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Note on Self-Maps Inducing the Identity Automorphisms of Homotopy Groups

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§1 Introduction

Let X be a connected CW-complex with the base point $*$. Then we can consider the group $\mathscr{E}(X)$ of all based homotopy classes of self homotopy equivalences of $(X, *)$. This group has been studied by several authors.

Also, we can consider the subgroup

$$(1.1) \qquad \mathscr{E}_{\#}(X) \qquad (\subset \mathscr{E}(X)),$$

formed by self-maps of $(X, *)$ inducing the identity automorphisms of all homotopy groups $\pi_i(X)$. Then $\mathscr{E}_{\#}(X)=1$ means that a (continuous) map $\psi\colon (X, *)\to (X, *)$ is homotopic rel $*$ to the identity map if and only if ψ induces the identity automorphisms, that is,

$$\psi_* = \mathrm{id}\colon \pi_i(X) \longrightarrow \pi_i(X) \qquad \text{for all} \quad i.$$

The purpose of this note is to study a sufficient condition for $\mathscr{E}_{\#}(X)=1$.

Let $\{X_n|n\geqq 1\}$ be a Postnikov system of X, that is, X_n be a CW-complex obtained by attaching $(i+1)$-cells $(i>n)$ to X so that X_n kills the homotopy groups $\pi_i(X)$ for $i>n$. Then we can consider the cohomology group

$$(1.2) \qquad H^n(X_{n-1}; \pi_n(X)), \text{ with the local coefficient,}$$

where $\pi_1(X_{n-1})\cong\pi_1(X)$ acts on the coefficient $\pi_n(X)$ by the usual action of the fundamental group in X.

Our main result is stated as follows:

THEOREM 1.3. *Assume that a connected CW-complex X satisfies*

$$\pi_i(X) = 0 \ (i > N) \quad or \quad \dim X = N, \qquad for\ some\ integer\ N,$$

and that the cohomology groups of (1.2) *are*

$$H^n(X_{n-1}; \pi_n(X)) = 0 \qquad (1 < n \leqq N).$$

Then, the group $\mathscr{E}_{\#}(X)$ of (1.1) *consists only of the identity* 1.

Our proof is based on the considerations of D. H. Gottlieb [4], who studies fibre homotopy equivalences of a Hurewicz fibering by using the classifying space of J. Stasheff [11], and on the usual obstruction theory. After studying $K(\pi, n)$-fibrations in § 2, we prove the above theorem in § 3. Also, we give in § 4 some examples for $\mathscr{E}_{\#}(X)=1$.

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§ 2. K(π, n)-fibrations

In this section, we consider a Hurewicz fibration

$$p\colon E \longrightarrow B, \qquad F = p^{-1}(*) \qquad (* \in B), \tag{2.1}$$

where the base space B is a connected CW-complex and the fibre F has the homotopy type of a connected CW-complex.

Let $\mathscr{L}(E)$ be the group of all fibre homotopy classes of fibre homotopy equivalences of (2.1), and F^F be the space of all homotopy equivalences of F with the compact-open topology. Then, the homomorphism

$$r\colon \mathscr{L}(E) \longrightarrow \pi_0(F^F) \tag{2.2}$$

is defined as follows: For any fibre homotopy equivalence $\varphi\colon E \to E$ of (2.1), $r[\varphi]$ is represented by the restriction $\varphi|F \in F^F$.

The above homomorphism r is restated as follows. Let

$$p_\infty\colon E_\infty \longrightarrow B_\infty, \text{ with the fibre } F, \tag{2.3}$$

be a universal fibration in the sence of J. Stasheff [11], and

$$k\colon B \longrightarrow B_\infty \tag{2.4}$$

be a classifying map for the given fibration (2.1). In fact, we can choose (2.3) so that k is an inclusion and (2.1) is the restriction of (2.3) to B (cf. [4, p. 45]). Also let $L(B, B_\infty; k) \in k$ be the path component of the space $L(B, B_\infty)$ of all maps from B to B_∞ with the compact-open topology, and consider the evaluation map

$$\omega\colon L(B, B_\infty; k) \longrightarrow B_\infty, \qquad \omega(f) = f(*). \tag{2.5}$$

Then, the following lemma is easily seen by the considerations of [4, §§ 3–4].

Lemma 2.6. *The homomorphism r of (2.2) is equal to the composition*

$$\mathscr{L}(E) \cong \pi_1(L(B, B_\infty; k)) \xrightarrow{\omega_*} \pi_1(B_\infty) \cong \pi_0(F^F),$$

where the first isomorphism is the one given by [4, Th. 1] *and the last is the special one for* $B=*$.

Now, we consider the case that

(2.7) the fibre F of (2.1) is an Eilenberg-MacLane space $K(\pi, n)$ for some integer $n \geqq 2$.

Then the following lemma is proved by C. A. Robinson [7, § 2].

LEMMA 2.8. *For the case* (2.7), *the following are valid.*

(i) *The homotopy groups of the classifying space* B_∞ *of* (2.3) *are given by*

$$\pi_1(B_\infty) = \operatorname{aut}\pi, \quad \pi_{n+1}(B_\infty) = \pi, \quad \pi_i(B_\infty) = 0 \qquad (i \neq 1, n+1),$$

and the usual action of π_1 *on* π_{n+1} *coincides with the usual action of the automorphism group* $\operatorname{aut}\pi$ *on* π.

(ii) *Furthermore, the usual action of* $\alpha \in \pi_1(B)$ *on* $\pi = \pi_n(F)$ *in the fibration* (2.1) *coincides with the action of* $k_*(\alpha) \in \pi_1(B_\infty)$ *on* π *in* (i), *where* k *is the classifying map of* (2.4).

By these considerations, we can prove the following

PROPOSITION 2.9. *Assume that a fibration* (2.1) *satisfies* (2.7), *and*

$$H^n(B; \pi) \qquad (\text{with the local coefficient } \pi) = 0,$$

where the action of $\pi_1(B)$ *on* π *is the one in* (ii) *of the above lemma.*

Then, the homomorphism ω_* *in Lemma* 2.6 *is monic.*

PROOF. Since the range $\pi_1(L(B, B_\infty; k))$ of ω_* is equal to the quasi-homotopy group $Q_1(L(B, B_\infty; k))$ by [4, p. 44, Cor.], it is sufficient to prove the following fact by the definition (2.5) of ω:

Any map $h: S^1 \times B \to B_\infty$, satisfying

$$h(*, x) = x\ (= k(x)), \qquad h(t, *) = *\ (x \in B, t \in S^1)$$

is homotopic rel $S^1 \vee B$ to the map

$$\bar{k}: S^1 \times B \longrightarrow B_\infty, \qquad \bar{k}(t, x) = x.$$

For any $x \in B^0$ (the 0-skeleton of B), there is a path l_x of B connecting x with $*$. Then, $h|S^1 \times x$ and $\bar{k}|S^1 \times x$ are homotopic to $h|S^1 \times * = \bar{k}|S^1 \times *$ by the homotopies $h(t, l_x(s))$ and $\bar{k}(t, l_x(s))$ $(s \in I)$, respectively, and hence $h|S^1 \times x$ is homotopic rel $(*, x)$ to $\bar{k}|S^1 \times x$. Therefore, we see that h is homotopic rel $S^1 \vee B$ to $\bar{k}$ on

$$(S^1 \vee B) \cup (S^1 \times B^0) = (S^1 \vee B) \cup (S^1 \times B)^1,$$

where $(S^1 \times B)^i$ is the i-skeleton of the product complex $S^1 \times B$.

Furthermore, we consider the obstructions to the skeleton-wise extensions of homotopies between h and $\bar{k}$ rel $S^1 \vee B$. Then, by Lemma 2.8 (i) and [6, (16.3)], we see that the only obstruction lies in the cohomology group

$$H^{n+1}(S^1 \times B, S^1 \vee B; \pi)$$

with the local coefficient π, where $\beta \in \pi_1(S^1 \times B)$ acts on $\pi = \pi_{n+1}(B_\infty)$ by the action of $\bar{k}_*(\beta) \in \pi_1(B_\infty)$. Since the above cohomology group is isomorophic to $H^n(B; \pi)$ of the proposition by [6, (4.8)], we have the desired result. *q.e.d.*

For the case that

(2.10) $F = K(\pi, n)$, $\pi_i(B) = 0$ $(i \geqq n)$ for some $n \geqq 2$, it is easy to see that

(2.11) the local coefficient of the cohomology group $H^n(B; \pi)$ in the above proposition is given by the usual action of $\pi_1(E) \cong \pi_1(B)$ on $\pi_n(E) \cong \pi_n(F) = \pi$.

By Lemma 2.6 and Proposition 2.9, we have the following

PROPOSITION 2.12. *Assume* (2.10) *and let the cohomology group* $H^n(B; \pi)$ *of* (2.11) *vanish. If a fibre map*

$$\varphi: (E, *) \longrightarrow (E, *), \quad p \circ \varphi = p, \quad (p(*) = *)$$

of (2.1) *induces the identity automorphisms of all homotopy groups* $\pi_i(E)$, *then* φ *is homotopic rel* $*$ *to the identity map* id_E.

PROOF. It is well known that E has the homotopy type of a CW-complex, (cf. [11, Prop. (0)]). Therefore the assumption $\varphi_* = \mathrm{id}$ implies that φ is a homotopy equivalence by the theorem of J. H. C. Whitehead, and so φ is a fibre homotopy equivalence by the result of A. Dold [1, Th. 6.1]. Also, the restriction $\varphi|F$ is homotopic to id_F, since $F = K(\pi, n)$ and $(\varphi|F)_* = \mathrm{id}: \pi \to \pi$. On the other hand, the homomorphism r of (2.2) is monic by Lemma 2.6 and Proposition 2.9. These show that φ is homotopic to id_E by a fibre homotopy φ_t. Then the loop $\varphi_t(*)$ $(t \in I)$ of F is homotopic to the constant loop since $\pi_1(F) = 0$, and so φ is homotopic rel $*$ to id_E by the homotopy extension theorem. *q.e.d.*

§ 3. Proof of Theorem 1.3

Now, we consider a connected CW-complex X and its Postnikov system

$$\{X_n | n \geqq 1\},$$

defined as follows: Let X_n be a CW-complex obtained by attaching $(i+1)$-cells

$(i>n)$ to X, so that X_n kills the homotopy groups $\pi_i(X)$ for $i>n$, that is,

(3.1) $$j_{n*}\colon \pi_i(X) = \pi_i(X_n) \quad (i \leqq n), \qquad \pi_i(X_n) = 0 \quad (i > n),$$

where $j_n\colon X \to X_n$ is the inclusion.

Then, we can consider the cohomology group

(3.2) $H^n(X_{n-1}; \pi_n(X))$, with the local coefficient,

where $\pi_1(X_{n-1}) = \pi_1(X)$ acts on $\pi_n(X)$ as usual. It is a subgroup of the cohomology group $H^n(X; \pi_n(X))$ with the local coefficient.

THEOREM 3.3. *Assume that a connected CW-complex X satisfies*

$$\pi_i(X) = 0 \quad (i > N) \quad \textit{for some integer } N.$$

If the cohomology groups of (3.2) *are*

$$H^n(X_{n-1}; \pi_n(X)) = 0 \qquad (1 < n \leqq N),$$

then any map $\psi\colon (X, *) \to (X, *)$, *satisfying*

$$\psi_* = \mathrm{id}\colon \pi_i(X) \longrightarrow \pi_i(X) \qquad (1 \leqq i \leqq N),$$

is homotopic rel $*$ *to the identity map* id_X.

PROOF. By the assumption, we take $X_N = X$. By the obstruction theory and the definition of X_n, it is easy to see that there is an extension

(3.4) $$f_n\colon X_n \longrightarrow X_{n-1}, \qquad f_n \circ j_n = j_{n-1} \quad (1 < n \leqq N),$$

and we have a Hurewicz fibration

$$p_n\colon X'_n \longrightarrow X_{n-1}$$

induced from f_n, i.e., there is a based homotopy equivalence

$$h_n\colon X_n \longrightarrow X'_n \text{ such that } p_n \circ h_n \sim f_n \text{ rel} *.$$

Then, p_n satisfies (2.10) for $\pi = \pi_n(X)$ by (3.1) and (3.4).

Also for a given map ψ, we see easily by the obstruction theory that there is a map

$$\psi_n\colon X_n \longrightarrow X_n, \qquad \psi_n \circ j_n = j_n \circ \psi \quad (1 \leqq n \leqq N),$$

and it holds $f_n \circ \psi_n \sim \psi_{n-1} \circ f_n$ rel $*$. Therefore, we have a map $\psi'_n\colon (X'_n, *) \to (X'_n, *)$ such that

$$\psi'_n \circ h_n \sim h_n \circ \psi_n \text{ rel} *, \qquad p_n \circ \psi'_n = \psi_{n-1} \circ p_n.$$

By the assumptions $\psi_*=\mathrm{id}$ and (3.1), we see $\psi_{n*}=\mathrm{id}$ and so $\psi'_{n*}=\mathrm{id}$ since h_n is a based homotopy equivalence.

Now, we prove $\psi=\psi_N\sim\mathrm{id}$ rel $*$ by showing $\psi_n\sim\mathrm{id}$ rel $*$ inductively. ψ_1 is so since $X_1=K(\pi_1(X), 1)$ and $\psi_{1*}=\mathrm{id}$. Assume $\psi_{n-1}\sim\mathrm{id}$ rel $*$, then ψ'_n is homotopic rel $*$ to a fibre map ψ''_n of the fibration p_n and $\psi''_{n*}=\psi'_{n*}=\mathrm{id}$. Therefore, we see $\psi''_n\sim\mathrm{id}$ rel $*$ by Proposition 2.12, and so $\psi_n\sim\mathrm{id}$ rel $*$ as desired. *q.e.d.*

PROOF OF THEOREM 1.3. The case $\pi_i(X)=0$ $(i>N)$ is the above theorem. For the case $\dim X=N$, we consider a CW-complex X_N defined as in (3.1), Then, we see easily by the elementary homotopy theory that the inclusion j_N: $X\to X_N$ induces an isomorphism

$$\mathscr{E}(X)\cong\mathscr{E}(X_N)$$

(cf. [8, Lemma 7.1]). Therefore, we have the desired result by the above theorem. *q.e.d.*

As a special case, we consider a connected CW-complex X satisfying

$$\pi_i(X)=0 \qquad (i\neq n, m) \tag{3.4}$$

or

$$\dim X=m, \qquad \pi_i(X)=0 \quad (i=n, i<m), \tag{3.5}$$

for some $m>n\geqq 1$. Then, we see $X_{m-1}=K(\pi_n(X), n)$ and the following corollary, which is shown in [9, Cor. 2] for the case $n>1$.

COROLLARY 3.6. *For the case* (3.4) *or* (3.5), *we have* $\mathscr{E}_\#(X)=1$ *if*

$$H^m(\pi_n(X), n; \pi_m(X))=0$$

where the cohomology group is the one with the local coefficient for $n=1$ *by the usual action* $\pi_1(X)$ *on* $\pi_m(X)$.

§4. Some examples $\mathscr{E}_\#(X)=1$

EXAMPLE 4.1. (Cf. [10, Lemma 3.2].) *Let G be a finite group, which acts freely on the odd dimensional sphere S^{2n-1}. Then $\mathscr{E}_\#(S^{2n-1}/G)=1$.*

PROOF. We have the covering $S^m\to S^m/G=X$, $m=2n-1$, and so $\pi_1(X)=G$,

$$\pi_1(X)=G,\quad \pi_m(X)=\pi_m(S^m)=Z,\quad \pi_i(X)=0 \quad (1<i<m).$$

The degree of the action $g: S^m\to S^m$ of $g\in G$ is $(-1)^{m+1}=1$ the Lefshetz fixed point theorem, and the usual action of $g \in \pi_1(X)$ on $\pi_m(X)$ coincides with g_*: $\pi_m(S^m) \to \pi_m(S^m)$, and so we see that $\pi_1(X)$ acts trivially on $\pi_m(X)$. Also G has the periodic cohomology and

$$H^m(G; Z) = H^m(\pi_1(X), 1; \pi_m(X)) = 0$$

since m is odd (cf. [5]). Therefore we have the desired result by Corollary 3.6 for the case (3.5). *q.e.d.*

EXAMPLE 4.2. *$\mathscr{E}_\sharp(RP^n)=1$, where RP^n is the real projective n-space.*

PROOF. For odd n, the result is contained in the above example.

For even n, by the same way as in the proof of the above example, we see that $\pi_1(RP^n)=Z_2$ acts non-trivially on $\pi_n(RP^n)=Z$. Also, we see easily by definition that the cohomology group $H^n(Z_2; Z)$ with the non trivial local coefficient is 0 (cf. [3, § 16]). *q.e.d.*

EXAMPLE 4.3. *The condition of Corollary 3.6 holds, if $n=1$ and $\pi_1(X)$ is a free group, by* [3, Th. 7.1].

EXAMPLE 4.4. *Let X be simple and acyclic, i.e., $\tilde{H}_*(X)=0$. Then, $\mathscr{E}_\sharp(X) = \mathscr{E}_\sharp(X_n)=1$ for all n, where X_n is the n-th Postnikov complex of X defined as in* (3.1).

PROOF. By [2, Th. 4.2], we see that $\mathscr{E}(X)=\operatorname{aut}\pi_1(X)$ and so $\mathscr{E}_\sharp(X)=1$. Since $j_m{}^*$: $H_i(X) \to H_i(X_m)$ is epic for $i \leqq m+1$, by the theorem of J. H. C. Whitehead, we see $H_i(X_m)=0$ for $i \leqq m+1$, and so $H^{m+1}(X_m; \pi_{m+1}(X))=0$. Thus we have $\mathscr{E}_\sharp(X_n)=1$ by Theorem 1.3. *q.e.d.*

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