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# *Some Counterexamples Related to Prime Chains in Integral Domains*

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In this paper all rings are assumed to be commutative with identity. If $A$ is a noetherian Hilbert ring which satisfies the second chain condition for prime ideals, then the polynomial ring $A[X]$ in an indeterminate $X$ over $A$ has the second chain condition for prime ideals ([11], Theorem 1.14). However, in Section 1, we show that $A[X]$ does not necessarily satisfy the first chain condition for prime ideals, even though $A$ is a noetherian Hilbert ring which satisfies the first chain condition for prime ideals. If a ring $A$ satisfies the first chain condition for prime ideals, then as we know, for each prime ideal $\mathfrak{p}$ in $A$, $ht(\mathfrak{p}) + \dim(A/\mathfrak{p}) = \dim(A)$. However, it is unknown whether the converse of this statement is true or not ([7], Remark 2.25). Moreover, in Section 1, we give a noetherian integral domain such that the converse is false. Let $A$ be a noetherian semi local ring such that $ht(\mathfrak{p}) + \dim(A/\mathfrak{p}) = \dim(A)$ for any non maximal prime ideal $\mathfrak{p}$ in $A$. Then it is known that $ht(\mathfrak{m}) = \dim(A)$ or $ht(\mathfrak{m}) = 1$ for any maximal ideal $\mathfrak{m}$ in $A$. But it is unknown whether this assertion is true or not for a general noetherian ring ([7], Remark 2.6). In Section 2, we give a noetherian integral domain such that the above assertion is false. This example shows besides that the statement b) and the statement c) of Remark 2.25 of Ratliff's paper [7] are not equivalent: Even if $\dim(A/\mathfrak{p}) = \dim(A) - 1$ for each height one prime ideal $\mathfrak{p}$ in a noetherian integral domain $A$, the equality $ht(\mathfrak{P}) + \dim(A/\mathfrak{P}) = \dim(A)$ does not necessarily hold for any prime ideal $\mathfrak{P}$ in $A$. In Section 3, making use of the example given in Section 2, we construct a non-catenarian local integral domain $D$ such that for each height one prime ideal $\mathfrak{p}$ in $D$, $ht(\mathfrak{p}) + \dim(D/\mathfrak{p}) = \dim(D)$ (cf. [9], p. 232).

Throughout this paper the notation $M \subset N$ (or $N \supset M$) means that $M$ is a proper subset of $N$.

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**1.** It is known that if a ring $A$ satisfies the first chain condition for prime ideals, then for each prime ideal $\mathfrak{p}$ in $A$, $ht(\mathfrak{p}) + \dim(A/\mathfrak{p}) = \dim(A)$ ([7], p. 1083). Moreover, in [8], Ratliff proved that if $A$ is a noetherian local domain, then the converse of this assertion holds. However it is an open problem whether or not the converse holds in general case ([7], p. 1085). The purpose of this section is to give a noetherian integral domain such that the converse is false.

For the convenience of the reader we cite here the following lemma which was obtained by W. Heinzer ([4], p. 230).

LEMMA 1. *Let $D$, $B$ and $V$ be integral domains with the same quotient field $K$ such that $D=B\cap V$, $V$ is a rank one valuation ring with a rational value group, $D\subset B\subset K$, and $V$ is centered on a maximal ideal $\mathfrak{P}$ in $D$. Then $V=D_{\mathfrak{P}}$, so $\mathfrak{P}$ is a maximal ideal in $D$ of height one. Moreover, $B$ is a flat $D$-module. Hence the non-zero ideals in $B$ are in 1–1 inclusion preserving correspondence with the ideals in $D$ not contained in $\mathfrak{P}$, this correspondence being effected by extension and contraction. In particular, $B$ is a Hilbert ring if and only if $D$ is a Hilbert ring, and $D$ is noetherian if and only if $B$ and $V$ are noetherian.*

LEMMA 2. *Let $R$ be a noetherian integral domain, and let $R'$ be a finite integral extension over $R$. If there exists a prime ideal $\mathfrak{P}$ in $R'$ such that $ht(\mathfrak{P}\cap R)>ht(\mathfrak{P})$, then $R[Z]$ is not catenarian, where $Z$ is an indeterminate.*

PROOF. If $R[Z]$ is catenarian, then $R$ satisfies the altitude formula by Theorem 3.6 in [6]. Therefore, $ht(\mathfrak{P})+\mathrm{tr.deg}_{R/\mathfrak{P}\cap R}(R'/\mathfrak{P})=ht(\mathfrak{P}\cap R)+\mathrm{tr.deg}_R(R')$. Hence, $ht(\mathfrak{P})=ht(\mathfrak{P}\cap R)$ because $R'/\mathfrak{P}$ is integral over $R/(\mathfrak{P}\cap R)$ and $R'$ is integral over $R$. This is a contradiction.

A ring $R$ is said to be equicodimensional if every maximal ideal in $R$ has the same height $\dim(R)$.

LEMMA 3. *If $R$ is an equicodimensional neotherian Hilbert ring, then $R[Z]$ is equicodimensional.*

PROOF. Let $\mathfrak{M}$ be any maximal ideal in $R[Z]$. Since $R$ is a Hilbert ring, $\mathfrak{M}\cap R$ is maximal in $R$ by Theorem 5 in [2]. Therefore $ht(\mathfrak{M}\cap R)=\dim(R)$ by the assumption, and hence $\dim(R[Z])\geq ht(\mathfrak{M})\geq\dim(R)+1$ because $\mathfrak{M}\supset(\mathfrak{M}\cap R)R[Z]$. Thus $ht(\mathfrak{M})=\dim(R[Z])$.

NOTATION. We will retain the following notation for the remainder of this section.

(1) $K$ is a field of characteristic zero.

(2) $T$ is an algebraically independent variable over $K$.

(3) $X=T$, $Y=T+T^2/2!+T^3/3!+\cdots=e^T-1$. It is well-known that $X$ and $Y$ are algebraically independent over $K$.

(4) $A=K[X, Y]_{(X+2,Y)}$, $N=(X+2, Y)A$, $V=K[[T]]\cap K(X, Y)$, $v$ is a natural valuation of $K[[T]]$, $\mathfrak{M}=XV=TK[[T]]\cap K(X, Y)$, $D_1=A\cap V$, $\mathfrak{n}=D_1\cap\mathfrak{N}=V\cap\mathfrak{N}$, $\mathfrak{m}=D_1\cap\mathfrak{M}=A\cap\mathfrak{M}$, $\mathfrak{i}=\mathfrak{n}\cap\mathfrak{m}=\mathfrak{N}\cap\mathfrak{M}$, $R_1=K+\mathfrak{i}$, $B=K[X, Y, 1/X]$, $D=B\cap D_1=B\cap V$, $R=B\cap R_1$, $\mathfrak{q}=\mathfrak{M}\cap D$,

$\mathfrak{h}=D\cap\mathfrak{n}=D\cap\mathfrak{N}$, and $\mathfrak{j}=R\cap\mathfrak{i}$.

Remark. $R_1$ is the same as Nagata's example ([5], Example 2, pp. 204–205) in case $m=0$ and $r=1$.

LEMMA 4. *The following statements hold.*

a) *$X$ is integral over $R$, and $R[X]=R+RX$.*

b) *$R+RX$ contains $Y/X$.*

c) *Let $m$, $n$ be positive integers such that $m\geq n$. Then $R+RX$ contains $X^{m-i}Y^i/X^n$, where $0\leq i\leq m$.*

d) *We denote by $f(m,k)$ the coefficient of $T^{m+k}$ in $Y^m$, where $k\geq 1$, namely $Y^m=T^m+f(m,1)T^{m+1}+\cdots+f(m,i)T^{m+i}+\cdots$. Let $b_{m,n}(X,Y)=\{Y^m-X^m-f(m,1)X^{m+1}-\cdots f(m,n-m)X^n\}/X^{n+1}$, where $n\geq m$. Then $R+RX$ contains $b_{m,n}(X,Y)$.*

PROOF. a) Let $u=(X+2)X$. As $u$ is an element of $\mathfrak{i}\,(=\mathfrak{M}\cap\mathfrak{N})$, $R$ contains $u$. Therefore $X$ is integral over $R$ and $R[X]=R+RX$ because $X^2+2X-u=0$.

b) Let $d=(Y-X-X^2/2!)/X^2$. Then $\mathfrak{N}$ contains $d$. Since $d=(T^3/3!+T^4/4!+\cdots)/T^2$, $v(d)>0$. Therefore $\mathfrak{i}$ contains $d$, and hence $R+RX$ contains $Y/X$ because $Y/X=1+\{(1/2)+\mathrm{d}\}X$.

c) If $i\geq n$, then $(X^{m-i}Y^i)/X^n=X^{m-i}Y^{i-n}(Y/X)^n$, and if $i<n$, then $(X^{m-i}Y^i)/X^n=X^{m-n}(Y/X)^i$. Therefore $(X^{m-i}Y^i)/X^n$ is an element of $R+RX$ by our assertion *b*).

d) Set $g_{m,n}(X,Y)=Y^m-X^m-f(m,1)X^{m+1}-\cdots-f(m,n-m)X^n-f(m,n-m+1)X^{n+1}$. Since $b_{m,n}(X,Y)=f(m,n-m+1)+(g_{m,n}(X,Y)/X^{n+1})=f(m,n-m+1)+\{(g_{m,n}(X,Y)-(g_{m,n}(-2,0)/(-2)^{n+2})X^{n+2}\}/X^{n+1}+\{g_{m,n}(-2,0)/(-2)^{n+2}\}X$ and since $\mathfrak{i}$ contains $\{(g_{m,n}(X,Y)-(g_{m,n}(-2,0)/(-2)^{n+2})X^{n+2}\}/X^{n+1}$, we have $b_{m,n}(X,Y)\in R+RX$.

LEMMA 5. $D=R+RX$.

PROOF. Let $f/X^n$ be an arbitrary element of $D$, where $f\in K[X,Y]$. We may assume that the monomials whose degree is greater than $n-1$ don't appear in $f$ by the assertion c) of Lemma 4. Namely $f$ is of the form $a_{1,0}X+a_{0,1}Y+\cdots+a_{i,j}X^iY^j+\cdots+a_{n-1,0}X^{n-1}+a_{n-2,1}X^{n-2}Y+\cdots+a_{0,n-1}Y^{n-1}$. The value of $f/X^n$ is non-negative. Therefore if we replace $X$, $Y$ by $T$, $T+T^2/2!+T^3/3!+\cdots$ respectively in $f$, then for every $i=1,2,\ldots,n-1$, the coefficient of $T^i$ is zero, namely $a_{1,0}+a_{0,1}=0$, $a_{2,0}+a_{1,1}+a_{0,2}+a_{0,1}f(1,1)=0$, $a_{3,0}+a_{2,1}+a_{1,2}+a_{0,3}+a_{1,1}f(1,1)+a_{0,2}f(2,1)+a_{0,1}f(1,2)=0,\ldots,\sum_{i=0}^{m}a_{i,m-i}+\sum_{i=0}^{m-2}a_{i,m-1-i}f(m-1-i,1)+\cdots+\sum_{i=0}^{m-j-1}a_{i,m-j-i}f(m-j-i,j)+\cdots+\sum_{i=0}^{1}a_{i,2-i}f(2-i,m-2)+a_{0,1}f(1,m-1)=0,\ldots$. Therefore $a_{1,0}=-a_{0,1}$, $a_{2,0}=-a_{1,1}-a_{0,2}-a_{0,1}f(1,1)$, $a_{3,0}=-a_{2,1}$

$-a_{1,2}-a_{0,3}-a_{1,1}f(1,1)-a_{0,2}f(2,1)-a_{0,1}f(1,2),\dots, a_{m,0}=-\sum_{i=0}^{m-1}a_{i,m-i}-\sum_{i=0}^{m-2}a_{i,m-1-i}f(m-1-i,1)-\cdots-\sum_{i=0}^{m-j-1}a_{i,m-j-i}f(m-j-i,j)-\cdots-\sum_{i=0}^{1}a_{i,2-i}f(2-i,m-2)-a_{0,1}f(1,m-1),\dots$. We substitute $-\sum_{i=0}^{m-1}a_{i,m-1}-\sum_{i=0}^{m-2}a_{i,m-1-i}f(m-1-i,1)-\cdots-a_{0,1}f(1,m-1)$ for the coefficient $a_{m,0}$ of $X^m$ in $f$. Then we obtain $f=a_{0,1}(Y-X-f(1,1)X^2-\cdots-f(1,n-3)X^{n-2}-f(1,n-2)X^{n-1})+a_{1,1}(XY-X^2-f(1,1)X^3-\cdots-f(1,n-3)X^{n-1})+\cdots+a_{i-j,j+1}(X^{i-j}Y^{j+1}-X^{i+1}-f(j+1,1)X^{i+2}-\cdots-f(j+1,n-i-2)X^{n-1})+\cdots+a_{n-2,1}(X^{n-2}Y-X^{n-1})+\cdots+a_{0,n-1}(Y^{n-1}-X^{n-1})$. Therefore $f/X^n=a_{0,1}b_{1,n-1}(X,Y)+a_{1,1}b_{1,n-2}(X,Y)+a_{0,2}b_{2,n-1}(X,Y)+a_{2,1}b_{1,n-3}(X,Y)+\cdots+a_{\iota,1}b_{1,n-i-1}(X,Y)+\cdots+a_{i-j,j+1}b_{j+1,n-i+j-1}(X,Y)+\cdots+a_{n-2,1}b_{1,1}(X,Y)+\cdots+a_{0,n-1}b_{n-1,n-1}(X,Y)$, and hence $f/X^n$ is an element of $R+RX$ by our assertion d) of Lemma 4. Thus $D=R+RX$.

LEMMA 6. *The following statements hold.*

a) $\mathfrak{q}$ *is a maximal ideal in* $D$, *and* $ht(\mathfrak{q})=1$.

b) $D$ *and* $R$ *are noetherian Hilbert rings.*

c) $ht(\mathfrak{h})=2$, $ht(\mathfrak{j})=2$ *and* $\mathfrak{q}\cap R=\mathfrak{j}$.

d) $R[Z]$ *is not catenarian, where* $Z$ *is an indeterminate. In particular,* $R[Z]$ *does not satisfy the first chain condition for prime ideals.*

e) $R$ *satisfies the first chain condition for prime ideals.*

PROOF. a) As $K\subseteq D/\mathfrak{q}\subseteq V/\mathfrak{M}=K$, $D/\mathfrak{q}=K$. Hence $\mathfrak{q}$ is a maximal ideal in $D$, and hence Lemma 1 implies that $D_{\mathfrak{q}}=V$ and that $ht(\mathfrak{q})=1$.

b) Since $B$ is a noetherian Hilbert ring and $V$ is noetherian, $D$ is a noetherian Hilbert ring by the assertion of Lemma 1. Since $D$ is a finite integral extension of $R$ by Lemma 5, $R$ is a Hilbert ring and is noetherian by Eakin-Nagata's theorem.

c) $\mathfrak{q}\cap R=\mathfrak{M}\cap D\cap R=\mathfrak{M}\cap D_1\cap D\cap R=\mathfrak{m}\cap D\cap R=\mathfrak{m}\cap R=B\cap\mathfrak{m}\cap R_1$ and $\mathfrak{m}\cap R_1=\mathfrak{i}$. Hence we have $\mathfrak{q}\cap R=\mathfrak{j}$. Since $\mathfrak{h}\cap K[X,Y]=\mathfrak{N}\cap D\cap K[X,Y]=\mathfrak{N}\cap K[X,Y]=(X+2,Y)K[X,Y]$ and $K[X,Y]\subseteq D\subseteq A$, we have $A=K[X,Y]_{(X+2,Y)}\subseteq D_{\mathfrak{h}}\subseteq A_{\mathfrak{N}}=A$, and hence $A=D_{\mathfrak{h}}$. Therefore $ht(\mathfrak{h})=2$. Since $\mathfrak{h}\cap R=D\cap\mathfrak{n}\cap R=\mathfrak{n}\cap R=B\cap R_1\cap\mathfrak{n}$ and $R_1\cap\mathfrak{n}=\mathfrak{i}$, $\mathfrak{h}\cap R=\mathfrak{j}$, and hence $ht(\mathfrak{j})=2$ because $D$ is integral over $R$ and because $ht(\mathfrak{h})=\dim(R)=2$.

d) The fact that $2=ht(\mathfrak{q}\cap R)>ht(\mathfrak{q})=1$ implies that $R[Z]$ is not catenarian by the assertion of Lemma 2.

e) By Lemma 1, the canonical mapping $\mathrm{Max}(B)\to\mathrm{Max}(D)-\{\mathfrak{q}\}$ is bijection, where $\mathrm{Max}(*)$ means the maximal spectrum of a ring $*$, i.e., the set of the maximal ideals in a ring $*$. Since $B$ is equicodimensional, the height of each element of $\mathrm{Max}(D)-\{\mathfrak{q}\}$ is 2. Moreover since the canonical mapping $\mathrm{Max}(D)\to\mathrm{Max}(R)$ is surjection and $ht(\mathfrak{j})=2$, $R$ is an equicodimensional ring of dimension 2 because a maximal ideal in $D$ except $\mathfrak{q}$ is height 2 and $ht(\mathfrak{j})=2$. Thus $R$ satisfies the first chain condition for prime ideals because $R$ is two-dimensional.

REMARK. $R$ is a Hilbert ring which satisfies the first chain condition for prime ideals, but $R[Z]$ does not satisfy the first chain condiiton for prime ideals. However, for the second chain condition for prime ideals, the following statement was obtained by H. Seydi ([11], Theorem 1.14): Let $C$ be a noetherian Hilbert ring. If $C$ satisfies the second chain condition for prime ideals, then so does $C[X]$, where $X$ is an indeterminate.

In the remainder of this section, we assume that $K$ is algebraically closed.

LEMMA 7. *Let* $\mathfrak{P}_{a,b}=D_1\cap(X-a, Y-b)B$, $a\neq 0$, *and let* $p_{a,b}=R\cap\mathfrak{P}_{a,b}$. *If* $(a, b)\neq(-2, 0)$, *then* $R_{p_{a,b}}$ *is a regular local ring.*

PROOF. If $b\neq 0$, then $Y$ is an element of $R-\mathfrak{p}_{a,b}$. Hence $R_{p_{a,b}}$ contains $X$ because $X=XY/Y$. Therefore $R_{\mathfrak{p}_{a,b}}\supseteq K[X, Y]$. Since $(X-a, Y-b)K[X, Y]=\mathfrak{P}_{a,b}\cap K[X, Y]=\mathfrak{P}_{a,b}\cap R_{\mathfrak{p}_{a,b}}\cap K[X, Y]=\mathfrak{p}_{a,b}R_{\mathfrak{p}_{a,b}}\cap K[X, Y]$, we have $K[X, Y]_{(X-a,Y-b)}\subseteq R_{\mathfrak{p}_{a,b}}\subseteq D_{\mathfrak{P}_{a,b}}=K[X, Y]_{(X-a,Y-b)}$, which implies that $R_{p_{a,b}}=K[X, Y]_{(X-a,Y-b)}$. If $a\neq 2$, then $(X+2)X$ is an element of $R$ but not of $\mathfrak{p}_{a,b}$ because $\mathfrak{i}$ contains $(X+2)X$. Hence $R_{\mathfrak{p}_{a,b}}$ contains $X$ because $X=X^2(X+2)/(X(X+2))$. Therefore we see similarly that $R_{\mathfrak{p}_{a,b}}=K[X, Y]_{(X-a,Y-b)}$. Thus if $(a, b)\neq(-2, 0)$, $R_{\mathfrak{p}_{a,b}}$ is a regular local ring.

LEMMA 8. *For each prime ideal* $\mathfrak{Q}$ *in* $R[Z]$, $ht(\mathfrak{Q})+\dim(R[Z]/\mathfrak{Q}=\dim(R[Z])$.

PROOF. Since $R$ is an equicodimensional Hilbert ring of dimension 2, every maximal ideal in $R[Z]$ has the same height 3 by Lemma 3. Therefore we may assume that $\mathfrak{Q}$ is not maximal. Suppose that there exists a maximal ideal $\mathfrak{N}''$ in $D[Z]$ such that $\mathfrak{Q}D[Z]\subseteq\mathfrak{N}''$ and $\mathfrak{N}''\cap D=\mathfrak{P}_{a,b}$, where $(a, b)\neq(-2, 0)$. As $\mathfrak{N}''D[Z]_{\mathfrak{P}_{a,b}}\supseteq\mathfrak{Q}R[Z]_{\mathfrak{p}_{a,b}}=\mathfrak{Q}R_{\mathfrak{p}_{a,b}}[Z]$ and $R_{\mathfrak{p}_{a,b}}$ is a regular local ring by the assertion of Lemma 7, $R[Z]_{\mathfrak{N}'}$ is a regular local ring, where $\mathfrak{N}'=\mathfrak{N}''\cap R[Z]$. Since $R[Z]$ is equicodimensional and $\mathfrak{N}'$ is maximal in $R[Z]$, the height of $\mathfrak{N}'$ is 3. Therefore $3=\dim(R[Z]_{\mathfrak{N}'})=ht(\mathfrak{Q}R[Z]_{\mathfrak{N}'})+\dim(R]Z]_{\mathfrak{N}'}/\mathfrak{Q}R[Z]_{\mathfrak{N}'})\leq ht(\mathfrak{Q})+\dim(R[Z]/\mathfrak{Q})\leq\dim(R[Z])=3$. Therefore $\dim(R[Z])=ht(\mathfrak{Q})+\dim(R[Z]/\mathfrak{Q})$. Now suppose that there does not exist a maximal ideal $\mathfrak{N}''$ in $D[Z]$ such that $\mathfrak{Q}D[Z]\subseteq\mathfrak{N}''$ and $\mathfrak{N}''\cap D=\mathfrak{P}_{a,b}$, where $(a, b)\neq(-2, 0)$. Since $R[Z]$ is a Hilbert ring, $\mathfrak{Q}=\cap\mathfrak{N}'_\lambda$, where $\mathfrak{N}'_\lambda$ is maximal in $R[Z]$. By our assumption, for any $\lambda$, $\mathfrak{N}'_\lambda\cap R=\mathfrak{j}$ because $\mathfrak{N}'_\lambda\cap R$ is maximal in $R$ by the fact that $R$ is a Hilbert ring (cf. [2], Theorem 5). Since $\Lambda$ is an infinite set, $\mathfrak{Q}=\mathfrak{j}R[Z]$. As $ht(\mathfrak{j}R[Z])=ht(\mathfrak{j})=2$ and $\dim(R[Z])/\mathfrak{j}R[Z])=\dim(K[Z])=1$, $\dim(R[Z])=ht(\mathfrak{Q})+\dim(R[Z]/\mathfrak{Q})$. Thus for each prime ideal $\mathfrak{Q}$ in $R[Z]$, $ht(\mathfrak{Q})+\dim(R[Z]/\mathfrak{Q})=\dim(R[Z])$.

By the above arguement, we obtain the following proposition which implies that $R[Z]$ is a counterexample to the assertion at the beginning of this Section.

PROPOSITION. *For each prime ideal* $\mathfrak{Q}$ *in* $R[Z]$, $\dim(R[Z])=ht(\mathfrak{Q})+\dim(R[Z]/\mathfrak{Q})$, *but* $R[Z]$ *does not satisfy the first chain condition for prime ideals.*

**2.** Let $A$ be a ring. Consider the following properties of $A$.

1) For each non-maximal prime ideal $\mathfrak{p}$ in $A$, $ht(\mathfrak{p})+\dim(A/\mathfrak{p})=\dim(A)$.

2) For each prime ideal $\mathfrak{p}$ in $A$, either $ht(\mathfrak{p})+\dim(A/\mathfrak{p})=\dim(A)$ or $\mathfrak{p}$ is a maximal ideal of height one.

In [7], pp. 1076–1077, Ratliff has considered the following statements.

a) The statement 1) implies (in the noetherian case) that $\dim(A)<\infty$.

b) 1) and 2) are equivalent in general (noetherian) case.

In this section, we construct a counterexample to the statement b).

REMARK. If $A$ is noetherian, then the statement a) is true. In fact, we suppose that $\dim(A)=\infty$. Then for each non-maximal prime ideal $\mathfrak{p}$ in $A$, $\dim(A/\mathfrak{p})=\infty$ by the assumption. Let $\mathfrak{p}_1$ be a non-maximal prime ideal in $A$. There exists a maximal ideal $\mathfrak{m}_2$ in $A$ such that $\mathfrak{m}_2\supset\mathfrak{p}_1$ and $ht(\mathfrak{m}_2/\mathfrak{p}_1)\geq 2$ because $\dim(A/\mathfrak{p}_1)=\infty$. Therefore, there exists a prime ideal $\mathfrak{p}_2$ in $A$ such that $\mathfrak{m}_2\supset\mathfrak{p}_2\supset\mathfrak{p}_1$.

Similarly we can take prime ideals $\mathfrak{p}_3, \mathfrak{p}_4,\dots$ such that $\mathfrak{p}_1\subset\mathfrak{p}_2\subset\mathfrak{p}_3\subset\mathfrak{p}_4\subset\cdots$, which contradicts the fact that $A$ is noetherian.

LEMMA 1. *Let $C$ be a locally noetherian ring, and $\mathfrak{a}_1\subseteq\mathfrak{a}_2\subseteq\mathfrak{a}_3\subseteq\cdots$ be an ascending chain of ideals in $C$. If there exist only a finite number of maximal ideals in $C$ which contain $\mathfrak{a}_1$, then $\mathfrak{a}_n=\mathfrak{a}_{n+1}=\cdots$ for some $n$.*

PROOF. Let $\mathfrak{m}_1, \mathfrak{m}_2,\dots, \mathfrak{m}_r$ be the maximal ideals in $C$ which contain $\mathfrak{a}_1$. Since $C_{\mathfrak{m}_i}$ is noetherian for each $i=1, 2,\dots, r$, $\mathfrak{a}_nC_{\mathfrak{m}_i}=\mathfrak{a}_{n+1}C_{\mathfrak{m}_i}=\cdots$ for a sufficiently large $n$. Let $\mathfrak{m}$ be any maximal ideal in $C$ other than $\mathfrak{m}_1,\dots, \mathfrak{m}_r$. Since $\mathfrak{a}_1C_{\mathfrak{m}}=C_{\mathfrak{m}}$, $\mathfrak{a}_nC_{\mathfrak{m}}=\mathfrak{a}_{n+1}C_{\mathfrak{m}}=\cdots$. Thus $\mathfrak{a}_n=\mathfrak{a}_{n+1}=\cdots$.

NOTATION. 1) $K$ is a field with cardinality $\leq\aleph_0$.

2) $Y_1, Y_2, Y_3, X_1, X_2, X_3,\dots$ are algebraically independent variables over $K$.

3) $A=K[Y_1, Y_2, Y_3, X_1, X_2,\dots]$,
$P=(Y_1, Y_2, Y_3)A$.

4) $F=\{f\in P; f$ is a prime element such that $fA\neq Y_1A$ and $X_1$ does not appear in $f\}$. Since $\mathrm{card}(K)\leq\aleph_0$, $\mathrm{card}(A)=\aleph_0$, and hence $\mathrm{card}(F)=\aleph_0$. Therefore we may set $F=\{f_i; i=1, 2, 3,\dots\}$, where $f_iA\neq f_jA$ if $i\neq j$.

5) Let $e(1, 1)$ and $e(1, 2)$ be two positive integers such that $e(1, 1)\neq 1$ and $e(1, 2)\neq 1$, and that $X_{e(1,1)}$ and $X_{e(1,2)}$ don't appear in $f_1$. Let $e(2, 1)$ and $e(2, 2)$ be two positive integers such that $\{1, e(1, 1), e(1, 2)\}\not\ni e(2, 1), e(2, 2)$, and that

$X_{e(2,1)}$ and $X_{e(2,2)}$ don't appear in $f_2$. By the same way as above, for each integer $n>2$ we proceed inductively to choose two positive integers $e(n,1)$ and $e(n,2)$ such that $\{1, e(1,1), e(1,2),\ldots, e(n-1,1), e(n-1,2)\} \not\ni e(n,1), e(n,2)$, and that $X_{e(n,1)}$ and $X_{e(n,2)}$ don't appear in $f_n$.

6) We replace $X_{e(1,1)}, X_{e(1,2)}, X_{e(2,1)},\ldots$ by $X_2, X_3, X_4,\ldots$ respectively, and denote by $Z_1, Z_2,\ldots$ the rest of $X_i$'s.

LEMMA 2. *Let* $P_i=(f_iX_1, X_{2i}, X_{2i+1})A$. *Let* $\phi: A\to R=K[Y_1, Y_2, Y_3, X_2, X_3,\ldots, Z_1, Z_2,\ldots]\simeq A/(X_1A)$ *be the canonical homomorphism. Then the following statements hold.*

a) $P_i$ *is a prime ideal in* $A$.

b) *For each non-zero element* $a$ *of* $R$, *there exist only a finite number of* $\phi(P_i)$'s *which contain* $a$. *In particular, for each element* $g$ *of* $A$ *but not of* $X_1A$, *there exist only a finite number of* $P_i$'s *which contain* $g$.

c) *Let* $P'$ *be the prime ideal in* $A$ *generated by* $Y_1, Y_2, Y_3$ *and* $X_1$. *If* $\mathfrak{a}$ *is an ideal in* $A$ *such that* $\mathfrak{a}\subseteq P'\cup\bigcup_{i=1}^{\infty}P_i$, *then* $\mathfrak{a}\subseteq P'$ *or* $\mathfrak{a}\subseteq P_i$ *for some* $i$. *In particular, if* $\mathfrak{a}$ *is an ideal in* $A$ *such that* $\mathfrak{a}\subseteq P\cup\bigcup_{i=1}^{\infty}P_i$, *then* $\mathfrak{a}\subseteq P'$ *or* $\mathfrak{a}\subseteq P_i$ *for some* $i$.

d) *Let* $T=A-P'\cup\bigcup_{i=1}^{\infty}P_i$, *and let* $S=A-P\cup\bigcup_{i=1}^{\infty}P_i$. *Then,* $T^{-1}A$ *and* $S^{-1}A$ *are noetherian.*

PROOF. a) As $f_i$ is a prime element and $X_1$, $X_{2i}$ and $X_{2i+1}$ don't appear in $f_i$, $P_i$ is a prime ideal in $A$.

b) Let $R_0=K[Y_1, Y_2, Y_3, Z_1, Z_2,\ldots]$. Let $m$ be a positive integer such that $R_0[X_1, X_2,\ldots, X_m]$ contains $a$. It suffices to show that there exist only a finite number of $i$'s such that $m<2i$ and $\phi(P_i)$ contains $a$. Suppose that $\phi(P_i)$ $(=(f_i, X_{2i}, X_{2i+1})R)$ contains $a$, where $m<2i$. Therefore $a=h_1f_i+h_2X_{2i}+h_3X_{2i+1}$, where $h_1, h_2, h_3\in R$. Since $X_{2i}$ and $X_{2i+1}$ don't appear in $a$ and $f_i$, by substituting 0 for $X_{2i}$ and $X_{2i+1}$, we see that $f_i$ devides $a$. This implies that our assertion holds by the facts that $f_i$ is a prime element and that $f_i$ and $f_j$ are relatively prime if $i\neq j$.

c) $\phi(\mathfrak{a})\subseteq\phi(P')\cup\bigcup_{i=1}^{\infty}\phi(P_i)=(Y_1, Y_2, Y_3)R\cup\bigcup_{i=1}^{\infty}(f_i, X_{2i}, X_{2i+1})R$. Suppose that $\phi(\mathfrak{a})$ is finitely generated, namely $\phi(\mathfrak{a})=(h_1,\ldots, h_s)R$. Let $r$ and $t$ be two positive integers such that $R_1=K[Y_1, Y_2, Y_3, X_2, X_3,\ldots, X_r, Z_1,\ldots, Z_t]$ contains $h_1,\ldots, h_s$, and let $N$ be a positive integer satsifying $r<2N$. Since $(Y_1, Y_2, Y_3)R$ contains $f_i$ for any $i$, $\phi(\mathfrak{a})\cap R_1\subseteq(Y_1, Y_2, Y_3)R_1\cup\bigcup_{i=1}^{N}(R_1\cap(f_i, X_{2i}, X_{2i+1})R)$. Therefore $\phi(\mathfrak{a})\cap R_1\subseteq(Y_1, Y_2, Y_3)R_1$ or $\phi(\mathfrak{a})\cap R_1\subseteq R_1\cap(f_i, X_{2i}, X_{2i+1})R$ so that $(Y_1, Y_2, Y_3)R\ni h_1,\ldots, h_s$ or $(f_i, X_{2i}, X_{2i+1})R\ni h_1,\ldots, h_s$. Hence $\phi(\mathfrak{a})\subseteq\phi(P')$ or $\phi(\mathfrak{a})\subseteq\phi(P_i)$ for some $i$. Next suppose that $\phi(\mathfrak{a})$ is not finitely generated. Let $\phi(\mathfrak{a})=(h_1, h_2,\ldots)R$ (Note that $\phi(\mathfrak{a})$ is generated by a countable number of the elements of $R$). Let $\mathfrak{b}_n=(h_1,\dots, h_n)R$. If $(Y_1, Y_2, Y_3)R\not\subseteq\phi(\mathfrak{a})$, there exists a positive integer $n_0$ such that for each $n\geq n_0$ $(Y_1, Y_2, Y_3)R\not\subseteq\mathfrak{b}_n$, whence there exists a positive integer $i(n)$ such that $\mathfrak{b}_n\subseteq\phi(P_{i(n)})$ for each $n\geq n_0$. The set $\{i(n); n=n_0, n_0+1,\dots\}$ is finite since there exist only a finite number of $\phi(P_i)$'s which contain $h_1$ by our assertion b). Hence $\phi(\mathfrak{a})\ (=\bigcup_{i=1}^{\infty}\mathfrak{b}_n)$ is contained in $\phi(P_i)$ for some $i$. Thus for any ideal $\mathfrak{a}$ in $A$ satisfying $\mathfrak{a}\subseteq P'\cup\bigcup_{i=1}^{\infty}P_i$, we have $\phi(\mathfrak{a})\subseteq\phi(P')$ or $\phi(\mathfrak{a})\subseteq\phi(P_i)$ for some $i$ so that $\mathfrak{a}\subseteq P'$ or $\mathfrak{a}\subseteq P_i$ for some $i$.

d) Every maximal ideal in $T^{-1}A$ is of the form $T^{-1}P'$ or $T^{-1}P_i$ for some $i$ by our assertion c), and hence $T^{-1}A$ is locally noetherian. Let $\mathfrak{B}_1\subseteq\mathfrak{B}_2\subseteq\cdots$ be an ascending chain of ideals in $T^{-1}A$, and let $\mathfrak{b}_i=\mathfrak{B}_i\cap A$. If there exists a positive integer $n_0$ satisfying $\mathfrak{b}_{n_0}\not\subseteq X_1A$, then by our assertion b) there exist only a finite number of maximal ideals in $T^{-1}A$ which contain $\mathfrak{B}_{n_0}$. Therefore $\mathfrak{B}_n=\mathfrak{B}_{n+1}=\cdots$ for some $n$ by Lemma 1. If $\mathfrak{b}_n\subseteq X_1A$ for any $n$, $\mathfrak{b}_n$ is of the form $X^{a(n)}\mathfrak{c}_n$, where $\mathfrak{c}_n$ is the ideal in $A$ such that $\mathfrak{c}_n\not\subseteq X_1A$. As $a(1)\geq a(2)\geq a(3)\geq\cdots$, $a(m)=a(m+1)=\cdots$ for some $m$, whence $T^{-1}\mathfrak{c}_m\subseteq T^{-1}\mathfrak{c}_{m+1}\subseteq\cdots$. Since $\mathfrak{c}_m\not\subseteq X_1A$, by applying the similar method as before, we see that $T^{-1}\mathfrak{c}_r=T^{-1}\mathfrak{c}_{r+1}\equiv\cdots$ for some $r$. Therefore $\mathfrak{B}_r=\mathfrak{B}_{r+1}=\cdots$. Thus we conclude that $T^{-1}A$ is noetherian. Also $S^{-1}A$ is noetherian since $S^{-1}A=S^{-1}(T^{-1}A)$.

LEMMA 3. *Let $B=S^{-1}A$. Let $H=\{e_1Y_1^m+e_2X_1; e_1\in S, m\in\boldsymbol{N}$ and $e_2\in A\}$ and let $S_1$ be the multiplicatively closed set generated by $X_1$ and all the elements of $H$. Let $Q=(Y_2, Y_3)A$ and let $U_i=\{\mathfrak{p}\in\operatorname{Spec}(A); P_i\supset\mathfrak{p}, ht(P_i/\mathfrak{p})=1$ and $\mathfrak{p}\not\ni X_1\}$. Then the following statements hold.*

a) $(Y_2, Y_3, X_1)A\cap S=\emptyset$.

b) $Q\cap S_1=\emptyset$.

c) *Let $g$ be any element of $K[Y_1, Y_2, Y_3]$ such that $g$ and $f_i$ are relatively prime. Then, $P_i\not\ni g$.*

d) $P_i\cap H=\emptyset$ *for any* $i$.

e) *Let $\mathfrak{p}$ be any element of $\bigcup_{i=1}^{\infty}U_i$. Then $S_1^{-1}(S^{-1}\mathfrak{p})$ is a maximal ideal of height 3 in $S_1^{-1}B$.*

f) *$S_1^{-1}(S^{-1}Q)$ is a maximal ideal of height 2 in $S_1^{-1}B$.*

PROOF. a) Let $h$ be an arbitrary element of $(Y_2, Y_3, X_1)A$. We can express $h=X_1h_1+h_2$, where $h_1\in A$, $h_2\in(Y_2, Y_3)A$ and $X_1$ doesn't appear in $h_2$. If $h_2=0$, then $h\in P_i$ for any $i$, whence $S\not\ni h$. If $h_2=0$, then there exists at least one prime divisor of $h_2$ which is of the form $f_i$, so that $P_i$ contains $h$, and hence $S\not\ni h$. Thus $(Y_2, Y_3, X_1)A\cap S=\emptyset$.

b) Suppose that $Q$ contains an element $e_1Y_1^m+e_2X_1$ of $H$. Since $\phi(e_1Y_1^m+e_2X_1)\in(Y_2, Y_3)R$, $\phi(e_1)Y_1^m\in(Y_2, Y_3)R$. Therefore $\phi(e_1)\in(Y_2. Y_3)R$, and hence $e_1\in(Y_2, Y_3, X_1)A$. However this contradicts our assertion a).

c) Suppose that $P_i$ contains $g$. Then we can write $g=h_1f_i+h_2X_1+h_3X_2$,

$+h_4X_{2i+1}$, where $h_1,\dots, h_4\in A$. Since $X_1$, $X_{2i}$ and $X_{2i+1}$ don't appear in $f_i$ and $g$, by substituting 0 for $X_1$, $X_{2i}$ and $X_{2i+1}$, we see that $f_i$ divides $g$. This is a contradiction.

d) Suppose that $P_i$ contains an element $e_1Y_1^m+e_2X_1$ of $H$. Then $P_i\ni e_1Y_1^m$ since $P_i$ contains $X_1$. This is impossible because $P_i$ does not contain $e_1$ and $Y_1$. Thus $P_i\cap H=\emptyset$ for any $i$.

e) Since $\mathfrak{p}\cap H=\emptyset$ and $\mathfrak{p}\not\ni X_1$ for each element $\mathfrak{p}$ of $U_i$, $\mathfrak{p}\cap S_1=\emptyset$. $ht(\mathfrak{p})=ht(P_i)-1=3$ because $A$ is catenarian. Since $\dim(B)=4$ and since every maximal ideal of height 4 in $B$ is of the form $S^{-1}P_i$ for some $i$ by our assertion c) of Lemma 2, we have $\dim(S_1^{-1}B)=3$. Thus $S_1^{-1}(S^{-1}\mathfrak{p})$ is a maximal ideal of height 3 in $S_1^{-1}B$.

f) If $P_i\supset Q$, then $P_i$ contains $Y_2$, $Y_3$, $X_1$, $X_{2i}$, $X_{2i+1}$, whence $ht(P_i)\geq 5$. This contradicts $ht(P_i)=4$. Therefore to prove that $S_1^{-1}(S^{-1}Q)$ is a maximal ideal in $S_1^{-1}B$, it suffices to show that for any prime ideal $Q'$ in $A$ such that $Q\subset Q'\subseteq P'$, we have $Q'\cap S_1\neq\emptyset$. Let $g$ be an element of $Q'$ but not of $Q$. We may assume that $Y_2$ and $Y_3$ don't appear in $g$. If $Y_1$ does not appear in $g$, then $g$ is of the form $g_1X_1^n$, where $g_1\notin P'$. Hence $Q'\ni X_1$. Thus $Q'\cap S_1\neq\emptyset$. Now suppose that $Y_1$ appears in $g$. Then $g$ is of the form $u_1Y_1^n+u_2X_1$, where $u_1, u_2\in A$. We may assume that $X_1$ does not appear in $u_1$ and that $Y_1A\not\ni u_1$. Therefore $P$ does not contain $u_1$. By our assertion $b$) of Lemma 2, we may assume that $P_{i(1)},\dots, P_{i(m)}$ are totality of $P_i$'s which contain $u_1$. Let $P_{j(1)},\dots, P_{j(s)}$ be the totality of $P_i$'s such that $X_{2i}$ or $X_{2i+1}$ appears in $u_1$. Let $r$ be a positive integer such that $Y_2+Y_3^r$ is relatively prime to each $f_{i(1)},\dots, f_{i(m)}, f_{j(1)},\dots, f_{j(s)}$. Then $P_{i(1)}\cup\cdots\cup P_{i(m)}\cup P_{j(1)}\cup\cdots\cup P_{j(s)}$ does not contain $Y_2+Y_3^r$ by our assertion c). Let $t$ be a positive integer such that $P_{j(1)}\cup\cdots\cup P_{j(s)}$ does not contain $Y_1^t(Y_2+Y_3^r)+u_1$. (Proof of the existence of such an integer $t$: Suppose that $P_{j(1)}\cup\cdots\cup P_{j(s)}$ contains $Y_1^t(Y_2+Y_3^r)+u_1$ for any positive integer $t$. Then some $P_{j(k)}$ contains $Y_1^{t(1)}(Y_2+Y_3^r)+u_1$ and $Y_1^{t(2)}(Y_2+Y_3^r)+u_1$, where $t(1)<t(2)$. Hence $P_{j(k)}$ contains $Y_1^{t(1)}(1-Y_1^{t(2)-t(1)})(Y_2+Y_3^r)$ so that $P_{j(k)}$ contains $1-Y_1^{t(2)-t(1)}$ because $Y_1$ and $Y_2+Y_3^r$ are not contained in $P_{j(k)}$. This contradicts the fact that $P_{j(k)}\subseteq(Y_1, Y_2, Y_3, X_1, X_2,\dots)A$.) Then $P\cup\bigcup_{i=1}^{\infty}P_i$ does not contain $Y_1^t(Y_2+Y_3^r)+u_1$. Indeed, $P\not\ni Y_1^t(Y_2+Y_3^r)+u_1$ since $P\ni Y_1$ and $P\not\ni u_1$. And if for some $i\neq i(1), i(2),\dots, i(m), j(1),\dots, j(s)$, $P_i$ contains $Y_1^t(Y_2+Y_3^r)+u_1$, then $Y_1^t(Y_2+Y_3^r)+u_1=h_1f_i+h_2X_1+h_3X_{2i}+h_4X_{2i+1}$, where $h_1,\dots, h_4\in A$. As $X_1$, $X_{2i}$ and $X_{2i+1}$ don't appear in $u_1$, by substituting 0 for $X_1$, $X_{2i}$ and $X_{2i+1}$, we see that $u_1$ is of the form $f_i\bar{h}_1-Y_1^t(Y_2+Y_3^r)$ so that $P\ni u_1$. This is a contradiction. For each $i(k)$, $P_{i(k)}\not\ni Y_1^t(Y_2+Y_3^r)+u_1$ since $P_{i(k)}\ni u_1$ and $P_{i(k)}\not\ni Y_1^t(Y_2+Y_3^r)$. Thus $S$ contains $Y_1^t(Y_2+Y_3^r)+u_1$. Therefore $H\cap Q'$ contains $(Y_1^t(Y_2+Y_3^r)+u_1)Y_1^n+u_2X_1$. Thus $S_1\cap Q'\neq\emptyset$. Since $ht(Q)=2$, the height of $S_1^{-1}(S^{-1}Q)$ is 2. Thus the proof is completed.

Now we obtain the following proposition which gives our desired example.

PROPOSITION. *For each non-maximal ideal* $\mathfrak{q}$ *in* $S_1^{-1}B$, $ht(\mathfrak{q})+\dim(S_1^{-1}B/\mathfrak{q})=3$. *However,* $S_1^{-1}B$ *has a maximal ideal of height* 2.

PROOF. Let $\mathfrak{q}$ $(=S_1^{-1}(S^{-1}\mathfrak{p})$, where $\mathfrak{p}\in\mathrm{Spec}(A))$ be a non-maximal prime ideal in $S_1^{-1}B$. To prove that $ht(\mathfrak{q})+\dim(S_1^{-1}B/\mathfrak{q})=3$, we may assume that $ht(\mathfrak{q})=1$ because $S_1^{-1}B$ is three-dimensional. Since $A$ is a unique factorization domain, $\mathfrak{p}=Af$ for a suitable prime element $f$ of $A$. If $f\in P_i$ for some $i$, then Lemma 5 of [1] implies that a maximal element with respect to the inclusion relation in the family $\{\mathfrak{p}'\in\mathrm{Spec}(A); \mathfrak{p}'\subset P_i, \mathfrak{p}'\ni f$ and $\mathfrak{p}'\not\ni X_1\}$ has the height 3, and hence $\dim(S_1^{-1}B/\mathfrak{q})=2$ by our assertion e) of Lemma 3 and by the fact that $B$ is catenarian. If $f\in P'$, then we can express $f=g+hX_1$, where $g$, $h\in A$. Since $f$ is a prime element and $f$ is an element of $\mathfrak{q}$, $g$ is not zero. We may assume that $X_1$ does not appear in $g$. Therefore, $g\in P$. If $g$ has a prime divisor $f_i$ for some $i$, $P_i$ contains $f$, whence $\dim(S_1^{-1}B/q)=2$. If any $f_i$ isn't a prime divisor of $g$, then $g$ is of the form $Y_1^m g_1$, where $g_1\in A-P$. Since $f$ is not an element of $H$, $S$ does not contain $g$. Therefore $P_i\ni g_1$ for some $i$ because $g_1\in A-P$. Hence $P_i\ni f$, so that $\dim(S_1^{-1}B/q)=2$. Thus for each non-maximal prime ideal $\mathfrak{q}$ in $S_1^{-1}B$, $ht(\mathfrak{q})+\dim(S_1^{-1}B/\mathfrak{q})=3$. $S_1^{-1}(S^{-1}Q)$ is a maximal ideal of height 2 in $S_1^{-1}B$ by $f$) of Lemma 3. Thus our assertion is proved.

REMARK 1. Every prime ideal of height one in $S_1^{-1}B$ is contained in some maximal ideal of height 3 by the proof of the above Proposition. Therefore $S_1^{-1}B$ does not have a maximal ideal of height one. Moreover, we see that for a noetherian ring $E$ the following statements of Remark 2.25 in [7] are not equivalent: b) For each prime ideal $\mathfrak{p}$ in $E$, $ht(\mathfrak{p})+\dim(E/\mathfrak{p})=\dim(E)$. c) For each height one prime ideal $\mathfrak{p}$ in $E$, $\dim(E/\mathfrak{p})=\dim(E)-1$.

REMARK 2. If every maximal ideal of height 3 in $S_1^{-1}B$ is of the form $S_1^{-1}(S^{-1}\mathfrak{p})$ for some element $\mathfrak{p}$ of $\bigcup_{i=1}^{\infty} U_i$, then by using Corollary 10.5.8 in [3], p. 106, we see that $S_1^{-1}B$ is a Hilbert ring.

**3.** In [9], p. 232, Ratliff gave the following conjecture.

$H$-conjecture: If $R$ is a noetherian local domain such that $ht(\mathfrak{p})+\dim(R/\mathfrak{p})=\dim(R)$ for each height one prime ideal $\mathfrak{p}$ in $R$, then $R$ is catenarian.

In this section, making use of the example constructed in the previous section, we give a non-noetherian local domain $D$ such that $D$ is not catenarian, but for each height one prime ideal $\mathfrak{n}$ in $D$, $ht(\mathfrak{n})+\dim(D/\mathfrak{n})=\dim(D)$.

LEMMA 1. *Let $K$ be a field and let $C$ be a noetherian integral domain over $K$. Let $D=K+ZC[[Z]]$, where $Z$ is an indeterminate, and let $\mathfrak{N}=ZC[[Z]]$.*

*Then the following statements hold.*

a) *$D$ is a local ring whose unique maximal ideal is $\mathfrak{N}$.*

b) *$\mathfrak{N}=\sqrt{DZ}$. In particular, $\mathfrak{N}$ is a minimal prime ideal of $DZ$.*

c) *Let $V=\{\mathfrak{n}\in \mathrm{Spec}(D); \mathfrak{n}\subset\mathfrak{N}\}$. Let $\rho(\mathfrak{p})=\mathfrak{p}C[[Z]]\cap D$ for each prime ideal $\mathfrak{p}$ in $C$. Then $\rho$: $\mathrm{Spec}(C)\to V$ is injective.*

d) *$ht(\mathfrak{p})=ht(\rho(\mathfrak{p}))$ for each prime ideal $\mathfrak{p}$ in $C$.*

e) *Let $\mu(\mathfrak{n})=\{g\in C[[Z]]; Zg\in\mathfrak{n}\}$ for each element $\mathfrak{n}$ of $V$. Then $\mu(\mathfrak{n})$ is a prime ideal in $C[[Z]]$.*

f) *For each element $\mathfrak{n}$ of $V$, $Z\rho(\mathfrak{n})=\mathfrak{n}$ and $nD_Z=\rho(\mathfrak{n})C[[Z]][1/Z]$. In particular, $\mu$: $V\to\mathrm{Spec}(C[[Z]])$ is injective, and $ht(\mathfrak{n})=ht(\mu(\mathfrak{n}))$ for each element $\mathfrak{n}$ of $V$.*

g) *$\mu\rho(\mathfrak{p})=\mathfrak{p}C[[Z]]$ for each prime ideal $\mathfrak{p}$ in $C$.*

h) *Let $\mathfrak{n}'$ be a prime ideal in $C[[Z]]$. Then $Z\mathfrak{n}'$ is prime in $D$ if and only if $\mathfrak{n}'$ does not contain $Z$. In particular, for each maximal ideal $\mathfrak{M}$ in $C[[Z]]$, $Z\mathfrak{M}$ is not prime in $D$.*

i) *$ht(\mathfrak{N}/\rho(\mathfrak{m}))=1$ for each maximal ideal $\mathfrak{m}$ in $C$.*

j) $\dim(D)=\dim(C)+1$.

PROOF. We see obviously that the assertion a), b), c), f) and g) hold.

d) Since $D_Z=C[[Z]][1/Z]$, $\rho(\mathfrak{p})_Z=\mathfrak{p}C[[Z]][1/Z]\cap D_Z=\mathfrak{p}C[[Z]][1/Z]$, and hence $ht(\rho(\mathfrak{p}))=ht(\rho(\mathfrak{p})_Z)=ht(\mathfrak{p})$ because $C$ is noetherian.

e) Suppose that $fg$ belongs to $\mu(\mathfrak{n})$, where $f, g\in C[[Z]]$. Then $\mathfrak{n}\ni Zfg$, whence $\mathfrak{n}\ni(Zf)(Zg)$. But $\mathfrak{n}$ is prime in $D$. Consequently either $Zf\in\mathfrak{n}$ or $Zg\in\mathfrak{n}$. It follows that either $f\in\mu(\mathfrak{n})$ or $g\in\mu(\mathfrak{n})$. Hence $\mu(\mathfrak{n})$ is prime in $C[[Z]]$.

h) First suppose that $Z\mathfrak{n}'$ is prime in $D$. If $\mathfrak{n}'$ contains $Z$, then $Z\mathfrak{n}'$ contains $Z^2$, whence $Z\mathfrak{n}'\ni Z$ because $Z\mathfrak{n}'$ is prime in $D$. Hence $\mathfrak{n}'\ni 1$. This is a contradiction. Next suppose that $\mathfrak{n}'$ does not contain $Z$. Let $(Zf)(Zg)$ be an element of $Z\mathfrak{n}'$, where $f, g\in C[[Z]]$. Then $Zfg$ belongs to $\mathfrak{n}'$, whence $\mathfrak{n}'\ni fg$ by our assumption. Therefore either $\mathfrak{n}'\ni f$ or $\mathfrak{n}'\ni g$. It follows that either $Z\mathfrak{n}'\ni Zf$ or $Z\mathfrak{n}'\ni Zg$. Thus $Z\mathfrak{n}'$ is prime in $D$. Finally, the radical of $C[[Z]]$ contains $Z$ so that the last assertion is obvious.

i) Suppose that there exists a prime ideal $\mathfrak{n}$ in $D$ such that $\rho(\mathfrak{m})\subset\mathfrak{n}\subset\mathfrak{N}$. Then $\mathfrak{m}C[[Z]]=\mu\rho(\mathfrak{m})\subset\mu(\mathfrak{n})$ by our assertion e), g). It follows that $\mu(\mathfrak{n})$ is maximal in $C[[Z]]$ since $\mathfrak{m}$ is maximal in $C$. On the other hand, $Z\mu(\mathfrak{n})=\mathfrak{n}$ by the assertion f), this contradicts the assertion h).

j) We may assume that $\dim(C)<\infty$ by the assertion d). Set $n=\dim(C)$. Let $\mathfrak{m}$ be a maximal ideal of height $n$ in $C$. Since $\mathfrak{N}\supset\rho(\mathfrak{m})$, $ht(\mathfrak{N})\geq n+1$ by the assertion d). Let $\mathfrak{n}$ be any element of $V$. The assertion f) and h) imply that $\mu(\mathfrak{n})$ is not maximal in $C[[Z]]$, whence $ht(\mathfrak{n})<\dim(C[[Z]])=n+1$ by the assertion f). Hence $ht(\mathfrak{N})\leq n+1$. Thus $ht(\mathfrak{N})=n+1$.

LEMMA 2. *Let $C$ be a noetherian integral domain and let $\mathfrak{P}$ be a prime* *ideal in C. Let a be a non-zero element of* $\mathfrak{P}$. *Then there exists a prime ideal* $\mathfrak{p}$ *in C such that* $ht(\mathfrak{p})=ht(\mathfrak{P})-1$ *and* $\mathfrak{p}\not\ni a$.

PROOF. We prove the assertion by induction on $ht(\mathfrak{P})$. Set $n=ht(\mathfrak{P})$. If $n=2$, then $\cap\mathfrak{p}_\lambda=0$, where $\mathfrak{p}_\lambda$ is a prime ideal of height one contained in $\mathfrak{P}$, whence $\mathfrak{p}_\lambda\not\ni a$ for some $\lambda$. Assume that $n>2$. Let $\mathfrak{P}=\mathfrak{p}_0\supset\mathfrak{p}_1\supset\cdots\supset\mathfrak{p}_{n-2}\supset\mathfrak{p}_{n-1}\supset 0$ be a chain of prime ideals in $C$. Similarly, we may assume that $\mathfrak{p}_{n-1}$ does not contain $a$. Then, applying the induction assumption to $\mathfrak{P}/\mathfrak{p}_{n-1}$, we obtain a prime ideal $\mathfrak{p}$ such that $ht(\mathfrak{p})=n-1$ and $\mathfrak{p}$ does not contain $a$.

LEMMA 3. (Samuel, [10], Theorem 2.1) *Let C be a regular unique factorization domain. Then C[[Z]] is also a regular unique factorization domain.*

We are now able to state:

PROPOSITION. *Let the notation be the same as in Section 2. Let* $C=S_1^{-1}B$ *and let* $D=K+ZC[[Z]]$, *where Z is an indeterminate. Then D is a non-catenarian local domain, and* $\dim(D/\mathfrak{n})=3$ *for each height one prime ideal* $\mathfrak{n}$ *in D.*

PROOF. $C$ has a maximal ideal of height 2 and a maximal ideal of height 3 by the assertion e) and f) of Lemma 3 of Section 2. Hence the assertions d) and i) of Lemma 1 imply that $D$ is not catenarian. Let $\mathfrak{n}$ be a prime ideal of height one in $D$. Since $C$ is a regular unique factorization domain, so is $C[[Z]]$ by Lemma 3. Hence $\mu(\mathfrak{n})=(c+Zg(Z))C[[Z]]$, where $c\in C$ and $g(Z)\in C[[Z]]$. Since $c+Zg(Z)$ is a prime element and since $\mathfrak{n}$ does not contain $Z$, $c$ is not zero. Let $\mathfrak{m}$ be a maximal ideal in $C$ of height 3 containing $c$. The existence of such $\mathfrak{m}$ follows from the proof of Proposition of Section 2. Let $\mathfrak{M}=\mathfrak{m}C[[Z]]+ZC[[Z]]$. Since $ht(\mathfrak{M})=4$ and $C[[Z]]$ is catenarian (cf. [11], p. 24), Lemma 2 implies that there exists a prime ideal $\mathfrak{n}'_1$ in $C[[Z]]$ such that $ht(\mathfrak{n}'_1)=3$, $\mathfrak{n}'_1\supset\mu(\mathfrak{n})$ and $\mathfrak{n}'_1$ does not contain $Z$. Hence there exists a chain of prime ideals $0\subset\mu(\mathfrak{n})\subset\mathfrak{n}'_2\subset\mathfrak{n}'_1$ in $C[[Z]]$ by the fact that $C[[Z]]$ is catenarian. Therefore $0\subset\mathfrak{n}\subset Z\mathfrak{n}'_2\subset Z\mathfrak{n}'_1\subset\mathfrak{N}$ is a chain of prime ideals in $D$ by our assertion f), h) of Lemma 1. Thus $\dim(D/\mathfrak{n})=3$.

REMARK. Since $K$ is algebraically closed in $C$, $D$ is a normal integral domain.

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