W; \boldsymbol{Q}_{AB}) = 0 < 1 = M^*(W; \boldsymbol{Q}_{AB}^{(f)}).$$

In fact, $w \in F_0(A, B)$ satisfies $|w_j| \leqq W_j$ on Y if and only if $w=0$, so that $M(W; F_0(A, B))=0$. Since $Q=\{3n+1; n \geqq 0\} \in \boldsymbol{Q}_{AB}$, we obtain $M^*(W; \boldsymbol{Q}_{AB})=0$. If $Q \in \boldsymbol{Q}_{AB}^{(f)}$, then Q must contain either arc $3n+2$ or arc $3n+3$ for some n. Hence $M^*(W; \boldsymbol{Q}_{AB}^{(f)})=1$. Note that

$$\sum_{n=1}^{\infty} \mu_n^{-1} = \infty$$

for the elementary exhaustion of (G, r) starting from A.

Next we show that the condition $\sum_{n=1}^{\infty} \mu_n^{-1} = \infty$ can not be omitted in Theorem 7.

EXAMPLE 4. Consider the infinite graph G defined in Example 1 and define $r \in L(Y)$ by

$$r_{3n+1} = r_{3n+2} = r_{3n+3} = 2^{-n} \qquad (n \geqq 0).$$

Consider $W \in L(Y)$ defined by (6.8). Then $W \in L_2^+(Y; r)$ and (6.9) holds with $A=\{0\}$ and $B=\{1\}$. We remark that $\sum_{n=1}^{\infty} \mu_n^{-1} < \infty$ for any exhaustion of (G, r).

Finally we show that each of problems (6.2) and (6.3) has no optimal solution in general.

EXAMPLE 5. Let (G, r) be the infinite network defined in Example 1 and let $A=\{0\}$ and $B=\{1\}$. Define $W \in L^+(Y)$ by

$$W_{3n+1} = 4^{-n} \quad \text{and} \quad W_{3n+2} = W_{3n+3} = 2^{-n} \qquad (n \geqq 0).$$

Then $M(W; G(A, B))=M^*(W; \boldsymbol{Q}_{AB}^{(f)})=4/3$. It is easily seen that $I(w)<4/3<\sum_Q W_j$ for any $w \in G(A, B)$ such that $|w_j| \leqq W_j$ on Y and any $Q \in \boldsymbol{Q}_{AB}^{(f)}$.

§7. The extremal widths of a network

Let A and B be mutually disjoint nonempty finite subsets of X and define the extremal widths $EW=EW(A, B)$, $EW_f=EW_f(A, B)$ and $EW_f^*=EW_f^*(A, B)$ of an infinite network $<X, Y, r>$ relative to two sets A and B as the values of the following extremum problems.

(7.1) Find $$EW^{-1} = \inf\{H(W);\ W\in E_{AB}^*\},$$

where

$$E_{AB}^* = \{W\in L_2^+(Y; r);\ \sum_Q W_j \geqq 1 \text{ for all } Q\in \boldsymbol{Q}_{AB}\}.$$

(7.2) Find $$EW_f^{-1} = \inf\{H(W);\ W\in E_{AB}^{*(f)}\},$$

where

$$E_{AB}^{*(f)} = \{W\in L_2^+(Y; r);\ \sum_Q W_j \geqq 1 \text{ for all } Q\in \boldsymbol{Q}_{AB}^{(f)}\}.$$

(7.3) Find $$EW_f^{*-1} = \inf\{H(W);\ W\in G_{AB}^*\},$$

where G_{AB}^* denotes the closure of the intersection of $E_{AB}^{*(f)}$ and $L_0(Y)$ in $L_2(Y; r)$.

It is clear that $EW\leqq EW_f$ and $EW_f^*\leqq EW_f$. Since $E_{AB}^{*(f)}$ and G_{AB}^* are nonempty closed convex subsets of $L_2(Y; r)$, each of problems (7.2) and (7.3) has a unique optimal solution by Theorem A.

In connection with the above problems, we consider the following two extremum problems.

(7.4) Find

$$d_0^* = \inf\{H(w);\ w\in F_0(A, B) \text{ and } I(w) = 1\}.$$

(7.5) Find

$$d^* = \inf\{H(w);\ w\in F(A, B) \text{ and } I(w) = 1\}.$$

It is clear that $d^*\leqq d_0^*$. Since the sets $\{w\in F_0(A, B); I(w)=1\}$ and $\{w\in F(A, B); H(w)<\infty \text{ and } I(w)=1\}$ are nonempty, convex and closed in $L_2(Y; r)$, each of problems (7.4) and (7.5) has a unique optimal solution by Theorem A.

We shall prove

THEOREM 8. $d_0^*=EW_f^{*-1}$.

PROOF. Let $\hat{w}$ be the optimal solution of problem (7.4), i.e., $\hat{w}\in F_0(A, B)$, $I(\hat{w})=1$ and $d_0^*=H(\hat{w})$. There exists a sequence $\{w^{(n)}\}$ in $F(A, B)$ such that $w^{(n)}\in L_0(Y)$ and $H(\hat{w}-w^{(n)})\to 0$ as $n\to\infty$. We note that $I(w^{(n)})\to I(\hat{w})$ as $n\to\infty$. Hence we may assume that $I(w^{(n)})>0$. Define $\hat{W}$ and $W^{(n)}$ by

$$\hat{W}_j = |\hat{w}_j| \quad \text{and} \quad W_j^{(n)} = |w_j^{(n)}|/I(w^{(n)}).$$

Then $W^{(n)} \in L_0^+(Y)$ and $W^{(n)} \in E_{AB}^{*(f)}$ by Corollary 2 of Theorem 5. Since

$$H(\hat{W}-W^{(n)}) \leqq 2H(\hat{w}-w^{(n)})+2(1-1/I(w^{(n)}))^2 H(w^{(n)}) \to 0$$

as $n\to\infty$, we derive that $\hat{W} \in G_{AB}^*$ and

$$EW_f^{*-1} \leqq H(\hat{W}) = H(\hat{w}) = d_0^*.$$

On the other hand, let $\overline{W}$ be the optimal solution of problem (7.3), i.e., $\overline{W} \in G_{AB}^*$ and $EW_f^{*-1} = H(\overline{W})$. There exists a sequence $\{\overline{W}^{(n)}\}$ in $E_{AB}^{*(f)}$ such that $\overline{W}^{(n)} \in L_0^+(Y)$ and $H(\overline{W}-\overline{W}^{(n)}) \to 0$ as $n\to\infty$. We have by Corollary 3 of Lemma 9

$$M(\overline{W}^{(n)}; F_0(A, B)) = M^*(\overline{W}^{(n)}; \boldsymbol{Q}_{AB}^{(f)}) \geqq 1.$$

There exists $\overline{w}^{(n)} \in F_0(A, B)$ such that $|\overline{w}_j^{(n)}| \leqq \overline{W}_j^{(n)}$ on Y and $I(\overline{w}^{(n)}) = M(\overline{W}^{(n)}; F_0(A, B))$ by Lemma 6. Then

$$d_0^* \leqq H(\overline{w}^{(n)}/I(\overline{w}^{(n)})) \leqq H(\overline{w}^{(n)}) \leqq H(\overline{W}^{(n)}).$$

Therefore we have

$$d_0^* \leqq \lim_{n\to\infty} H(\overline{W}^{(n)}) = H(\overline{W}) = EW_f^{*-1},$$

and hence $d_0^* = EW_f^{*-1}$.

For the relation among EW_f^*, EW_f and EW, we have first of all

THEOREM 9. $EW = EW_f^*$.

PROOF. First we shall prove $E_{AB}^{*(f)} \cap L_0(Y) \subset E_{AB}^*$. Let W be an element of $E_{AB}^{*(f)} \cap L_0(Y)$. Consider a finite subnetwork $<X', Y'>$ of $<X, Y>$ such that $A \cup B \subset X'$ and $SW \subset Y'$. Let $Q \in \boldsymbol{Q}_{AB}$ and $Q = Q(A) \ominus Q(B)$. In case $Q \in \boldsymbol{Q}_{AB}^{(f)}$, we have $\sum_Q W_j \geqq 1$ by our assumption that $W \in E_{AB}^{*(f)}$. In case $Q \notin \boldsymbol{Q}_{AB}^{(f)}$, we put

$$Q'(A) = Q(A) \cap X',$$

$$Q'(B) = (Q(B) \cap X') \cup (X - X').$$

Then $Q' = Q'(A) \ominus Q'(B) \in \boldsymbol{Q}_{AB}^{(f)}$ and $Q \cap Y' = Q' \cap Y'$. We have

$$1 \leqq \sum_{Q'} W_j = \sum_{Q' \cap Y'} W_j = \sum_{Q \cap Y'} W_j = \sum_{Q} W_j.$$

Therefore $W \in E_{AB}^*$. Now we shall show that $EW_f^* \leqq EW$. Let $\overline{W}$ be the optimal solution of (7.3). There exists a sequence $\{W^{(n)}\}$ in $E_{AB}^{*(f)} \cap L_0(Y)$ such that $H(\overline{W}-W^{(n)}) \to 0$ as $n\to\infty$. Then $W^{(n)} \in E_{AB}^*$ by the above observation, so that $EW^{-1} \leqq H(W^{(n)})$. Thus

$$EW^{-1} \leqq \lim_{n\to\infty} H(W^{(n)}) = H(\overline{W}) = EW_f^{*-1}.$$

On the other hand, let $W \in E^*_{AB}$. There exists $w \in F_0(A, B)$ such that $|w_j| \leqq W_j$ on Y and $I(w) = M(W; F_0(A, B))$ by Lemma 6. Since $M(W; F_0(A, B)) \geqq M^*(W; \boldsymbol{Q}_{AB}) \geqq 1$ by the Corollary of Theorem 6 and our assumption that $W \in E^*_{AB}$, we have

$$EW_f^{*-1} = d_0^* \leqq H(w/I(w)) \leqq H(w) \leqq H(W)$$

by Theorem 8. By the arbitrariness of $W \in E^*_{AB}$, we obtain $EW_f^{*-1} \leqq EW^{-1}$. This completes the proof.

THEOREM 10. *Let $\hat{W}$ be the optimal solution of* (7.2), *i.e.,* $\hat{W} \in E^{*(f)}_{AB}$ *and* $EW_f^{-1} = H(\hat{W})$. *If* $M(\hat{W}; F_0(A, B)) = M^*(\hat{W}; \boldsymbol{Q}^{(f)}_{AB})$, *then* $EW_f^* = EW = EW_f$.

PROOF. By Theorem 8, $d_0^* = EW_f^{*-1} \geqq EW_f^{-1}$. On the other hand there exists $w \in F_0(A, B)$ such that $|w_j| \leqq \hat{W}_j$ on Y and $I(w) = M(\hat{W}; F_0(A, B))$ by Lemma 6. Since $M(\hat{W}; F_0(A, B)) = M^*(\hat{W}; \boldsymbol{Q}^{(f)}_{AB}) = 1$ by our assumption, we have $I(w) = 1$ and

$$d_0^* \leqq H(w) \leqq H(\hat{W}) = EW_f^{-1}.$$

Therefore $d_0^* = EW_f^{-1}$. Thus $EW_f^* = EW = EW_f$ by Theorem 9.

By the Corollary of Theorem 7 and Theorem 10, we have

COROLLARY 1. *If there exists an exhaustion* $\{\langle X^{(n)}, Y^{(n)} \rangle\}$ *of* $\langle X, Y \rangle$ *such that*

$$\sum_{n=1}^{\infty} \mu_n^{-1} = \infty,$$

then $EW_f^* = EW = EW_f$.

By Corollary 4 of Lemma 9 and Theorem 10, we have

COROLLARY 2. If $\boldsymbol{Q}_{AB} = \boldsymbol{Q}^{(f)}_{AB}$, then $EW_f^* = EW = EW_f$.

Next we shall investigate relations among d, d_0^* and d^*.

LEMMA 11. *Let $\hat{w}$ be the optimal solution of problem* (7.4) *or* (7.5). *Then $\hat{w}$ is a passive flow, i.e.,*

$$\sum_{j=1}^{\infty} r_j \hat{w}_j C_{kj} = 0 \tag{7.6}$$

for any circuit C_k, where C_{kj} is the circuit-arc incidence matrix defined in §1.

PROOF. Let C_k be a circuit and define $w^{(k)} \in L_0(Y)$ by $w_j^{(k)} = C_{kj}$. Then $\hat{w} \pm w^{(k)}$ belong to $F_0(A, B)$ or $F(A, B)$ and $I(\hat{w} \pm w^{(k)}) = 1$, so that $\langle \hat{w}, w^{(k)} \rangle = 0$ by Theorem A. This leads to (7.6).

Let $\hat{w}$ be the optimal solution of problem (7.4). For $\alpha \in A$, we define $v^{(\alpha)} \in L(X)$ by

$$v_\alpha^{(\alpha)} = 0 \quad \text{and} \quad v_\nu^{(\alpha)} = \sum_{j=1}^{\infty} r_j p_j(P) \hat{w}_j \qquad (\nu \neq \alpha) \tag{7.7}$$

for some $P \in \boldsymbol{P}_{\alpha\nu}$, where $p(P)$ is the index of P. It follows from Lemma 11 that $v^{(\alpha)}$ is uniquely determined by $\hat{w}$ and independent of the choice of $P \in \boldsymbol{P}_{\alpha\nu}$. Define $\hat{v} \in L(X)$ by

$$\hat{v}_\nu = \inf\{|v_\nu^{(\alpha)}|;\, \alpha \in A\}. \tag{7.8}$$

We have

LEMMA 12. *Let $\hat{w}$ be the optimal solution of problem* (7.4) *and $\hat{v}$ be the function defined by* (7.7) *and* (7.8). *Then $\hat{v}=0$ on A, $\hat{v}=d_0^*$ on B and*

$$\left|\sum_{\nu=0}^{\infty} K_{\nu j} \hat{v}_\nu\right| \leqq r_j |\hat{w}_j| \tag{7.9}$$

for each $j \in Y$.

PROOF. It is clear that $\hat{v}=0$ on A, since $v_\alpha^{(\alpha)}=0$ for any $\alpha \in A$. Let $\alpha \in A$, $\beta \in B$ and $P \in \boldsymbol{P}_{\alpha\beta}$. Then $p(P) \in F_0(A, B)$ and $I(p(P))=1$. We have by Theorem A and (7.7)

$$d_0^* = H(\hat{w}) = \langle \hat{w}, p(P) \rangle = \sum_{j=1}^{\infty} r_j p_j(P) \hat{w}_j = v_\beta^{(\alpha)}, \tag{7.10}$$

and hence $\hat{v}_\beta = d_0^*$. Therefore $\hat{v}=d_0^*$ on B. Next we show (7.9). Let $k \in Y$, $K_{ak}=-1$ and $K_{bk}=1$. In case $a \in A$ and $b \in A$, we have $\hat{v}_a=\hat{v}_b=0$, and hence $|\hat{v}_a-\hat{v}_b|=0 \leqq r_k|\hat{w}_k|$. In case $a \in A$ and $b \notin A$, we consider $P \in \boldsymbol{P}_{ab}$ defined by $C_X(P)=\{a, b\}$, $C_Y(P)=\{k\}$, $p_j(P)=0$ if $j \neq k$ and $p_k(P)=1$. Then we have $v_b^{(a)}= r_k\hat{w}_k$, and hence $0 \leqq \hat{v}_b \leqq r_k|\hat{w}_k|$. Thus $|\hat{v}_a-\hat{v}_b|=\hat{v}_b \leqq r_k|\hat{w}_k|$. In case $a \notin A$ and $b \in A$, we have (7.9) similarly. Finally we consider the case where $a \notin A$ and $b \notin A$. There exists $\alpha \in A$ such that $\hat{v}_a=|v_a^{(\alpha)}|$, since A is a finite set. Consider $P \in \boldsymbol{P}_{\alpha a}$ and let $\bar{P}$ be the path from node α to node b which is generated by P and $\{k\}$ (cf. the proof of Lemma 9). Then

$$v_b^{(\alpha)} = \sum_{j=1}^{\infty} r_j p_j(\bar{P}) \hat{w}_j = \sum_{j=1}^{\infty} r_j p_j(P) \hat{w}_j + r_k \hat{w}_k = v_a^{(\alpha)} + r_k \hat{w}_k,$$

so that

$$\hat{v}_b \leqq |v_a^{(\alpha)} + r_k \hat{w}_k| \leqq |v_a^{(\alpha)}| + r_k |\hat{w}_k| = \hat{v}_a + r_k |\hat{w}_k|.$$

Interchanging the roles of node a and node b in the above discussion, we obtain

$\hat{v}_a \leqq \hat{v}_b + r_k|\hat{w}_k|$. Therefore $|\hat{v}_a - \hat{v}_b| \leqq r_k|\hat{w}_k|$. This completes the proof.

We shall prove

THEOREM 11. $d d_0^* = 1$.

PROOF. First we show $1 \leqq d d_0^*$. For any $v \in \boldsymbol{D}^A$ such that $v=1$ on B and any $w \in F_0(A, B)$ such that $I(w)=1$, we have by Theorem 5 and Lemma 4

$$(7.11) \qquad 1 = I(w) = \sum_{\nu=0}^{\infty} v_\nu \Big(\sum_{j=1}^{\infty} K_{\nu j} w_j \Big) = \sum_{j=1}^{\infty} w_j \Big(\sum_{\nu=0}^{\infty} K_{\nu j} v_\nu \Big) \leqq D(v) H(w),$$

which leads to the desired inequality. Next we show $d d_0^* \leqq 1$. Let $\hat{w}$ be the optimal solution of problem (7.4) and $\hat{v}$ be the function defined by (7.7) and (7.8). Then we have by (7.9)

$$(7.12) \qquad D(\hat{v}) = \sum_{j=1}^{\infty} r_j^{-1} \Big(\sum_{\nu=0}^{\infty} K_{\nu j} \hat{v}_\nu \Big)^2 \leqq \sum_{j=1}^{\infty} r_j \hat{w}_j^2 = H(\hat{w}) = d_0^*.$$

Writing $\hat{u} = \hat{v}/d_0^*$, we have $\hat{u}=0$ on A and $\hat{u}=1$ on B by Lemma 12, so that

$$d \leqq D(\hat{u}) = D(\hat{v})/(d_0^*)^2 \leqq d_0^*/(d_0^*)^2 = 1/d_0^*$$

by (7.12). Hence $d d_0^* \leqq 1$. This completes the proof.

On account of Theorems 4, 8, 9 and 11, we have

THEOREM 12. $(EL)(EW)=1$.

For the relation between d^* and d_0^*, we have

THEOREM 13. *If* $\boldsymbol{D}^A = \boldsymbol{D}_0^A$, *then* $d^* = d_0^*$.

PROOF. In view of the inequality $d^* \leqq d_0^*$ and Theorem 11, it suffices to prove that $d d^* \geqq 1$. For any $v \in \boldsymbol{D}^A$ such that $v=1$ on B and any $w \in F(A, B)$ such that $I(w)=1$ and $H(w) < \infty$, we have (7.11) by Lemma 4 and Remark 1 (iii), since $\boldsymbol{D}^A = \boldsymbol{D}_0^A$ and A and B are finite sets.

It is not always valid that $d^* = d_0^*$. This is shown by

EXAMPLE 6. Let us consider the infinite graph G defined in Example 2. Let $A=\{0\}$ and $B=\{1\}$ and define $r \in L(Y)$ by

$$r_1 = 1 \quad \text{and} \quad r_{2n} = r_{2n+1} = 4^{-n} \qquad (n \geqq 1).$$

We see easily that $w \in F_0(A, B)$ and $I(w)=1$ if and only if $w_1=1$ and $w_j=0$ $(j \neq 1)$. Therefore $d_0^*=1$. On the other hand, consider $\bar{w} \in L(Y)$ defined by

$$\bar{w}_1 = 0 \quad \text{and} \quad \bar{w}_{2n} = \bar{w}_{2n+1} = 1 \qquad (n \geqq 1).$$

Then $\bar{w} \in F(A, B)$, $I(\bar{w})=1$ and

$$d^* \leqq H(\bar{w}) = 2\sum_{n=1}^{\infty} 4^{-n} = 2/3.$$

§8. The path-cut inequality

Duffin showed a path-cut inequality on a finite network by using his theorem which states that the extremal length and the extremal width are reciprocals to each other. We are concerned with the inequality on an infinite network in this section.

Let A and B be mutually disjoint nonempty finite subsets of X, $W \in L^+(Y)$ and $V \in L^+(Y)$.

We have

THEOREM 14. *Assume that* $W \in L_2^+(Y; r)$ *and* $\sum_{j=1}^{\infty} r_j^{-1} V_j^2 < \infty$. *Then*

$$\sum_{j=1}^{\infty} W_j V_j \geqq N(A, B; V) M^*(W; \boldsymbol{Q}_{AB}), \tag{8.1}$$

or equivalently,

$$\sum_{j=1}^{\infty} W_j V_j \geqq (\inf\{\sum_P V_j; P \in \boldsymbol{P}_{AB}\})(\inf\{\sum_Q W_j; Q \in \boldsymbol{Q}_{AB}\}).$$

PROOF. There exists $u \in L(X)$ such that

$$u = 0 \quad \text{on} \quad A, \ |\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu| \leqq V_j \quad \text{on} \quad Y \quad \text{and}$$

$$N(A, B; V) = N^*(A, B; V) = \inf\{u_\nu; \nu \in B\}$$

by Theorem 3. Let $v = \min(u, N(A, B; V))$. Then $v = 0$ on A, $v = N(A, B; V)$ on B and

$$|\sum_{\nu=0}^{\infty} K_{\nu j} v_\nu| \leqq |\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu| \leqq V_j \quad \text{on} \quad Y.$$

Note that

$$D(v) = \sum_{j=0}^{\infty} r_j^{-1} (\sum_{\nu=0}^{\infty} K_{\nu j} v_\nu)^2 \leqq \sum_{j=1}^{\infty} r_j^{-1} V_j^2 < \infty.$$

There exists $w \in F_0(A, B)$ such that $|w_j| \leqq W_j$ on Y and $M(W; F_0(A, B)) = I(w)$ by Lemma 6. We have by Theorem 5 and the Corollary of Theorem 6

$$\begin{aligned} N(A, B; V) M^*(W; \boldsymbol{Q}_{AB}) &\leqq N(A, B; V) M(W; F_0(A, B)) \\ &= \sum_{\nu \in B} v_\nu (\sum_{j=1}^{\infty} K_{\nu j} w_j) \\ &= \sum_{j=1}^{\infty} w_j (\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu) \leqq \sum_{j=1}^{\infty} W_j V_j. \end{aligned} \tag{8.2}$$

This completes the proof.

By using Remark 1 (ii) instead of Theorem 5 in the proof of (8.2), we have

THEOREM 15. *If* $\sum_{j=1}^{\infty} W_j < \infty$, *then* (8.1) *holds.*

Similarly we can prove

THEOREM 16. *If* $W \in L_0^+(Y)$, *then*

$$\sum_{j=1}^{\infty} W_j V_j \geqq N(A, B; V) M^*(W; Q_{AB}^{(f)}). \tag{8.3}$$

THEOREM 17. *Assume that* $W \in L_2^+(Y; r)$ *and* $\sum_{j=1}^{\infty} r_j^{-1} V_j^2 < \infty$. *If* $M(W; F_0(A, B)) = M^*(W; Q_{AB}^{(f)})$, *then* (8.3) *holds.*

We remark that (8.3) does not hold in general. In fact, we give

EXAMPLE 7. Consider the infinite network (G, r) defined in Example 1 and let $A = \{0\}$ and $B = \{1\}$. Define W and V by

$$W_{3n+1} = 0, \quad W_{3n+2} = W_{3n+3} = 1,$$

$$V_{3n+1} = 1, \quad V_{3n+2} = V_{3n+3} = 0.$$

Then we have

$$\sum_{j=1}^{\infty} W_j V_j = 0 < 1 = N(A, B; V) M^*(W; Q_{AB}^{(f)}).$$

Note that $M(W; F_0(A, B)) = 0 < 1 = M^*(W; Q_{AB}^{(f)})$ (cf. Example 3).

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School of Engineering,
Okayama University