Generalized Extremal Length of an Infinite Network

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(Received September 2, 1975)

Introduction

The extremal length of a network, which is the reciprocal of the value of a quadratic programming problem, was first investigated by R. J. Duffin [4] on a finite graph and next by the second author [7] on an infinite graph. In this paper we shall be concerned with a generalized form of the extremal length as in [5] along the same lines as in [4] and [7]. The generalized extremal length of an infinite network may be regarded as the reciprocal of the value of a convex programming problem. One of our main purposes is to establish a reciprocal relation between the generalized extremal distance and the generalized extremal width of an infinite network which was established by M. Ohtsuka [5] for the continuous case. We shall also study the generalized extremal length of an infinite network relative to a finite set and the ideal boundary of the network. A concept of non-linear flows which was studied in [1] and [3] will appear in §3 and §4 in connection with the extremal width of a network.

§1. Preliminaries

Let \( X \) be a set of nodes and let \( Y \) be a set of directed arcs. Since we always consider the case where \( X \) and \( Y \) consist of a countably infinite number of elements, we put

\[
X = \{0, 1, 2, \ldots, n, \ldots\},
\]
\[
Y = \{1, 2, \ldots, n, \ldots\}.
\]

Let \( K = (K_{vj}) \) be the node-arc incidence matrix. Namely \( K_{vj} = 1 \) if arc \( j \) is directed toward node \( v \), \( K_{vj} = -1 \) if arc \( j \) is directed away from node \( v \) and \( K_{vj} = 0 \) if arc \( j \) and node \( v \) do not meet.

We assume that \( X, Y \) and \( K \) satisfy the following conditions:

\[(1.1)\ \ \{j \in Y; K_{vj} \neq 0\} \text{ is a nonempty finite set for each } v \in X.\]

\[(1.2)\ \ e(j) = \{v \in X; K_{vj} \neq 0\} \text{ consists of exactly two nodes for each } j \in Y.\]

\[(1.3)\ \ \text{For any } \alpha, \beta \in X, \text{ there are } v_1, \ldots, v_n \in X \text{ and } j_1, \ldots, j_{n+1} \in Y \text{ such that } e(j_i) = \{v_{i-1}, v_i\}, i = 1, \ldots, n+1 \text{ with } v_0 = \alpha \text{ and } v_{n+1} = \beta.\]
Given a strictly positive function \( r \) on \( Y \), the quartet \(<X, Y, K, r>\) is then called an infinite network. For simplicity denote by \(<X, Y>\) a network \(<X, Y, K, r>\) if there is no confusion from the context.

Let \( X' \) and \( Y' \) be subsets of \( X \) and \( Y \) respectively and let \( K' \) and \( r' \) be the restrictions of \( K \) and \( r \) onto \( X' \times Y' \) and \( Y' \) respectively. We say that \(<X', Y', K', r'>\) is a subnetwork of \(<X, Y, K, r>\) if it is a network in itself. In case \( X' \) (or \( Y' \)) is a finite set, we call \(<X', Y'>\) a finite subnetwork of \(<X, Y>\).

We say that a sequence \( \{<X_n, Y_n>\} \) of finite subnetworks of \(<X, Y>\) is an exhaustion of \(<X, Y>\) if

\[
X = \bigcup_{n=1}^{\infty} X_n \quad \text{and} \quad Y = \bigcup_{n=1}^{\infty} Y_n,
\]

\[
\{j \in Y; K_{vj} \neq 0\} \subseteq Y_{n+1} \quad \text{for each} \quad v \in X_n.
\]

Let \( p \) and \( q \) be positive numbers such that

\[
1/p + 1/q = 1 \quad \text{and} \quad p > 1.
\]

Let \( L(X) \) and \( L(Y) \) be the sets of all real functions on \( X \) and \( Y \) respectively. For \( u \in L(X) \) and \( w \in L(Y) \), we put

\[
u_v = u(v), \quad w_j = w(j),
\]

\[
Su = \{v \in X; u_v \neq 0\}, \quad Sw = \{j \in Y; w_j \neq 0\},
\]

\[
D_p(u) = \sum_{j=1}^{\infty} r_j |u_j|^p,
\]

\[
H_p(w) = \sum_{j=1}^{\infty} r_j |w_j|^p.
\]

We shall use the following classes of functions on \( X \) and \( Y \):

\[
L_0(X) = \{u \in L(X); Su \text{ is a finite set}\},
\]

\[
L_0(Y) = \{w \in L(Y); Sw \text{ is a finite set}\},
\]

\[
L^+(Y) = \{w \in L(Y); w_j \geq 0 \quad \text{on} \quad Y\},
\]

\[
L_p(Y; r) = \{w \in L(Y); H_p(w) < \infty\},
\]

\[
L_p^+(Y; r) = \{w \in L^+(Y); H_p(w) < \infty\}.
\]

Note that \( L_p(Y; r) \) is a reflexive Banach space with respect to the norm \([H_p(w)]^{1/p}\).

If \( H_p(w - w^{(n)}) \to 0 \) as \( n \to \infty \), then \( w_j^{(n)} \to w_j \) as \( n \to \infty \) for each \( j \).

For a nonempty subset \( A \) of \( X \), let us put
\[ D^{(p)} = D^{(p),A} = \{ u \in L(X); D_p(u) < \infty \text{ and } u = 0 \text{ on } A \}. \]

We have

**Lemma 1.1.** For any \( n \), there exists a constant \( M_n \) such that

\[ \sum_{i=0}^{n} |u_i| \leq M_n[D_p(u)]^{1/p} \]

for all \( u \in D^{(p)} \).

**Proposition 1.1.** \( D^{(p)} \) is a reflexive Banach space with respect to the norm \([D_p(u)]^{1/p}\).

**Proof.** It follows from Lemma 1.1 and the Minkowski inequality that \([D_p(u)]^{1/p}\) is a norm on \( D^{(p)} \). We can prove by a standard argument that \( D^{(p)} \) is a Banach space. Let \( E \) be the linear transformation from \( L(X) \) into \( L(Y) \) defined by

\[ w_j = (Eu)_j = r_j^{-1} \sum_{r=0}^{\infty} K_{rj} u_r \]

and denote by \( E(D^{(p)}) \) the image of \( D^{(p)} \) under \( E \). From the relation \( H_p(Eu) = D_p(u) \), it follows that \( E \) is a Banach space isomorphism from \( D^{(p)} \) onto \( E(D^{(p)}) \). It is easily seen that \( E(D^{(p)}) \) is a closed linear subspace of \( L_p(Y; r) \). Since \( L_p(Y; r) \) is a reflexive Banach space, \( E(D^{(p)}) \) is also a reflexive Banach space (cf. [2], p. 116, Proposition 11). Therefore \( D^{(p)} \) is reflexive.

**Lemma 1.2.** Let \( T \) be a normal contraction of the real line \( R \) and \( u \in D^{(p)} \). Then \( Tu \in D^{(p)} \) and \( D_p(Tu) \leq D_p(u) \).

We often use the following theorem to assure the existence of an optimal solution of an extremum problem.

**Theorem A.** Let \( Z \) be a reflexive Banach space with the norm \( \|z\| \) and \( C \) be a nonempty closed convex set in \( Z \). Then there exists a point \( \hat{z} \in C \) such that \( \|\hat{z}\| = \min \{ \|z\|; z \in C \} \). This minimizing point is unique if every boundary point of the ball \( \|z\| \leq 1 \) is an extreme point.

**§2. Generalized extremal length of a network**

A path \( P \) from node \( \alpha \) to node \( \beta \) is the triple \((C_X(P), C_Y(P), p(P))\) of a finite

1) Cf. Lemma 1 in [7].
2) Cf. Lemma 2 in [7].
3) [2], p. 117, Exercise 1.
ordered set \( C_x(P) = \{v_0, v_1, \ldots, v_n\} \) of nodes, a finite ordered set \( C_y(P) = \{j_1, j_2, \ldots, j_s\} \) of arcs and a function \( p(P) \) on \( Y \) called the index of \( P \) such that

\[
\begin{align*}
v_0 &= \alpha, \quad v_n = \beta, \quad v_i \neq v_k \quad (i \neq k), \\
e(j_i) &= \{v_{i-1}, v_i\} \quad \text{if} \quad j \in C_y(P), \\
p_j(P) &= 0 \quad \text{if} \quad j \notin C_y(P), \\
p_j(P) &= -K_{v_j} \text{ with } v = v_{i-1} \quad \text{if} \quad j = j_i.
\end{align*}
\]

\( (P) \)

A path \( P \) from node \( \alpha \) to the ideal boundary \( \infty \) of \( \langle X, \gamma \rangle \) is the triple \((C_x(P), C_y(P), p(P))\) of an infinite ordered set \( C_x(P) = \{v_0, v_1, \ldots\} \) of nodes, an infinite ordered set \( C_y(P) = \{j_1, j_2, \ldots\} \) of arcs and a function \( p(P) \) on \( Y \) which satisfy condition \( (P) \) except the terminal condition \( v_n = \beta \).

Denote by \( P_{\alpha \beta} \) (resp. \( P_{\alpha \infty} \)) the set of all paths from node \( \alpha \) to node \( \beta \) (resp. \( \infty \)). Note that condition (1.3) means \( P_{\alpha \beta} \neq \emptyset \) for any \( \alpha, \beta \in X \). For mutually disjoint nonempty subsets \( A \) and \( B \) of \( X \), denote by \( P_{A,B} \) the set of all paths \( P \) such that \( P \in P_{\alpha \beta} \), \( C_x(P) \cap A = \{\alpha\} \) and \( C_y(P) \cap B = \{\beta\} \) for some \( \alpha \in A \) and \( \beta \in B \). Let \( P_{A,\infty} \) be the set of all paths \( P \) such that \( P \in P_{\alpha \infty} \) and \( C_x(P) \cap A = \{\alpha\} \) for some \( \alpha \in A \).

Let \( \Gamma \) be a set of paths in an infinite network \( \langle X, \gamma, \delta \rangle \). For every \( W \in L^+(Y) \), a value \( t(W; \Gamma) \) is defined by

\[
(2.1) \quad t(W; \Gamma) = \inf \{ \sum r_j W_j; P \in \Gamma \},
\]

where \( \sum r_j W_j \) is an abbreviation of \( \sum_{j \in C_y(P)} r_j W_j \).

We define the extremal length \( \lambda_p(\Gamma) \) of \( \Gamma \) of order \( p \) by

\[
(2.2) \quad \lambda_p(\Gamma)^{-1} = \inf \{ H_p(W); W \in E_p(\Gamma) \},
\]

where

\[
E_p(\Gamma) = \{ W \in L^+_p(Y; r); t(W; \Gamma) \geq 1 \}.
\]

We use the convention in this paper that the infimum of a real function on the empty set \( \phi \) is equal to \( \infty \). We shall study some properties of the extremal length which are analogous to the continuous case (cf. [6]).

Let \( \Gamma_1 \) and \( \Gamma_2 \) be sets of paths in \( \langle X, \gamma \rangle \). We shall write \( \Gamma_1 < \Gamma_2 \) if for any \( P^{(2)} \in \Gamma_2 \) there is a \( P^{(1)} \in \Gamma_1 \) such that \( C_y(P^{(1)}) \subset C_y(P^{(2)}) \).

We easily obtain

**Lemma 2.1.** If \( \Gamma_1 \) and \( \Gamma_2 \) are sets of paths in \( \langle X, \gamma \rangle \) such that \( \Gamma_1 < \Gamma_2 \), then \( \lambda_p(\Gamma_1) \leq \lambda_p(\Gamma_2) \).

**Proposition 2.1.** Let \( P \) be a path and set \( R(P) = \sum r_j \). Then \( \lambda_p(\{P\}) = R(P)^{p^{-1}} \).
PROOF. Let \( W \in E_p(\{P\}) \). Then \( \sum p r_j W_j \geq 1 \). It follows from Hölder’s inequality that \( 1 \leq R(\{P\})^1/\sigma H_p(W)^{1/\sigma} \). Thus we have \( \lambda_p(\{P\}) \leq R(\{P\})^{\sigma - 1} \). Next we show the converse inequality. Let \( \{<X_\alpha, Y_\alpha>\} \) be an exhaustion of \( <X, Y> \) such that \( C_\gamma(P) \cap Y_1 \neq \phi \). Set \( Y'_n = C_\gamma(P) \cap Y_n \) and define \( W'(n) \in L(Y) \) by \( W'_j(n) = (\sum r_j)^{-1} \) if \( j \in Y'_n \) and \( W'_j(n) = 0 \) if \( j \notin Y'_n \). Then \( W'(n) \in E_p(\{P\}) \) and

\[
\lambda_p(\{P\}) \geq H_p(W'(n))^{-1} = (\sum r_j)^{\sigma - 1}.
\]

By letting \( n \rightarrow \infty \), we conclude that \( \lambda_p(\{P\}) \geq R(\{P\})^{\sigma - 1} \). This completes the proof.

Let \( \Gamma_1 \) and \( \Gamma_2 \) be sets of paths in \( <X, Y> \). We say that \( \Gamma_1 \) and \( \Gamma_2 \) are mutually disjoint if \( C_\gamma(P^{(1)}) \cap C_\gamma(P^{(2)}) = \phi \) for every \( P^{(1)} \in \Gamma_1 \) and \( P^{(2)} \in \Gamma_2 \).

LEMMA 2.2.4) Let \( \{\Gamma_n; n = 1, 2, \ldots\} \) be mutually disjoint sets of paths and \( \Gamma \) be a set of paths. If \( \Gamma_n \subset \Gamma \) for each \( n \), then

\[
\lambda_p(\Gamma)^{\sigma - 1} \geq \sum_{n=1}^{\infty} \lambda_p(\Gamma_n)^{\sigma - 1}.
\]

PROOF. If \( \lambda_p(\Gamma_n) = \infty \) for at least one \( n \), our inequality is valid by Lemma 2.1. Therefore we may assume that \( \lambda_p(\Gamma_n) < \infty \) for each \( n \). Moreover we may assume that \( \lambda_p(\Gamma_n) > 0 \), i.e., \( E_p(\Gamma_n) \neq \phi \) for each \( n \). Let \( Y_n = \cup \{C_\gamma(P); P \in \Gamma_n\} \). Then

\[
\lambda_p(\Gamma_n)^{-1} = \inf \{H_p(W); W \in E_p(\Gamma_n) \text{ and } W = 0 \text{ on } Y_n \}.
\]

Choose any positive integer \( m \) and fix it. Let \( t_1, t_2, \ldots, t_m \) be non-negative numbers such that \( \sum_{n=1}^{m} t_n = 1 \); they will be determined below. Taking \( W_j = \sum_{n=1}^{m} t_n W_j(n) \) with \( W(n) \in E_p(\Gamma_n) \) such that \( W(n) = 0 \) on \( Y - Y(n) \), we have \( W_j = t_n W_j(n) \) for each \( j \in Y_n \) and

\[
\sum p r_j W_j = \sum_{n=1}^{m} t_n \sum p r_j W_j(n) \geq \sum_{n=1}^{m} t_n = 1
\]

for every \( P \in \Gamma \), so that \( W \in E_p(\Gamma) \). Therefore

\[
\lambda_p(\Gamma)^{-1} \leq \sum_{j=1}^{\infty} r_j \left( \sum_{n=1}^{m} t_n W_j(n) \right)^{\sigma} = \sum_{n=1}^{m} t_n^P H_p(W(n)).
\]

It follows that

\[
\lambda_p(\Gamma)^{-1} \leq \sum_{n=1}^{m} t_n^P \lambda_p(\Gamma_n)^{-1}.
\]

4) Cf. [6], p. 79, Theorem 2.10.
Now we choose \( t_n = \lambda_p(\Gamma_n)^{q-1} \left( \sum_{n=1}^{m} \lambda_p(\Gamma_n)^{q-1} \right)^{-1} \) and obtain
\[
\lambda_p(\Gamma)^{-1} \leq \left[ \sum_{n=1}^{m} \lambda_p(\Gamma_n)^{q-1} \right]^{1-p},
\]
which leads to the desired inequality.

Let \( A \) and \( B \) be mutually disjoint nonempty subsets of \( X \). We define the extremal distance \( EL_p(A, B) \) (resp. \( EL_p(A, \infty) \)) of order \( p \) of an infinite network \(<X, Y, K, r> \) relative to \( A \) and \( B \) (resp. \( A \) and \( \infty \)) by
\[
(2.3) \quad EL_p(A, B) = \lambda_p(P_{A,B}),
\]
\[
(2.4) \quad EL_p(A, \infty) = \lambda_p(P_{A,\infty}).
\]

Next we consider the following extremum problem:
\[
(2.5) \quad \text{Find } d_p(A, B) = \inf \{ D_p(u); u \in L(X), u = 0 \text{ on } A \text{ and } u = 1 \text{ on } B \}.
\]

We have

**LEMMA 2.3.** Let \( V \in L^+(Y) \). There exists \( u \in L(X) \) such that \( u = 0 \) on \( A \),
\[
(2.6) \quad \sum_{v=0}^{t_v} K_{uv} u_v \leq V_j \quad \text{for each } j \in Y,
\]
and
\[
(2.7) \quad \inf \{ \sum_{j \in Y} V_j; P \in P_{A,B} \} = \inf \{ u_v; v \in B \}.
\]

**THEOREM 2.1.** \( d_p(A, B) = EL_p(A, B)^{-1} \).

**PROOF.** We set \( d_p = d_p(A, B) \) and \( EL_p = EL_p(A, B) \). First we shall prove \( d_p \leq EL_p^{-1} \) in case \( EL_p^{-1} < \infty \). Let \( W \in E_p(P_{A,B}) \) and put \( V_j = r_j W_j \). Then \( \inf \{ \sum_{j \in Y} V_j; P \in P_{A,B} \} = t(W; P_{A,B}) \geq 1 \). We can find \( u \in L(X) \) by Lemma 2.3 such that \( u = 0 \) on \( A \) and \( u \) satisfies (2.6) and (2.7). Then \( u \geq 1 \) on \( B \) and
\[
D_p(u) = \sum_{j=1}^{t_v} r_j^{-p} \sum_{v=0}^{t_v} K_{uv} u_v \leq \sum_{j=1}^{t_v} r_j^{-p} V_j = H_p(W) < \infty.
\]
Let \( v = \min \{ u, 1 \} \). Then \( v = 0 \) on \( A \) and \( v = 1 \) on \( B \), so that
\[
d_p \leq D_p(v) \leq D_p(u) \leq H_p(W)
\]

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5) Cf. Theorem 3 in [7].
by Lemma 1.2. By the arbitrariness of \( W \), we obtain \( d_p \leq EL_p^{-1} \). Next we shall show that \( EL_p^{-1} \leq d_p \) in case \( d_p < \infty \). Let \( u \in L(X) \) satisfy \( u = 0 \) on \( A \), \( u = 1 \) on \( B \) and \( D_p(u) < \infty \). Define \( W \in L^+(Y) \) by \( W_j = r_j^{-1} \sum_{i=0}^{\infty} K_{ij} u_i \). Then it is easily seen that \( W \in E_p(\mathcal{P}_{A,B}) \) (cf. the proof of Theorem 4 in [7]). Hence \( EL_p^{-1} \leq H_p(W) = D_p(u) \) and \( EL_p^{-1} \leq d_p \). Thus we have \( d_p = EL_p^{-1} \).

By the aid of Theorem A, we have

**Proposition 2.2.** In case \( E_p(\mathcal{P}_{A,B}) \neq \emptyset \), there exists a unique \( \hat{W} \in E_p(\mathcal{P}_{A,B}) \) such that \( EL_p(A, B) = H_p(\hat{W}) \).

**Proposition 2.3.** In case \( \{ u \in D^p(A); u = 1 \text{ on } B \} \neq \emptyset \), there exists a unique optimal solution \( \hat{u} \) of problem (2.5), i.e., \( \hat{u} \in \{ u \in D^p(A); u = 1 \text{ on } B \} \) such that \( d_p(A, B) = D_p(\hat{u}) \).

Hereafter in this section, we always assume that \( A \) is a nonempty finite subset of \( X \) and that \( \{ <X_n, Y_n> \} \) is an exhaustion of \( <X, Y> \) such that \( A \subseteq X_1 \). We shall be concerned with the relation between \( EL_p(A, X - X_n) \) and \( EL_p(A, \infty) \).

We prepare

**Lemma 2.4.** Let \( W \in L^+(Y) \) and set \( t_n(W) = t(W; \mathcal{P}_{A,X-X_n}) \) and \( t(W) = t(W; \mathcal{P}_{A,\infty}) \). Then \( t_n(W) \leq t_{n+1}(W) \leq t(W) \) and \( t_n(W) \to t(W) \) as \( n \to \infty \). Furthermore there exists \( P \in \mathcal{P}_{A,\infty} \) such that \( t(W) = \sum r_j W_j \).

**Proof.** Since \( \mathcal{P}_{A,X-X_n} \subseteq \mathcal{P}_{A,X-X_{n+1}} \subseteq \mathcal{P}_{A,\infty} \), we have \( t_n(W) \leq t_{n+1}(W) \leq t(W) \). For each \( n \) there exists \( P^{(n)} \in \mathcal{P}_{A,X-X_n} \) such that \( t_n(W) = \sum_{P^{(n)}} r_j W_j \). Since \( A \) is a finite set, there is \( x_0 \in A \) such that \( x_0 \in C_X(P^{(n)}) \) for infinitely many \( n \). For each \( x \in X \), we put

\[
Y(x) = \{ j \in Y; K_{x,j} \neq 0 \},
\]

\[
X(x) = \{ v \in X; v \neq x \text{ and } K_{v,j} \neq 0 \text{ for some } j \in Y(x) \}.
\]

Since \( X(x_0) \) is a finite subset of \( X \), there are \( x_1 \in X(x_0) \) and \( j_1 \in Y(x_0) \) such that \( e(j_1) = \{ x_0, x_1 \} \) and \( j_1 \in C_Y(P^{(n)}) \) for infinitely many \( n \). Similarly there are \( x_2 \in X(x_1) \) and \( j_2 \in Y(x_1) \) such that \( e(j_2) = \{ x_1, x_2 \} \) and \( \{ j_1, j_2 \} \subseteq C_Y(P^{(n)}) \) for infinitely many \( n \). Repeating this process, we can define ordered sets \( C_X(P) \) and \( C_Y(P) \) by

\[
C_X(P) = \{ x_0, x_1, x_2, \ldots \} \quad \text{and} \quad C_Y(P) = \{ j_1, j_2, \ldots \}.
\]

Define \( p(P) \in L(Y) \) by \( p(P) = -K_{x,j} \) with \( v = x_{i-1} \) if \( j = j_i \) and \( p(P) = 0 \) if \( j \notin C_Y(P) \). Then \( P \in \mathcal{P}_{a_0} \). For any \( m \), there are infinitely many \( n \) such that \( \{ j_1, j_2, \ldots, j_m \} \subseteq C_Y(P^{(n)}) \). Thereby we have
\[ \sum_{k=1}^{m} r_k W_k \leq \sum_{j \in \mathcal{F}(\mathcal{G})} r_j W_j = t_d(W) \leq \lim_{n \to \infty} t_d(W). \]

By letting \( m \to \infty \), we have

\[ t(W) = \sum_{j} r_j W_j \leq \lim_{n \to \infty} t_d(W). \]

This completes the proof.

We have

**Theorem 2.2.** \( \lim_{n \to \infty} EL_p(A, X - X_n) = EL_p(A, \infty) \).

**Proof.** Since \( P_{A,x-X_n} < P_{A,x-X_{n+1}} < P_{A,\infty} \), we have \( EL_p(A, X - X_n) \leq EL_p(A, X - X_{n+1}) \leq EL_p(A, \infty) \) by Lemma 2.1. Therefore

\[ \lim_{n \to \infty} EL_p(A, X - X_n) = EL_p(A, \infty). \]

Let \( W \in E_p(P_{A,\infty}) \). Then \( t(W) t(W; P_{A,\infty}) \geq 1 \). Since \( t_d(W) t(W; P_{A,x-X_n}) \to t(W) \) as \( n \to \infty \) by Lemma 2.4, we may assume that \( t_d(W) > 0 \) for all \( n \). Writing \( W^{(n)} = W/t_d(W) \), we see that \( W^{(n)} \in E_p(P_{A,x-X_n}) \) and \( EL_p(A, X - X_n) \geq H_p(W^{(n)})^{-1} = t_d(W) p(H_p(W))^{-1} \). It follows that

\[ \lim_{n \to \infty} EL_p(A, X - X_n) \geq t(W)p(H_p(W))^{-1} \geq H_p(W)^{-1} \]

for all \( W \in E_p(P_{A,\infty}) \). Hence \( \lim_{n \to \infty} EL_p(A, X - X_n) \geq EL_p(A, \infty) \). This completes the proof.

We shall give upper and lower bounds for \( EL_p(A, \infty) \).

**Proposition 2.4.** \( EL_p(A, \infty) \leq R(P)p^{-1} \) for every \( P \in P_{A,\infty} \).

**Proof.** Let \( P \in P_{A,\infty} \). Then

\[ EL_p(A, \infty) \leq \lambda_p(\{P\}) = R(P)p^{-1} \]

by Lemma 2.1 and Proposition 2.1.

By taking \( \Gamma = P_{X_n,X_{n+1}-X_n} \) and \( \Gamma = P_{A,\infty} \) in Lemma 2.2, we obtain

**Proposition 2.5.** \( EL_p(A, \infty)^{q^{-1}} \geq \sum_{n=1}^{\infty} EL_p(X_n, X_{n+1}-X_n)^{q^{-1}}. \)

We have

**Proposition 2.6.** Let \( Z_n = Y_{n+1} - Y_n \) and \( \mu_n = \sum_{Z_n} r_j^{-p} \). Then

\[ EL_p(A, \infty)^{q^{-1}} \geq \sum_{n=1}^{\infty} \mu_n^{1-q}. \]
PROOF. In view of Proposition 2.5, it suffices to show that $\lambda_p(\Gamma_n)^{-1} \leq \mu_n$ for all $n$, where $\Gamma_n = P_{X_n, X_{n+1}}^{-1}$. Put $U_n = \cup \{C_f(P); P \in \Gamma_n\}$. Then $U_n \subset Z_n$. Define $W^{(n)} \in L(Y)$ by $W^{(n)}_{j} = r_j^{-1}$ if $j \in Z_n$ and $W^{(n)}_{j} = 0$ if $j \notin Z_n$. Then $W^{(n)} \in E_p(\Gamma_n)$ and

$$\lambda_p(\Gamma_n)^{-1} \leq H_p(W^{(n)}) = \sum_{Z_n} r_j^{-p} = \mu_n.$$

§3. Max-flows and min-cuts

Let $A$ and $B$ be mutually disjoint nonempty subsets of $X$. We say that a subset $Q$ of $Y$ is a cut between $A$ and $B$ if there exist mutually disjoint subsets $Q(A)$ and $Q(B)$ of $X$ such that $A \subset Q(A)$, $B \subset Q(B)$, $X = Q(A) \cup Q(B)$ and the set

$$Q(A) \cap Q(B) = \{ j \in Y; K_{aj}K_{bj} = -1 \text{ for some } a \in Q(A) \text{ and } b \in Q(B) \}$$

is equal to $Q$.

Let $A$ be a nonempty finite subset of $X$. We say that a subset $Q$ of $Y$ is a cut between $A$ and the ideal boundary $\infty$ of $\langle X, Y \rangle$ if there exist mutually disjoint subsets $Q(A)$ and $Q(\infty)$ such that $A \subset Q(A)$, $Q(\infty) = X - Q(A)$, $Q(A)$ is a finite set and $Q = Q(A) \cap Q(\infty)$. Denote by $Q_{A,B}$ (resp. $Q_{A,\infty}$) the set of all cuts between $A$ and $B$ (resp. $\infty$). We define the characteristic function $u = u(Q) \in L(X)$ of $Q \in Q_{A,B}$ and the index $s = s(Q) \in L(Y)$ of $Q$ by

$$u_\nu = 0 \text{ if } \nu \in Q(A) \quad \text{and} \quad u_\nu = 1 \text{ if } \nu \in Q(B),$$

$$s_j = \sum_{\nu=0}^{\infty} K_{\nu j}u_\nu.$$

We have $s_j = 0$ if $j \notin Q$ and $|s_j| = 1$ if $j \in Q$.

Let $A$ and $B$ be mutually disjoint nonempty finite subsets of $X$. We say that $w \in L(Y)$ is a flow from $A$ to $B$ of strength $I(w)$ if

$$(3.1) \quad \sum_{j=1}^{\infty} K_{\nu j}w_j = 0 \quad (\nu \notin A \cup B),$$

$$(3.2) \quad I(w) = - \sum_{\nu \in A} \sum_{j=1}^{\infty} K_{\nu j}w_j = \sum_{\nu \in B} \sum_{j=1}^{\infty} K_{\nu j}w_j.$$

Denote by $F(A, B)$ the set of all flows from $A$ to $B$ and set

$$G(A, B) = F(A, B) \cap L_0(Y).$$

Let $F_q(A, B)$ be the closure of $G(A, B)$ in $L_q(Y; r)$. Thus for any $w \in F_q(A, B)$, there exists a sequence $\{w^{(n)}\}$ in $G(A, B)$ such that $H_q(w - w^{(n)}) \to 0$ as $n \to \infty$. It follows that $w \in F(A, B)$ and $I(w^{(n)}) \to I(w)$ as $n \to \infty$. 
Let $g_p(t)$ be the real function on the real line $\mathbb{R}$ defined by

$$g_p(t) = |t|^{p-1} \text{sign}(t).$$

It is clear that

$$tg_p(t) = |t|^p \quad \text{and} \quad \frac{d}{dt}|t|^p = pg_p(t).$$

We say that $w \in L(Y)$ is a $p$-flow from $A$ to $B$ of strength $I_p(w)$ if $g_p \circ w$ is a flow from $A$ to $B$ and $I_p(w) = I(g_p \circ w)$. Denote by $F(p)(A, B)$ the set of all $p$-flows from $A$ to $B$ and set

$$G(p)(A, B) = F(p)(A, B) \cap L_0(Y).$$

It is clear that $F(2)(A, B) = F(A, B)$ and $I_2(w) = I(w)$. We remark that a $p$-flow is a non-linear flow in the sense of Birkhoff [1] and Duffin [3].

**REMARK 3.1.** $w \in G(p)(A, B)$ if and only if $g_p \circ w \in G(A, B)$.

**REMARK 3.2.** Let $A$ and $B$ be mutually disjoint nonempty finite subsets of $X$ and let $\hat{u}$ be the optimal solution of problem (2.5). Define $\hat{w} \in L(Y)$ by

$$\hat{w}_j = r_j^{-1} \sum_{v=0}^{\infty} K_{vj} \hat{u}_v.$$  

Then it can be shown that $\hat{w} \in F(p)(A, B)$.

We prepare

**LEMMA 3.1.** Let $u \in L(X)$ and $w \in L(Y)$. Then

$$(3.3) \quad \sum_{v=0}^{\infty} u_v \left( \sum_{j=1}^{\infty} K_{vj} w_j \right) = \sum_{j=1}^{\infty} w_j \left( \sum_{v=0}^{\infty} K_{vj} u_v \right)$$

holds if any one of the following conditions is fulfilled:

(i) \hspace{1cm} $u \in L_0(X)$ or $w \in L_0(Y)$.

(ii) \hspace{1cm} $D_p(u) < \infty$ and $w \in F_q(A, B)$.

**PROOF.** If condition (i) is satisfied, then (3.3) is clear. Assume condition (ii). Then there exists a sequence $\{w^{(n)}\}$ in $G(A, B)$ such that $H_q(w - w^{(n)}) \to 0$ as $n \to \infty$. We have

$$\sum_{j=1}^{\infty} w_j^{(n)} \left( \sum_{v=0}^{\infty} K_{vj} u_v \right) = \sum_{v=0}^{\infty} u_v \left( \sum_{j=1}^{\infty} K_{vj} w_j^{(n)} \right)$$
as \( n \to \infty \), since \( w_j^{(n)} \to w_j \) as \( n \to \infty \) for each \( j \in Y \). On the other hand, we have
\[
\sum_{j=1}^{\infty} |w_j - w_j^{(n)}| \leq \left[ H_q(w-w^{(n)}) \right]^{1/q} \left[ D_p(u) \right]^{1/p}
\]
by Hölder's inequality, so that
\[
\sum_{j=1}^{\infty} w_j \left( \sum_{v=0}^{\infty} K_{v,j} u_v \right) = \lim_{n \to \infty} \sum_{j=1}^{\infty} w_j^{(n)} \left( \sum_{v=0}^{\infty} K_{v,j} u_v \right) = \sum_{v=0}^{\infty} u_v \left( \sum_{j=1}^{\infty} K_{v,j} w_j \right).
\]
This completes the proof.

Let \( W \in L^+(Y) \). Let us consider the following extremum problems which are generalizations of the max-flow problem in network theory on a finite graph.

(3.4) Find
\[
M(W; F_q(A, B)) = \sup \{ I(w); w \in F_q(A, B) \text{ and } |w_j| \leq W_j \text{ on } Y \}.
\]

(3.5) Find
\[
M(W; G(A, B)) = \sup \{ I(w); w \in G(A, B) \text{ and } |w_j| \leq W_j \text{ on } Y \}.
\]

(3.6) Find
\[
M_p(W; G(A, B)) = \sup \{ I_p(w); w \in G(A, B) \text{ and } |w_j| \leq W_j \text{ on } Y \}.
\]

For \( W \in L^+(Y) \) let us denote by \( W^p \) the function \( V \in L(Y) \) defined by \( V_j = W_j^p \) for each \( j \in Y \).

On account of Remark 3.1, we have

**Proposition 3.1.** \( M_p(W; G(A, B)) = M(W^{p-1}; G(A, B)) \).

We shall prove

**Lemma 3.2.** Let \( W \in L^+_p(Y; r) \). Then there exists \( \hat{w} \in F_q(A, B) \) such that
\[
|\hat{w}_j| \leq W_j^{p-1} \text{ on } Y \text{ and } I(\hat{w}) = M(W^{p-1}; G(A, B)).
\]

**Proof.** There exists a sequence \( \{w^{(n)}\} \) in \( G(A, B) \) such that \( |w^{(n)}_j| \leq W_j^{p-1} \) on \( Y \) and \( I(w^{(n)}) \) converges to \( M(W^{p-1}; G(A, B)) \). Since \( L_q(Y; r) \) is a reflexive Banach space and \( \{w \in F_q(A, B); |w_j| \leq W_j^{p-1} \text{ on } Y\} \) is a bounded closed convex set in \( L_q(Y; r) \), we may assume that \( \{w^{(n)}\} \) converges weakly to \( \hat{w} \in L_q(Y; r) \). Then \( w^{(n)} \to \hat{w} \) as \( n \to \infty \) for each \( j \). Hence \( \hat{w} \in F_q(A, B) \), \( |\hat{w}_j| \leq W_j^{p-1} \) on \( Y \) and
\[
I(\hat{w}) = \sum_{v \in B} \sum_{j=1}^{\infty} K_{v,j} \hat{w}_j = \lim_{n \to \infty} I(w^{(n)}) = M(W^{p-1}; G(A, B)).
\]
This completes the proof.

Let \( W \in L^+(Y) \) and consider the following extremum problem which is a generalization of the min-cut problem in (finite) network theory:

(3.7) Find

\[
M^*(W; Q_{A,B}) = \inf \{ \sum_{Q} W_j ; Q \in Q_{A,B} \}.
\]

We have

**Lemma 3.3.**

\[ M(W; G(A, B)) = M^*(W; Q_{A,B}). \]

By Lemma 3.3 and Proposition 3.1, we have

**Corollary.**

\[ M_p(W; G^{(p)}(A, B)) = M^*(W^{p-1}; Q_{A,B}). \]

§ 4. Generalized extremal width of a network

Let \( A \) and \( B \) be mutually disjoint nonempty subsets of \( X \). We define the extremal width \( EW_p(A, B) \) of order \( p \) of an infinite network \( <X, Y, K, r> \) relative to two sets \( A \) and \( B \) by the value of the following extremum problem.

(4.1) Find

\[
EW_p(A, B)^{-1} = \inf \{ H_p(W); W \in E_p^*(Q_{A,B}) \},
\]

where \( E_p^*(Q_{A,B}) = \{ W \in L^+_p(Y; r); \sum_{Q} W_j^{p-1} \geq 1 \text{ for all } Q \in Q_{A,B} \} \).

Hereafter in this section we always assume that \( A \) and \( B \) are finite subsets of \( X \). In connection with the above problem, we consider the following extremum problems.

(4.2) Find

\[
d_p^*(A, B) = \inf \{ H_q(w); w \in F_q(A, B) \text{ and } I(w) = 1 \}.
\]

(4.3) Find

\[
d_p^*(A, B) = \inf \{ H_{p}(w); w \in G^{(p)}(A, B) \text{ and } I_{p}(w) = 1 \}.
\]

We shall prove

**Proposition 4.1.**

\[
\hat{d}_p^*(A, B) = d_p^*(A, B) = \inf \{ H_q(w); w \in G(A, B) \text{ and } I(w) = 1 \}.
\]

**Proof.** We set \( \hat{d}_p^* = \hat{d}_p^*(A, B) \) and \( d_p^* = d_p^*(A, B) \). By Remark 3.1 and by the relations \( I(g_p^*w) = I_p^*(w) \) and \( H_q(g_p^*w) = H_p(w) \), we have

6) Cf. Theorem 6 in [7].
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(4.4) \[ \hat{d}_p^* = \inf \{ H_q(z); z \in G(A, B) \text{ and } I(z) = 1 \}, \]

so that \( \hat{d}_p^* \geq d_q^* \). On the other hand, let \( w \in F_q(A, B) \) and \( I(w) = 1 \). There exists a sequence \( \{ w^{(n)} \} \) in \( G(A, B) \) such that \( H_q(w-w^{(n)}) \to 0 \) as \( n \to \infty \). Since \( I(w^{(n)}) \to I(w) \) as \( n \to \infty \), we may suppose that \( I(w^{(n)}) > 0 \) for all \( n \). It follows from (4.4) that

\[ \hat{d}_p^* \leq H_q(w^{(n)}/I(w^{(n)})) = H_q(w^{(n)})/(I(w^{(n)}))^{q}. \]

By letting \( n \to \infty \), we have \( \hat{d}_p^* \leq H_q(w) \), so that \( \hat{d}_p^* \leq d_q^* \). Hence \( \hat{d}_p^* = d_q^* \).

**Theorem 4.1.** \( EW_p(A, B)^{-1} = d_q^*(A, B) \).

**Proof.** We set \( EW_p = EW_p(A, B) \) and \( d_q^* = d_q^*(A, B) \). For each \( w \in G(A, B) \) such that \( I(w) = 1 \), consider \( W \in L^1(Y) \) defined by \( W_j = |w_j|^{-1/p-1} \) on \( Y \). Then we show that \( W \in E^*_p(Q_{A,B}) \). Let \( u = u(Q) \) be the characteristic function of \( Q \in Q_{A,B} \). We have by Lemma 3.1

\[ 1 = I(w) = \sum \limits_{v=0}^{\infty} u_v \left( \sum \limits_{j=1}^{\infty} K_{j,v} w_j \right) = \sum \limits_{j=1}^{\infty} W_j \left( \sum \limits_{v=0}^{\infty} K_{j,v} u_v \right) \]

\[ \leq \sum \limits_{j=1}^{\infty} |w_j| \left( \sum \limits_{v=0}^{\infty} K_{j,v} u_v \right) = \sum \limits_{Q} W_j^{p-1}. \]

Therefore \( W \in E^*_p(Q_{A,B}) \) and

\[ EW_p^{-1} \leq H_p(W) = \sum \limits_{j=1}^{\infty} r_j |w_j|^{1/(p-1)} = H_q(w). \]

Thus we have \( EW_p^{-1} \leq d_q^* \) by Proposition 4.1. On the other hand, let \( W \in E^*_p(Q_{A,B}) \), i.e., \( W \in L^1(Y; r) \) and \( M(W^{p-1}; Q_{A,B}) \geq 1 \). We can find \( w \in F_q(A, B) \) such that \( |w_j| \leq W_j^{p-1} \) on \( Y \) and \( M(W^{p-1}; G(A, B)) = I(w) \) by Lemma 3.2. It follows from Lemma 3.3 that \( I(w) \geq 1 \). We have

\[ d_q^* \leq H_q(w/I(w)) \leq H_q(w) = \sum \limits_{j=1}^{\infty} r_j |w_j|^q \]

\[ \leq \sum \limits_{j=1}^{\infty} r_j W_j^{q/(p-1)} = H_p(W), \]

so that \( d_q^* \leq EW_p^{-1} \). Therefore \( d_q^* = EW_p^{-1} \).

By the aid of Theorem A, we have

**Proposition 4.2.** There exists a unique \( \hat{w} \in F_q(A, B) \) such that \( I(\hat{w}) = 1 \) and \( d_q^*(A, B) = H_q(\hat{w}) \), i.e., \( \hat{w} \) is the optimal solution of problem (4.2).

Let \( A \) be a nonempty finite subset of \( X \). We define the extremal width
\[ EW_p(A, \infty) \] of order \( p \) of an infinite network relative to \( A \) and \( \infty \) by the value of the following extremum problem.

(3.5) Find
\[
EW_p(A, \infty)^{-1} = \inf \{ H_p(W); W \in E^*_p(Q_{A, \infty}) \},
\]
where \( E^*_p(Q_{A, \infty}) = \{ W \in L^*_p(Y; r); \sum_Q W^r_{j-1} \geq 1 \text{ for all } Q \in Q_{A, \infty} \} \).

Let \( \{ <X_n, Y_n> \} \) be an exhaustion of \( <X, Y> \) such that \( A \subset X_1 \). We shall be concerned with the relation between \( EW_p(A, X-X_n) \) and \( EW_p(A, \infty) \).

We shall prove

**Theorem 4.2.** \( \lim_{n \to \infty} EW_p(A, X-X_n) = EW_p(A, \infty) \).

**Proof.** Since \( Q_{A,X-X_n} \subset Q_{A,X-X_{n+1}} \subset Q_{A,\infty} \), we have \( EW_p(A, X-X_n) \leq EW_p(A, X-X_{n+1}) \leq EW_p(A, X-X_0) \), and hence

\[
\lim_{n \to \infty} EW_p(A, X-X_n) \geq EW_p(A, \infty).
\]

To prove the converse inequality we may assume that \( \lim EW(A, X-X_n) > 0 \).

For each \( n \), there is \( W^{(n)} \in E^*_p(Q_{A,X-X_n}) \) such that \( EW_p(A, X-X_n) = H_p(W^{(n)})^{-1} \).

Since \( \{ H_p(W^{(n)}) \} \) is a bounded sequence and \( L_p(Y; r) \) is a reflexive Banach space, we can choose a weakly convergent subsequence of \( \{ W^{(n)} \} \). Denote by \( \{ W^{(n)} \} \) the subsequence again and let \( \tilde{W} \) be the weak limit. We show that \( \tilde{W} \in E^*_p(Q_{A,\infty}) \).

Let \( Q \in Q_{A,\infty} \) with \( Q = Q(A) \ominus Q(\infty) \). Since \( Q(A) \) is a finite set, there is a number \( n_0 \) such that \( Q(A) \subset X_{n_0} \). Then \( X-X_n \subset Q(\infty) \) and hence \( Q \in Q_{A,X-X_n} \) for all \( n \geq n_0 \). Therefore \( \sum_Q [W_j^{(n)}]^{r-1} \geq 1 \) for all \( n \geq n_0 \). Since \( \{ W^{(n)} \} \) converges weakly to \( \tilde{W} \) and \( Q \) is a finite set, we obtain \( \sum_Q \tilde{W}_j^{r-1} \geq 1 \). Thus \( \tilde{W} \in E^*_p(Q_{A,\infty}) \).

Since \( [H_p(w)]^{1/p} \) is weakly lower semicontinuous in \( L_p(Y; r) \), we have

\[
\lim_{n \to \infty} [EW_p(A, X-X_n)]^{-1} = \lim_{n \to \infty} H_p(W^{(n)})^{-1} \geq H_p(\tilde{W}) \geq [EW_p(A, \infty)]^{-1}.
\]

This completes the proof.

§ 5. A reciprocal relation between \( EL_p \) and \( EW_p \)

Let \( A \) and \( B \) be mutually disjoint nonempty finite subsets of \( X \).

We prepare

**Lemma 5.1.** Let \( \hat{w} \) be the optimal solution of problem (4.2). If \( w' \in F(A, B) \) and \( I(w') = 0 \), then...
(5.1) \[ \sum_{j=1}^{\infty} r_j w_j \varrho_q(\bar{w}_j) = 0. \]

**Proof.** For any real number \( t \), we have \( \bar{w} + tw' \in F_q(A, B) \) and \( I(\bar{w} + tw') = 1 \), so that \( d_q^*(A, B) = H_q(\bar{w}) \leq H_q(\bar{w} + tw') \). Thus the derivative of \( H_q(\bar{w} + tw') \) with respect to \( t \) vanishes at \( t = 0 \). Since \( H_q(\bar{w} + tw') \) can be differentiated term by term at \( t = 0 \), we obtain (5.1).

**Corollary 1.** Let \( \bar{w} \) be the optimal solution of problem (4.2) and \( P \) be a path from node \( \alpha \in A \) to node \( \beta \in B \). Then

(5.2) \[ d_q^*(A, B) = \sum_{j=1}^{\infty} r_j p_j(P) \varrho_q(\bar{w}_j). \]

**Proof.** Note that \( p(P) \) is a flow from \( \{ \alpha \} \) to \( \{ \beta \} \) such that \( I(p(P)) = 1 \). Taking \( w' = \bar{w} - p(P) \), we see that \( w' \in F_q(A, B) \) and \( I(w') = 0 \). Thus we have by (5.1)

\[ \sum_{j=1}^{\infty} r_j (\bar{w}_j - p_j(P)) \varrho_q(\bar{w}_j) = 0. \]

Therefore

\[ d_q^*(A, B) = H_q(\bar{w}) = \sum_{j=1}^{\infty} r_j \bar{w}_j \varrho_q(\bar{w}_j) = \sum_{j=1}^{\infty} r_j p_j(P) \varrho_q(\bar{w}_j). \]

**Corollary 2.** Let \( \bar{w} \) be the optimal solution of problem (4.2) and let \( \alpha, \nu \in X (\alpha \neq \nu) \). If \( P \) and \( P' \) are paths from node \( \alpha \) to node \( \nu \), then

(5.3) \[ \sum_{j=1}^{\infty} r_j p_j(P) \varrho_q(\bar{w}_j) = \sum_{j=1}^{\infty} r_j p_j(P') \varrho_q(\bar{w}_j). \]

**Proof.** Taking \( w' = p(P) - p(P') \), we see that \( w' \in F_q(A, B) \) and \( I(w') = 0 \). Then (5.3) follows from (5.1).

Let \( \bar{w} \) be the optimal solution of problem (4.2). For any \( \alpha \in A \), we define \( v^{(\alpha)} \in L(X) \) by

(5.4) \[ v^{(\alpha)}_\alpha = 0, \quad v^{(\alpha)}_\nu = \sum_{j=1}^{\infty} r_j p_j(P) \varrho_q(\bar{w}_j) \quad (\nu \neq \alpha) \]

for some path \( P \) from node \( \alpha \) to node \( \nu \). It follows from Corollary 2 of Lemma 5.1 that \( v^{(\alpha)} \) is uniquely determined by \( \bar{w} \). Define \( \delta \in L(X) \) by

(5.5) \[ \delta_\nu = \inf \{|v^{(\alpha)}_\nu|; \alpha \in A\}. \]

We have

**Lemma 5.2.** Let \( \delta \) be the function defined by (5.4) and (5.5). Then \( \delta \)
=0 on $A$, $\psi = d_\infty^*(A, B)$ on $B$ and

$$
(5.6) \quad \left| \sum_{v=0}^{\infty} K_{v'} \psi_v \right| \leq r_j |\hat{\psi}_j|^{q-1} \quad \text{on } Y.
$$

**Proof.** Since $v_\alpha(x) = 0$ for any $\alpha \in A$, we have $\psi = 0$ on $A$. We have $\psi = d_\infty^*(A, B)$ on $B$ by Corollary 1 of Lemma 5.1. The proof of (5.6) is carried out by the same reasoning as in the proof of Lemma 12 in [7].

We shall prove

**Theorem 5.1.** \([d_p(A, B)]^{1/p}[d_\infty^*(A, B)]^{1/q} = 1.\)

**Proof.** We set $d_p = d_p(A, B)$ and $d_\infty^* = d_\infty^*(A, B)$. First we show that $1 \leq (d_p)^{1/p}(d_\infty^*)^{1/q}$. For any $v \in L(X)$ such that $v = 0$ on $A$, $v = 1$ on $B$ and $D_p(v) < \infty$ and any $w \in F_q(A, B)$ such that $I(w) = 1$, we have by Lemma 3.1

$$
1 = I(w) = \sum_{v=0}^{\infty} v \left( \sum_{j=1}^{\infty} K_{v'} w_j \right) = \sum_{j=1}^{\infty} w_j \left( \sum_{v=0}^{\infty} K_{v'} v_r \right)
$$

$$
\leq [D_p(v)]^{1/p}[H_q(w)]^{1/q},
$$

which leads to the desired inequality. Next we show that $(d_p)^{1/p}(d_\infty^*)^{1/q} \leq 1$. Let $\hat{w}$ be the optimal solution of problem (4.2) and define $\psi \in L(X)$ by (5.4) and (5.5). Then we have by (5.6)

$$
D_p(\psi) = \sum_{j=1}^{\infty} r_j^{-p} \left| \sum_{v=0}^{\infty} K_{v'} \psi_v \right|^{p} \leq \sum_{j=1}^{\infty} r_j |\hat{\psi}_j|^{p(q-1)} = H_q(\hat{w}) = d_\infty^*.
$$

Writing $\check{\psi} = \psi/d_\infty^*$, we see by Lemma 5.2 that $\check{\psi} = 0$ on $A$ and $\check{\psi} = 1$ on $B$, so that

$$
d_p \leq D_p(\check{\psi}) = D_p(\psi) (d_\infty^*)^{-p} \leq (d_\infty^*)^{-p} = (d_\infty^*)^{-p/q},
$$

or $(d_p)^{1/p}(d_\infty^*)^{1/q} \leq 1$.

By Proposition 4.1 and Theorem 5.1, we have

**Corollary.** \([d_p(A, B)]^{1/p}[d_\infty^*(A, B)]^{1/q} = 1.\)

By Theorems 2.1, 4.1 and 5.1, we have

**Theorem 5.2.** \([EL_p(A, B)]^{1/p}[EW_p(A, B)]^{1/q} = 1.\)

Next we shall be concerned with the reciprocal relation between $EL_p(A, \infty)$ and $EW_p(A, \infty)$. Henceforth let $A$ be a nonempty finite subset of $X$ and \(\langle X_n, Y_n \rangle\) be an exhaustion of \(\langle X, Y \rangle\) such that $A \subseteq X_1$.

We prepare
LEMMA 5.3. For every \( Q \in \mathcal{Q}_{A, X_{n+1}-X_n} \) there exists \( Q' \in \mathcal{Q}_{A, X-X_n} \) such that \( Q' \subset Q \).

PROOF. Let \( Q \in \mathcal{Q}_{A, X_{n+1}-X_n} \) and \( Q = Q(A) \cup Q(X_{n+1}-X_n) \). Let us define \( Q'(A) \) and \( Q'(X-X_n) \) by

\[
Q'(A) = Q(A) \setminus (X-X_n) \quad \text{and} \quad Q'(X-X_n) = X-Q'(A).
\]

Since \( A \cap (X-X_n) = \emptyset \) and \( Q'(A) \cap (X-X_n) = \emptyset \), we see that \( A \subset Q'(A) \) and \( X-X_n \subset Q'(X-X_n) \), so that \( Q' = Q'(A) \cup Q'(X-X_n) \in \mathcal{Q}_{A, X-X_n} \). It can be easily shown that \( Q' \subset Q \).

We have

THEOREM 5.3. \( EW_p(A, \infty) = EL_p(A, \infty)^{1-q} \).

PROOF. Since \( P_{A, X-X_n} = P_{A, X_{n+1}-X_n} \), we have

\[
EL_p(A, X-X_n) = EL_p(A, X_{n+1}-X_n).
\]

It follows from Lemma 5.3 that

\[
EW_p(A, X-X_n) = EW_p(A, X_{n+1}-X_n).
\]

We have by Theorem 5.2

\[
EW_p(A, X-X_n) = EL_p(A, X-X_n)^{1-q}.
\]

Our assertion follows from Theorems 2.2 and 4.2.

References
