HIROSHIMA MATH. J.
6 (1976), 73–93

On Lorentz Spaces $l_{p,q}\{E\}$

Mikio KATO

(Received August 22, 1975)

Introduction

The Lorentz space $l_{p,q}\{E\}$ is the space of zero sequences $\{x_i\}$ with values in a Banach space E such that

$$\|\{x_i\}\|_{p,q}=\begin{cases}\left(\sum_{i=1}^{\infty} i^{q/p-1}\|x_i\|^{*q}\right)^{1/q} & \text{for } 1\leq p\leq\infty, \quad 1\leq q<\infty,\\ \sup_i i^{1/p}\|x_i\|^* & \text{for } 1\leq p<\infty, \quad q=\infty\end{cases}$$

is finite, where $\{\|x_i\|^*\}$ is the non-increasing rearrangement of $\{\|x_i\|\}$. In particular, $l_{p,p}\{E\}$ coincides with $l_p\{E\}$ (cf. [10]). Recently, the space $l_{p,q}\{E\}$ has been used to introduce and investigate several classes of operators, e.g., (p, q)-nuclear, $(p, q; r)$-absolutely summing and $(r; p, q)$-strongly summing operators ([6], [9], [10]). However, concerning $l_{p,q}\{E\}$ itself very little is known, although the Lorentz space $L_{p,q}(E)$ has been considerably investigated ([1], [5], [12]). Thus it seems to be significant to clarify fundamental and intrinsic properties of $l_{p,q}\{E\}$. The purpose of this paper is to establish a sequence of important properties of the space $l_{p,q}\{E\}$ and especially to characterize the dual space of $l_{p,q}\{E\}$.

We shall show that $l_{p,q}\{E\}'$ and $l_{p',q'}\{E'\}$ are isometrically isomorphic (resp. isomorphic) for $p\leq q$ (resp. $p>q$) where $1/p+1/p'=1/q+1/q'=1$. It should be noted that for $p>q$ $l_{p',q'}\{E'\}$ is not a normed space but a quasi-normed space. In this case, we shall introduce the space $l^0_{p',q'}\{E'\}$ as the Banach space of all E'-valued sequences $\{x'_i\}$ such that for each $\{x_i\}\in l_{p,q}\{E\}$ the series $\sum_{i=1}^{\infty}<x_i, x'_i>$ converges, where the norm is given by $\|\{x'_i\}\|^0_{p',q'}=\sup\{|\sum_{i=1}^{\infty}<x_i, x'_i>|;\ \|\{x_i\}\|_{p,q}\leq 1\}$, and show that $l_{p,q}\{E\}'$ is isometrically isomorphic to $l^0_{p',q'}\{E'\}$ and $l_{p',q'}\{E'\}$ is isomorphic to $l^0_{p',q'}\{E'\}$. As an application we shall refine the main result in [6], that is, we shall characterize the conjugates of $(p, q; r)$-absolutely summing operators ([9]) as $(r'; p', q')$-strongly summing operators where $1/p+1/p'=1/q+1/q'=1/r+1/r'=1$.

The author would like to thank Professor S. Tôgô and Professor M. Itano for their many valuable comments in preparing this paper.

§1. The space $l_{p,q}\{E\}$

Throughout the paper E and F will denote Banach spaces and E' and F' their continuous dual spaces. Let K be the real or complex field and I be the set of positive integers.

DEFINITION 1. *For $1\leq p\leq\infty$, $1\leq q<\infty$ or $1\leq p<\infty$, $q=\infty$ $l_{p,q}\{E\}$ is the space of all E-valued* 0*-sequences $\{x_i\}$ such that*

$$\|\{x_i\}\|_{p,q}=\begin{cases}\left(\sum_{i=1}^{\infty} i^{q/p-1}\|x_{\phi(i)}\|^q\right)^{1/q} & \text{for}\quad 1\leq p\leq\infty,\quad 1\leq q<\infty,\\ \sup_i i^{1/p}\|x_{\phi(i)}\| & \text{for}\quad 1\leq p<\infty,\quad q=\infty\end{cases}$$

is finite, where $\{\|x_{\phi(i)}\|\}$ is the non-increasing rearrangement of $\{\|x_i\|\}$. In particular, if $E=K$, $l_{p,q}\{K\}$ is denoted by $l_{p,q}$ (cf. [10]).

In case of $p=q$ $l_{p,p}\{E\}$ coincides with $l_p\{E\}$ and $\|\cdot\|_{p,p}=\|\cdot\|_p$.

REMARK. In the case where $1\leq p<q\leq\infty$, $\|\cdot\|_{p,q}$ is not a norm. Indeed, if $1\leq p<q<\infty$, we can take two positive numbers α and β such that

$$1<\frac{\alpha}{\beta}<(2^{q/p-1})^{1/(q-1)}.$$

By the mean value theorem of differential calculus there exist two positive numbers γ_1 and γ_2 such that

$$\left\{\left(\frac{\alpha+\beta}{2}\right)^q-\beta^q\right\}\Big/\frac{\alpha-\beta}{2}=q\gamma_1^{q-1},$$

$$\left\{\alpha^q-\left(\frac{\alpha+\beta}{2}\right)^q\right\}\Big/\frac{\alpha-\beta}{2}=q\gamma_2^{q-1}$$

and

$$\beta<\gamma_1<\frac{\alpha+\beta}{2}<\gamma_2<\alpha.$$

Then we have

$$\begin{aligned}\left\{\alpha^q-\left(\frac{\alpha+\beta}{2}\right)^q\right\}\Big/\left\{\left(\frac{\alpha+\beta}{2}\right)^q-\beta^q\right\}&=\left(\frac{\gamma_2}{\gamma_1}\right)^{q-1}\\&<\left(\frac{\alpha}{\beta}\right)^{q-1}\\&<2^{q/p-1},\end{aligned}$$

whence

$$\alpha^q + 2^{q/p-1}\beta^q < \left(\frac{\alpha+\beta}{2}\right)^q + 2^{q/p-1}\left(\frac{\alpha+\beta}{2}\right)^q.$$

Consequently, if we put

$$\boldsymbol{u} = (\alpha, \beta, 0, 0, 0,\dots),$$

$$\boldsymbol{v} = (\beta, \alpha, 0, 0, 0,\dots),$$

we have

$$\begin{aligned}\|\boldsymbol{u}\|_{p,q} + \|\boldsymbol{v}\|_{p,q} &= 2(\alpha^q + 2^{q/p-1}\beta^q)^{1/q} \\ &< \{(\alpha+\beta)^q + 2^{q/p-1}(\alpha+\beta)^q\}^{1/q} \\ &= \|\boldsymbol{u}+\boldsymbol{v}\|_{p,q},\end{aligned}$$

which implies that $\|\cdot\|_{p,q}$ is not a norm. If $1 \le p < q = \infty$, we can take two positive numbers α and β such that $1 < \alpha/\beta < 2^{1/p}$, and show that $\|\cdot\|_{p,\infty}$ does not satisfy the triangular inequality for $\boldsymbol{u}$ and $\boldsymbol{v}$, which implies that $\|\cdot\|_{p,\infty}$ is not a norm.

We now recall the following inequality (Hardy, Littlewood and Pólya [3]) which is one of the most useful tools in our subsequent discussions.

Let $\{c_i^*\}$ and $\{^*c_i\}$ be the non-increasing and non-decreasing rearrangements of a finite sequence $\{c_i\}$ of positive numbers. Then for two sequences $\{a_i\}_{1\le i\le n}$ and $\{b_i\}_{1\le i\le n}$ of positive numbers,

$$\sum_i a_i^{**}b_i \le \sum_i a_i b_i \le \sum_i a_i^* b_i^*. \tag{1}$$

LEMMA 1. *Let $\{x_i\}$ and $\{y_i\}$ be 0-sequences in E. Let $\{\|x_{\phi(i)}\|\}$, $\{\|y_{\psi(i)}\|\}$ and $\{\|x_{\omega(i)}+y_{\omega(i)}\|\}$ be the non-increasing rearrangements of $\{\|x_i\|\}$, $\{\|y_i\|\}$ and $\{\|x_i+y_i\|\}$ respectively. Then for any positive integer k*

$$\|x_{\omega(2k)}+y_{\omega(2k)}\| \le \|x_{\omega(2k-1)}+y_{\omega(2k-1)}\| \le \|x_{\phi(k)}\| + \|y_{\psi(k)}\|.$$

PROOF. The first inequality is clear. Since

$$\begin{aligned}&\{i\in I\colon \|x_i+y_i\| > \|x_{\phi(k)}\| + \|y_{\psi(k)}\|\} \\ &\qquad \subset \{i\in I\colon \|x_i\| > \|x_{\phi(k)}\|\} \cup \{i\in I\colon \|y_i\| > \|y_{\psi(k)}\|\},\end{aligned}$$

comparing the cardinal numbers of these sets we have

$$\operatorname{card}\{i\in I\colon \|x_i+y_i\| > \|x_{\phi(k)}\| + \|y_{\psi(k)}\|\}$$

$$\leq \operatorname{card}\{i \in I: \|x_i\| > \|x_{\phi(k)}\|\} + \operatorname{card}\{i \in I: \|y_i\| > \|y_{\psi(k)}\|\}$$

$$\leq 2(k-1),$$

which implies the second inequality.

PROPOSITION 1. *If* $1 \leq q \leq p \leq \infty$, $l_{p,q}\{E\}$ *is a normed space. If* $1 \leq p < q \leq \infty$, $l_{p,q}\{E\}$ *is not a normed space but a quasi-normed space; for any* $\{x_i\}$, $\{y_i\} \in l_{p,q}\{E\}$

$$\|\{x_i + y_i\}\|_{p,q} \leq 2^{1/p}(\|\{x_i\}\|_{p,q} + \|\{y_i\}\|_{p,q}).$$

PROOF. Let $\{x_i\}, \{y_i\} \in l_{p,q}\{E\}$. Let $\{\|x_{\phi(i)}\|\}$, $\{\|y_{\psi(i)}\|\}$ and $\{\|x_{\omega(i)} + y_{\omega(i)}\|\}$ be the non-increasing rearrangements of $\{\|x_i\|\}$, $\{\|y_i\|\}$ and $\{\|x_i + y_i\|\}$ respectively. We assume $p \neq q$. In the case where $1 \leq q < p \leq \infty$, $\{i^{q/p-1}\}$ is non-increasing and hence by (1) we have

$$\begin{aligned}
\|\{x_i + y_i\}\|_{p,q} &= \left(\sum_{i=1}^{\infty} i^{q/p-1}\|x_{\omega(i)} + y_{\omega(i)}\|^q\right)^{1/q} \\
&\leq \left\{\sum_{i=1}^{\infty} (i^{1/p-1/q}\|x_{\omega(i)}\| + i^{1/p-1/q}\|y_{\omega(i)}\|)^q\right\}^{1/q} \\
&\leq \left(\sum_{i=1}^{\infty} i^{q/p-1}\|x_{\omega(i)}\|^q\right)^{1/q} + \left(\sum_{i=1}^{\infty} i^{q/p-1}\|y_{\omega(i)}\|^q\right)^{1/q} \\
&\leq \left(\sum_{i=1}^{\infty} i^{q/p-1}\|x_{\phi(i)}\|^q\right)^{1/q} + \left(\sum_{i=1}^{\infty} i^{q/p-1}\|y_{\psi(i)}\|^q\right)^{1/q} \\
&= \|\{x_i\}\|_{p,q} + \|\{y_i\}\|_{p,q}.
\end{aligned}$$

Thus $l_{p,q}\{E\}$ is a normed space in this case. In Remark after Definition 1 we have shown that for $1 \leq p < q \leq \infty$ $l_{p,q}\{E\}$ is not a normed space. Let $1 \leq p < q < \infty$. Then, by Lemma 1 we have

$$\begin{aligned}
&\sum_{i=1}^{\infty} i^{q/p-1}\|x_{\omega(i)} + y_{\omega(i)}\|^q \\
&= \sum_{i=1}^{\infty} \{(2i-1)^{q/p-1}\|x_{\omega(2i-1)} + y_{\omega(2i-1)}\|^q + (2i)^{q/p-1}\|x_{\omega(2i)} + y_{\omega(2i)}\|^q\}^{1/q} \\
&\leq 2^{q/p}\sum_{i=1}^{\infty} i^{q/p-1}(\|x_{\phi(i)}\| + \|y_{\psi(i)}\|)^q.
\end{aligned}$$

Therefore,

$$\|\{x_i + y_i\}\|_{p,q} \leq 2^{1/p}\left\{\sum_{i=1}^{\infty} (i^{1/p-1/q}\|x_{\phi(i)}\| + i^{1/p-1/q}\|y_{\psi(i)}\|)^q\right\}^{1/q}$$

$$\leq 2^{1/p}\left\{\left(\sum_{i=1}^{\infty} i^{q/p-1}\|x_{\phi(i)}\|^q\right)^{1/q}+\left(\sum_{i=1}^{\infty} i^{q/p-1}\|y_{\psi(i)}\|^q\right)^{1/q}\right\}$$

$$= 2^{1/p}(\|\{x_i\}\|_{p,q}+\|\{y_i\}\|_{p,q}).$$

For $1\leq p<q=\infty$ we can show in a similar way

$$\|\{x_i+y_i\}\|_{p,\infty} \leq 2^{1/p}(\|\{x_i\}\|_{p,\infty}+\|\{y_i\}\|_{p,\infty}).$$

Thus $l_{p,q}\{E\}$ is a quasi-normed space if $1\leq p<q\leq\infty$.

LEMMA 2. *Let $1\leq p$, $q<\infty$. Let $\{x_i\}\in l_{p,q}\{E\}$. Then for every $i\in I$*

$$\|x_{\phi(i)}\| \leq \left(\frac{q}{p}\right)^{1/q} i^{-1/p}\|\{x_i\}\|_{p,q} \qquad \text{if } 1\leq p\leq q<\infty, \tag{2}$$

$$\|x_{\phi(i)}\| \leq i^{-1/p}\|\{x_i\}\|_{p,q} \qquad \text{if } 1\leq q<p<\infty. \tag{3}$$

PROOF. If $\{x_i\}\in l_{p,q}\{E\}$, for every $i\in I$

$$\|\{x_i\}\|_{p,q}^q \geq \sum_{k=1}^{i} k^{q/p-1}\|x_{\phi(k)}\|^q \tag{4}$$

$$\geq \|x_{\phi(i)}\|^q \sum_{k=1}^{i} k^{q/p-1}.$$

In case of $p=q$ (2) follows immediately from (4). If $1\leq p<q<\infty$, for each $k\in I$

$$\frac{q}{p}k^{q/p-1} \geq k^{q/p}-(k-1)^{q/p},$$

whence (2) follows from (4). In case of $1\leq q<p<\infty$, $\{i^{q/p-1}\}$ is non-increasing and therefore (3) is immediately from (4).

PROPOSITION 2. (i) *Let $1\leq p<\infty$, $1\leq q<q_1\leq\infty$. Then*

$$l_{p,q}\{E\} \subset l_{p,q_1}\{E\}$$

and for every $\{x_i\}\in l_{p,q}\{E\}$

$$\|\{x_i\}\|_{p,q_1} \leq \left(\frac{q}{p}\right)^{1/q-1/q_1}\|\{x_i\}\|_{p,q} \qquad \text{if } p<q, \tag{5}$$

$$\|\{x_i\}\|_{p,q_1} \leq \|\{x_i\}\|_{p,q} \qquad \text{if } p\geq q. \tag{6}$$

(ii) *Let either $1\leq p<p_1\leq\infty$, $1\leq q<\infty$ or $1\leq p<p_1<\infty$, $q=\infty$. Then*

$$l_{p,q}\{E\} \subset l_{p_1,q}\{E\}$$

and for every $\{x_i\}\in l_{p,q}\{E\}$

$$\|\{x_i\}\|_{p_1,q} \leq \|\{x_i\}\|_{p,q}.$$

PROOF. Let $\{x_i\} \in l_{p,q}\{E\}$. Let $1 \leq p < q < q_1 < \infty$. Then by using (2) we have

$$\begin{aligned}
\|\{x_i\}\|_{p,q_1}^{q_1} &= \sum_{i=1}^{\infty} i^{q_1/p-1}\|x_{\phi(i)}\|^{q_1-q}\|x_{\phi(i)}\|^q \\
&\leq \sum_{i=1}^{\infty} i^{q_1/p-1}\left(\frac{q}{p}\right)^{\frac{q_1-q}{q}} i^{-\frac{q_1-q}{p}} \|\{x_i\}\|_{p,q}^{q_1-q}\|x_{\phi(i)}\|^q \\
&= \left(\frac{q}{p}\right)^{q_1/q-1} \|\{x_i\}\|_{p,q}^{q_1-q} \sum_{i=1}^{\infty} i^{q/p-1}\|x_{\phi(i)}\|^q \\
&= \left(\frac{q}{p}\right)^{q_1/q-1} \|\{x_i\}\|_{p,q}^{q_1},
\end{aligned}$$

whence we obtain (5) and $l_{p,q}\{E\} \subset l_{p,q_1}\{E\}$. If $1 \leq p < q < q_1 = \infty$, (5) follows immediately from (2). In case of $p \geq q$, in a similar way we can deduce (6) from (3).

The proof of (ii) is easy and omitted.

COROLLARY. *Let $1 \leq p_1 \leq p \leq q \leq q_1 \leq \infty$ and let p, q be not both equal to ∞. Then*:

(i) $$l_{p_1}\{E\} \subset l_{p,q}\{E\} \subset l_{q_1}\{E\}$$

and for every $\{x_i\} \in l_{p_1}\{E\}$

$$\left(\frac{p}{q}\right)^{1/q-1/q_1}\|\{x_i\}\|_{q_1} \leq \|\{x_i\}\|_{p,q} \leq \|\{x_i\}\|_{p_1}.$$

(ii) $$l_{p_1}\{E\} \subset l_{q,p}\{E\} \subset l_{q_1}\{E\}$$

and for every $\{x_i\} \in l_{p_1}\{E\}$

$$\|\{x_i\}\|_{q_1} \leq \|\{x_i\}\|_{q,p} \leq \|\{x_i\}\|_{p_1}.$$

We shall now show in the following lemma a generalized form of Hölder's inequality which is stated in a more generalized form without proof in [10].

LEMMA 3. *Let $1 \leq p, q \leq \infty$ and $1/p + 1/p' = 1/q + 1/q' = 1$. Let $\{x_i\} \in l_{p,q}\{E\}$ and $\{x_i'\} \in l_{p',q'}\{E'\}$. Then, $\{<x_i, x_i'>\} \in l_1$ and*

$$\|\{<x_i, x_i'>\}\|_1 \leq \|\{x_i\}\|_{p,q}\|\{x_i'\}\|_{p',q'}.$$

PROOF. From (1) and the usual Hölder's inequality we have

$$\sum_{i=1}^{\infty} |<x_i, x'_i>| \leq \sum_{i=1}^{\infty} i^{1/p-1/q}\|x_{\phi(i)}\| \cdot i^{1/p'-1/q'}\|x'_{\psi(i)}\|$$

$$\leq \left(\sum_{i=1}^{\infty} i^{q/p-1}\|x_{\phi(i)}\|^q\right)^{1/q}\left(\sum_{i=1}^{\infty} i^{q'/p'-1}\|x'_{\psi(i)}\|^{q'}\right)^{1/q'}$$

$$= \|\{x_i\}\|_{p,q}\|\{x'_i\}\|_{p',q'}.$$

In the rest of the paper, we denote by $\mathscr{F}(E)$ the space of E-valued finite sequences.

PROPOSITION 3. *For* $1 \leq p \leq \infty$, $1 \leq q < \infty$, $\mathscr{F}(E)$ *is dense in* $l_{p,q}\{E\}$.

PROOF. Let $\{x_i\} \in l_{p,q}\{E\}$. When $1 \leq p \leq q < \infty$, let

$$u_n = \sum_{i=1}^{n} (0,\ldots, 0, \overbrace{x_{\phi(i)}}^{\phi(i)}, 0, 0, 0,\ldots). \tag{7}$$

Then $u_n \in \mathscr{F}(E)$ and

$$\|\{x_i\} - u_n\|_{p,q} = \left(\sum_{i=1}^{\infty} i^{q/p-1}\|x_{\phi(n+i)}\|^q\right)^{1/q}$$

$$\leq \left(\sum_{i=n+1}^{\infty} i^{q/p-1}\|x_{\phi(i)}\|^q\right)^{1/q}$$

$$\to 0 \qquad (n \to \infty).$$

When $1 \leq q < p \leq \infty$, let

$$I_n = \left\{i \in I\colon \|x_i\| > \frac{1}{n}\right\}$$

and

$$v_n = \sum_{i \in I_n} (0,\ldots, 0, \overbrace{x_i}^{i}, 0, 0, 0,\ldots). \tag{8}$$

Since I_n is finite, we put $k_n = \operatorname{card} I_n$. Then for any $j_0 \in I$ we have

$$\|\{x_i\} - v_n\|_{p,q} = \left(\sum_{i=1}^{\infty} i^{q/p-1}\|x_{\phi(k_n+i)}\|^q\right)^{1/q}$$

$$\leq \left(\sum_{i=1}^{j_0} i^{q/p-1}\|x_{\phi(k_n+i)}\|^q\right)^{1/q}$$

$$+\left(\sum_{i=j_0+1}^{\infty} i^{q/p-1}\|x_{\phi(k_n+i)}\|^q\right)^{1/q}$$

$$\leq \frac{1}{n}\left(\sum_{i=1}^{j_0} i^{q/p-1}\right)^{1/q} + \left(\sum_{i=j_0+1}^{\infty} i^{q/p-1}\|x_{\phi(i)}\|^q\right)^{1/q}.$$

Hence

$$\overline{\lim_{n\to\infty}} \|\{x_i\}-\boldsymbol{v}_n\|_{p,q} \le \left(\sum_{i=j_0+1}^{\infty} i^{q/p-1}\|x_{\phi(i)}\|^q\right)^{1/q}$$

Therefore, letting $j_0\to\infty$, we have

$$\lim_{n\to\infty} \|\{x_i\}-\boldsymbol{v}_n\|_{p,q} = 0.$$

LEMMA 4. *Let $\{x_i^{(\nu)}\}_{i,\nu}$ be an E-valued double sequence such that $\lim_{i\to\infty}x_i^{(\nu)}=0$ for each $\nu\in I$ and let $\{x_i\}$ be an E-valued sequence such that $\lim_{\nu\to\infty}x_i^{(\nu)}=x_i$ (uniformly in i). Then, $\lim_{i\to\infty}x_i=0$ and for each $i\in I$*

$$\|x_{\phi(i)}\| \le \underline{\lim}_{\nu\to\infty} \|x^{(\nu)}_{\phi_\nu(i)}\|, \tag{9}$$

where $\{\|x_{\phi(i)}\|\}$ and $\{\|x^{(\nu)}_{\phi_\nu(i)}\|\}_i$ are the non-increasing rearrangements of $\{\|x_i\|\}$ and $\{\|x_i^{(\nu)}\|\}_i$ respectively.

PROOF. It can be easily shown that $\lim_{i\to\infty}x_i^{(\nu)}=0$ (uniformly in ν). Therefore we have immediately $\lim_{i\to\infty}x_i=0$. Let i be an arbitrary positive integer and fixed. If there exists a positive number c_i such that

$$\|x_{\phi(i)}\| > c_i > \underline{\lim}_{\nu\to\infty} \|x^{(\nu)}_{\phi_\nu(i)}\|,$$

then the inequality

$$\|x_{\phi(i)}\| > c_i > \|x^{(\nu)}_{\phi_\nu(i)}\|$$

is valid for infinitely many $\nu\in I$. Since $\|x_{\phi(k)}\|>c_i$ and $\lim_{\nu\to\infty}\|x^{(\nu)}_{\phi(k)}\|=\|x_{\phi(k)}\|$ for $1\le k\le i$, there exists a $\nu_0\in I$ such that $\|x^{(\nu)}_{\phi(k)}\|>c_i$ for $\nu\ge\nu_0$ and $1\le k\le i$. Therefore, if we take a positive integer ν_1 such that $\nu_1\ge\nu_0$ and $c_i>\|x^{(\nu_1)}_{\phi_{\nu_1}(i)}\|$, we have

$$\|x^{(\nu_1)}_{\phi(k)}\| > c_i > \|x^{(\nu_1)}_{\phi_{\nu_1}(i)}\|$$

for $1\le k\le i$, which is a contradiction since the number of k such that $\|x_k^{(\nu_1)}\|>\|x^{(\nu_1)}_{\phi_{\nu_1}(i)}\|$ is $i-1$. Thus (9) holds for every $i\in I$.

THEOREM 1. *For $1\le p,\ q\le\infty$, $l_{p,q}\{E\}$ is complete.*

PROOF. Let $\{x_i^{(\nu)}\}_i\in l_{p,q}\{E\}$ $(\nu\in I)$ and

$$\lim_{\mu,\nu\to\infty} \|\{x_i^{(\mu)}-x_i^{(\nu)}\}\|_{p,q} = 0. \tag{10}$$

In the case where $q<\infty$, for any $\varepsilon>0$ there exists a $\nu_0\in I$ such that

(11) $$\left(\sum_{i=1}^{\infty} i^{q/p-1}\|x^{(\mu)}_{\psi_{\mu,\nu}(i)}-x^{(\nu)}_{\psi_{\mu,\nu}(i)}\|^q\right)^{1/q} < \varepsilon \qquad \text{for any } \mu,\ \nu \geq \nu_0$$

where $\{\|x^{(\mu)}_{\psi_{\mu,\nu}(i)}-x^{(\nu)}_{\psi_{\mu,\nu}(i)}\|\}_i$ denotes the non-increasing rearrangement of $\{\|x^{(\mu)}_i-x^{(\nu)}_i\|\}_i$. Then for any μ, $\nu \geq \nu_0$, from Proposition 2 we have

$$\begin{aligned}
&\sup_i \|x^{(\mu)}_i-x^{(\nu)}_i\| \\
&\leq \sup_i i^{1/p}\|x^{(\mu)}_{\psi_{\mu,\nu}(i)}-x^{(\nu)}_{\psi_{\mu,\nu}(i)}\| \\
&\leq \begin{cases} \left(\dfrac{q}{p}\right)^{1/q}\|\{x^{(\mu)}_i-x^{(\nu)}_i\}\|_{p,q} & \text{if } p<q, \\ \{\|x^{(\mu)}_i-x^{(\nu)}_i\}\|_{p,q} & \text{if } p\geq q \end{cases} \\
&\leq \begin{cases} \left(\dfrac{q}{p}\right)^{1/q}\varepsilon & \text{if } p<q, \\ \varepsilon & \text{if } p\geq q. \end{cases}
\end{aligned}$$

Therefore there exist $x_i \in E (i \in I)$ such that

(12) $$x_i = \lim_{\nu\to\infty} x^{(\nu)}_i \qquad \text{(uniformly in } i\text{)}.$$

Since $\lim_{i\to\infty} x^{(\nu)}_i = 0$ for each $\nu \in I$, we have by Lemma 4

(13) $$\lim_{i\to\infty} x_i = 0.$$

Hence we can take the non-increasing rearrangement $\{\|x_{\psi_\nu(i)}-x^{(\nu)}_{\psi_\nu(i)}\|\}_i$ of $\{\|x_i-x^{(\nu)}_i\|\}_i$. Let ν be an arbitrary positive integer with $\nu \geq \nu_0$ and fixed. If we put

$$y^{(\mu)}_i = x^{(\mu)}_i - x^{(\nu)}_i,$$

$$y_i = x_i - x^{(\nu)}_i,$$

then

$$\lim_{i\to\infty} y^{(\mu)}_i = 0 \qquad \text{for each } \mu \in I.$$

$$\lim_{\mu\to\infty} y^{(\mu)}_i = y_i \quad \text{(uniformly in } i\text{)}.$$

Therefore by Lemma 4 we have

$$\|y_{\phi(i)}\| \leq \varliminf_{\mu\to\infty} \|y^{(\mu)}_{\phi_\mu(i)}\| \qquad \text{for each } i \in I$$

that is,

$$\|x_{\psi_\nu(i)} - x^{(\nu)}_{\psi_\nu(i)}\| \leq \varliminf_{\mu\to\infty} \|x^{(\mu)}_{\psi_{\mu,\nu}(i)} - x^{(\nu)}_{\psi_{\mu,\nu}(i)}\| \qquad \text{for each } i \in I. \tag{14}$$

Consequently, by (11) and (14) we have for any $\nu \geq \nu_0$

$$\begin{aligned}\|\{x_i - x_i^{(\nu)}\}\|_{p,q} &= \left(\sum_{i=1}^{\infty} i^{q/p-1}\|x_{\psi_\nu(i)} - x^{(\nu)}_{\psi_\nu(i)}\|^q\right)^{1/q} \\ &\leq \left(\sum_{i=1}^{\infty} i^{q/p-1} \varliminf_{\mu\to\infty} \|x^{(\mu)}_{\psi_{\mu,\nu}(i)} - x^{(\nu)}_{\psi_{\mu,\nu}(i)}\|^q\right)^{1/q} \\ &\leq \varliminf_{\mu\to\infty}\left(\sum_{i=1}^{\infty} i^{q/p-1}\|x^{(\mu)}_{\psi_{\mu,\nu}(i)} - x^{(\nu)}_{\psi_{\mu,\nu}(i)}\|^q\right)^{1/q} \\ &\leq \varepsilon,\end{aligned}$$

and hence $\{x_i\} = \{x_i - x_i^{(\nu_0)}\} + \{x_i^{(\nu_0)}\} \in l_{p,q}\{E\}$, which completes the proof in case $q < \infty$.

In the case where $q = \infty$, by (10) for any $\varepsilon > 0$ there exists a $\nu_0 \in I$ such that

$$\sup_i i^{1/p}\|x^{(\mu)}_{\psi_{\mu,\nu}(i)} - x^{(\nu)}_{\psi_{\mu,\nu}(i)}\| < \varepsilon \qquad \text{for any } \mu, \nu \geq \nu_0. \tag{15}$$

Hence we can take in a similar way a sequence $\{x_i\}$ which satisfies (12) and (13). Then by (14) and (15) we have for any $\nu \geq \nu_0$

$$\begin{aligned}\|\{x_i - x_i^{(\nu)}\}\|_{p,\infty} &= \sup_i i^{1/p}\|x_{\psi_\nu(i)} - x^{(\nu)}_{\psi_\nu(i)}\| \\ &\leq \sup_i i^{1/p} \varliminf_{\mu\to\infty} \|x^{(\mu)}_{\psi_{\mu,\nu}(i)} - x^{(\nu)}_{\psi_{\mu,\nu}(i)}\| \\ &\leq \varliminf_{\mu\to\infty} \sup_i i^{1/p}\|x^{(\mu)}_{\psi_{\mu,\nu}(i)} - x^{(\nu)}_{\psi_{\mu,\nu}(i)}\| \\ &\leq \varepsilon\end{aligned}$$

and hence $\{x_i\} = \{x_i - x_i^{(\nu_0)}\} + \{x_i^{(\nu_0)}\} \in l_{p,\infty}\{E\}$, which completes the proof.

§2. The space $l^0_{p',q'}\{E'\}$

In this section we assume that $1 \leq q < p \leq \infty$, $1/p + 1/p' = 1/q + 1/q' = 1$. We now introduce the space $l^0_{p',q'}\{E'\}$ which will play an important role in the next section.

DEFINITION 2. *$l^0_{p',q'}\{E'\}$ is the space of E'-valued sequences $\{x'_i\}$ such that for every $\{x_i\} \in l_{p,q}\{E\}$ the series $\sum_{i=1}^{\infty} \langle x_i, x'_i \rangle$ converges. The norm $\|\cdot\|^0_{p',q'}$ on $l^0_{p',q'}\{E'\}$ is given by*

$$\|\{x'_i\}\|^0_{p',q'} = \sup_{\|\{x_i\}\|_{p,q} \leq 1} \left|\sum_{i=1}^{\infty} \langle x_i, x'_i \rangle\right|.$$

It should be noted that $\|\{x_i'\}\|_{p',q'}^0<\infty$ for all $\{x_i'\}\in l_{p',q'}^0\{E'\}$ and $\|\cdot\|_{p',q'}^0$ is really a norm. Indeed, if $\{x_i'\}\in l_{p',q'}^0\{E'\}$, then $\{x_i'\}$ can be considered as the linear form f on $l_{p,q}\{E\}$ defined by $f(\{x_i\})=\sum_{i=1}^{\infty}<x_i, x_i'>$. Define a sequence $\{f_n\}$ of linear forms on $l_{p,q}\{E\}$ by $f_n(\{x_i\})=\sum_{i=1}^{n}<x_i, x_i'>$. It is easy to see that each f_n is continuous. Furthermore $\{f_n\}$ converges to f at each point of $l_{p,q}\{E\}$. Since $l_{p,q}\{E\}$ is a Banach space by Proposition 1 and Theorem 1 ($p>q$), from the Banach-Steinhaus Theorem it follows that f is continuous and $\|\{x_i'\}\|_{p',q'}^0=\|f\|<\infty$. Hence $\|\cdot\|_{p',q'}^0$ is a norm.

The norm $\|\cdot\|_{p',q'}^0$ is also given by the following form

$$\|\{x_i'\}\|_{p',q'}^0 = \sup_{\|\{x_i\}\|_{p,q}\le 1} \sum_{i=1}^{\infty} |<x_i, x_i'>|, \tag{16}$$

as can be easily seen.

LEMMA 5. *Let* $\{x_i'\}\in l_{p',q'}^0\{E'\}$. *Let* $x_{\nu,i}'=x_i'$ *for* $1\le i\le\nu$ *and* $x_{\nu,i}'=0$ *for* $i>\nu$. *Then*

$$\lim_{\nu\to\infty}\|\{x_{\nu,i}'\}\|_{p',q'}^0 = \|\{x_i'\}\|_{p',q'}^0.$$

PROOF. By (16), for any $\varepsilon>0$ there exists an $\{x_i\}\in l_{p,q}\{E\}$ such that $\|\{x_i\}\|_{p,q}\le 1$ and

$$\|\{x_i'\}\|_{p',q'}^0 < \sum_{i=1}^{\infty} |<x_i, x_i'>| + \frac{\varepsilon}{2}.$$

Then there exists a $\nu_0\in I$ such that for any $\nu\ge\nu_0$

$$\sum_{i=1}^{\infty} |<x_i, x_i'>| < \sum_{i=1}^{\nu} |<x_i, x_i'>| + \frac{\varepsilon}{2}.$$

Therefore we have for any $\nu\ge\nu_0$

$$\begin{aligned}\|\{x_{\nu,i}'\}_i\|_{p',q'}^0 &\le \|\{x_i'\}\|_{p',q'}^0\\ &< \sum_{i=1}^{\nu}|<x_i, x_i'>|+\varepsilon\\ &\le \sup_{\|\{x_i\}\|_{p,q}\le 1}\sum_{i=1}^{\infty}|<x_i, x_{\nu,i}'>|+\varepsilon\\ &= \|\{x_{\nu,i}'\}_i\|_{p',q'}^0+\varepsilon,\end{aligned}$$

which shows that $\|\{x_{\nu,i}'\}_i\|_{p',q'}^0$ converges to $\|\{x_i'\}\|_{p',q'}^0$ as $\nu\to\infty$.

LEMMA 6. *Let* $\{x_{1,i}'\}, \{x_{2,i}'\}\in l_{p',q'}^0\{E'\}$. *If* $x_{1,i}'$ *or* $x_{2,i}'$ *is equal to* 0 *for each* $i\in I$, *then*

$$\|\{x'_{1,i}+x'_{2,i}\}\|^{0q'}_{p',q'} \geq \|\{x'_{1,i}\}\|^{0q'}_{p',q'}+\|\{x'_{2,i}\}\|^{0q'}_{p',q'}.$$

PROOF. We may suppose $0<\|\{x'_{k,i}\}_i\|^0_{p',q'}<\infty$ $(k=1, 2)$. For any $\varepsilon>0$ there exist $\{x_{k,i}\}_i\in l_{p,q}\{E\}$ $(k=1, 2)$ such that

$$\|\{x_{k,i}\}_i\|_{p,q} = \|\{x'_{k,i}\}_i\|^{0q'-1}_{p',q'}$$

and

$$\sum_{i=1}^{\infty} |<x_{k,i}, x'_{k,i}>| > \|\{x'_{k,i}\}_i\|^{0q'}_{p',q'}-\frac{\varepsilon}{2}.$$

Furthermore we may assume that $x_{k,i}=0$ if $x'_{k,i}=0$. Then we have

$$\begin{aligned}(17)\qquad &\sum_{i=1}^{\infty}|<x_{1,i}+x_{2,i}, x'_{1,i}+x'_{2,i}>| \\ &= \sum_{i=1}^{\infty}|<x_{1,i}, x'_{1,i}>+<x_{2,i}, x'_{2,i}>| \\ &= \sum_{i=1}^{\infty}|<x_{1,i}, x'_{1,i}>|+\sum_{i=1}^{\infty}|<x_{2,i}, x'_{2,i}>| \\ &> \|\{x'_{1,i}\}\|^{0q'}_{p',q'}+\|\{x'_{2,i}\}\|^{0q'}_{p',q'}-\varepsilon.\end{aligned}$$

On the other hand, denoting by $\{\|x_{1,\phi(i)}\|\}$, $\{\|x_{2,\psi(i)}\|\}$ and $\{\|x_{1,\omega(i)}+x_{2,\omega(i)}\|\}$ respectively the non-increasing rearrangements of $\{\|x_{1,i}\|\}$, $\{\|x_{2,i}\|\}$ and $\{\|x_{1,i}+x_{2,i}\|\}$ we have

$$\begin{aligned}\|\{x_{1,i}+x_{2,i}\}\|^q_{p,q} &= \sum_{i=1}^{\infty} i^{q/p-1}\|x_{1,\omega(i)}+x_{2,\omega(i)}\|^q \\ &= \sum{}' i^{q/p-1}\|x_{1,\omega(i)}\|^q+\sum{}'' i^{q/p-1}\|x_{2,\omega(i)}\|^q \\ &\leq \sum_{i=1}^{\infty} i^{q/p-1}\|x_{1,\phi(i)}\|^q+\sum_{i=1}^{\infty} i^{q/p-1}\|x_{2,\psi(i)}\|^q \\ &= \|\{x_{1,i}\}\|^q_{p,q}+\|\{x_{2,i}\}\|^q_{p,q} \\ &= \|\{x'_{1,i}\}\|^{0q'}_{p',q'}+\|\{x'_{2,i}\}\|^{0q'}_{p',q'}\end{aligned}$$

since $p>q$. Here $\sum'$ (resp. $\sum''$) denotes summation on those i for which $x_{2,\omega(i)}=0$ (resp. $x_{1,\omega(i)}=0$). Hence

$$(18)\qquad \|\{x_{1,i}+x_{2,i}\}\|_{p,q}\leq (\|\{x'_{1,i}\}\|^{0q'}_{p',q'}+\|\{x'_{2,i}\}\|^{0q'}_{p',q'})^{1/q}.$$

Consequently, from (16), (17) and (18) we have

$$\|\{x'_{1,i}\}\|^{0q'}_{p',q'}+\|\{x'_{2,i}\}\|^{0q'}_{p',q'}-\varepsilon$$

$$< (\|\{x'_{1,i}\}\|_{p',q'}^{0q'} + \|\{x'_{2,i}\}\|_{p',q'}^{0q'})^{1/q}\|\{x'_{1,i}+x'_{2,i}\}\|_{p',q'}^{0},$$

from which follows

$$\|\{x'_{1,i}\}\|_{p',q'}^{0q'} + \|\{x'_{2,i}\}\|_{p',q'}^{0q'} \leq \|\{x'_{1,i}+x'_{2,i}\}\|_{p',q'}^{0q'}.$$

PROPOSITION 4. *$\mathscr{F}(E')$ is dense in $l_{p',q'}^{0}\{E'\}$.*

PROOF. Let $\{x'_i\}\in l_{p',q'}^{0}\{E'\}\backslash\mathscr{F}(E')$. Then for any $\varepsilon>0$ there exists an $\{x_i\}\in l_{p,q}\{E\}$ such that $\|\{x_i\}\|_{p,q}\leq 1$ and

$$\sum_{i=1}^{\infty} |<x_i, x'_i>| \, \|\{x'_i\}\|_{p',q'}^{0q'-1} > \|\{x'_i\}\|_{p',q'}^{0q'} - \varepsilon^{q'}. \tag{19}$$

Since $\|\{x'_{\nu,i}\}_i\|_{p',q'}^{0} \to \|\{x'_i\}\|_{p',q'}^{0}$ $(\nu\to\infty)$ by Lemma 5, we have

$$\begin{aligned} &\sum_{i=1}^{\infty} |<x_i, x'_{\nu,i}>| \, \|\{x'_{\nu,i}\}_i\|_{p',q'}^{0q'-1} \\ &\to \sum_{i=1}^{\infty} |<x_i, x'_i>| \, \|\{x'_i\}\|_{p',q'}^{0q'-1} \qquad (\nu\to\infty). \end{aligned} \tag{20}$$

By (19) and (20), for a sufficiently large ν

$$\begin{aligned} \|\{x'_i\}\|_{p',q'}^{0q'} - \varepsilon^{q'} &< \sum_{i=1}^{\infty} |<x_i, x'_{\nu,i}>| \, \|\{x'_{\nu,i}\}\|_{p',q'}^{0q'-1} \\ &\leq \|\{x'_{\nu,i}\}_i\|_{p',q'}^{0q'}. \end{aligned}$$

This, combined with Lemma 6, implies

$$\begin{aligned} \|\{x'_i\} - \{x'_{\nu,i}\}\|_{p',q'}^{0q'} &\leq \|\{x'_i\}\|_{p',q'}^{0q'} - \|\{x'_{\nu,i}\}\|_{p',q'}^{0q'} \\ &< \varepsilon^{q'}, \end{aligned}$$

that is,

$$\|\{x'_i\} - \{x'_{\nu,i}\}\|_{p',q'}^{0} < \varepsilon.$$

PROPOSITION 5. *The dual space of $l_{p,q}\{E\}$ is isometrically isomorphic to $l_{p',q'}^{0}\{E'\}$, where a sequence $\{x'_i\}$ in $l_{p',q'}^{0}\{E'\}$ is identified with the linear form f defined by*

$$f(\{x_i\}) = \sum_{i=1}^{\infty} <x_i, x'_i> \qquad \textit{for each } \{x_i\}\in l_{p,q}\{E\}. \tag{21}$$

PROOF. Let $\{x'_i\}\in l_{p',q'}^{0}\{E'\}$. Then the linear form f defined by (21) is continuous and $\|f\| = \|\{x'_i\}\|_{p',q'}^{0}$, which we have already shown in the paragraph after Definition 2. Conversely, let $f\in l_{p,q}\{E\}'$. If for each $i\in I$ we define

$x_i' \in E'$ by

$$<x, x_i'> = f((0,\ldots, 0, \overset{i}{\smile}{x}, 0,\ldots)) \qquad \text{for each } x \in E,$$

then we have for any $\{x_i\} \in l_{p,q}\{E\}$

$$\begin{aligned}(22) \qquad \sum_{i=1}^{\infty} |<x_i, x_i'>| &= \sum_{i=1}^{\infty} |f((0,\ldots, 0, \overset{i}{\smile}{x_i}, 0,\ldots)) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i f((0,\ldots, 0, \overset{i}{\smile}{x_i}, 0,\ldots)) \\ &= \lim_{n\to\infty} f((\alpha_1 x_1,\ldots, \alpha_n x_n, 0,\ldots)) \\ &\leq \|f\| \lim_{n\to\infty}\left(\sum_{i=1}^{n} i^{q/p-1}\|x_i\|^{*q}\right)^{1/q} \\ &\leq \|f\| \lim_{n\to\infty}\left(\sum_{i=1}^{n} i^{q/p-1}\|x_{\phi(i)}\|^{q}\right)^{1/q} \\ &= \|f\| \, \|\{x_i\}\|_{p,q},\end{aligned}$$

where α_i $(i \in I)$ are the complex numbers such that $|\alpha_i| = 1$ and $|f((0,\ldots, 0, x_i, 0,\ldots))| = \alpha_i f((0,\ldots, 0, x_i, 0,\ldots))$ for each $i \in I$, and where $\{\|x_i\|^*\}_{1\leq i\leq n}$, $\{\|x_{\phi(i)}\|\}$ are the non-increasing rearrangements of $\{\|x_i\|\}_{1\leq i\leq n}$, $\{\|x_i\|\}_{1\leq i<\infty}$ respectively. Therefore we have $\{x_i'\} \in l^0_{p',q'}\{E'\}$ and $\|\{x_i'\}\|^0_{p',q'} \leq \|f\|$. On the other hand, if for any $\{x_i\} \in l_{p,q}\{E\}$ we put $\boldsymbol{v}_n$ as in (8), then

$$\boldsymbol{v}_n \to \{x_i\} \quad (n \to \infty) \quad \text{in} \quad l_{p,q}\{E\}$$

by Proposition 3. Hence we have

$$\begin{aligned} f(\{x_i\}) &= f(\lim_{n\to\infty} \boldsymbol{v}_n) \\ &= \lim_{n\to\infty} f(\boldsymbol{v}_n) \\ &= \lim_{n\to\infty} \sum_{i\in I_n} f((0,\ldots, 0, \overset{i}{\smile}{x_i}, 0,\ldots)) \\ &= \lim_{n\to\infty} \sum_{i\in I_n} <x_i, x_i'> \\ &= \sum_{i=1}^{\infty} <x_i, x_i'>, \end{aligned}$$

where the last inequality follows from the fact that the series $\sum_{i=1}^{\infty} <x_i, x_i'>$ converges unconditionally by (22). This completes the proof.

COROLLARY. *$l^0_{p',q'}\{E'\}$ is a Banach space.*

§3. The dual space of $l_{p,q}\{E\}$

THEOREM 2. *Let* $1<p\leq\infty$, $1<q<\infty$ *and* $1/p+1/p'=1/q+1/q'=1$. *The dual space of* $l_{p,q}\{E\}$ *is isometrically isomorphic (resp. isomorphic) to* $l_{p',q'}\{E'\}$ *if* $p\leq q$ *(resp.* $p>q$*). In both cases, a sequence* $\{x_i'\}$ *in* $l_{p',q'}\{E'\}$ *is identified with the linear form f defined by*

$$f(\{x_i\}) = \sum_{i=1}^{\infty} <x_i, x_i'> \qquad \textit{for each } \{x_i\}\in l_{p,q}\{E\}. \tag{21}$$

In the latter case, there exists a certain positive number $M_{p,q}$ *such that* $M_{p,q}>1$ *and*

$$\|f\| \leq \|\{x_i'\}\|_{p',q'} \leq M_{p,q}\|f\| \qquad \textit{for every } \{x_i'\}\in l_{p',q'}\{E'\}. \tag{23}$$

PROOF. In case of $p=q$, the statement is well known and proved in [2].

(i) Let $1<p<q<\infty$. Let $\{x_i'\}\in l_{p',q'}\{E'\}$. Then the linear form f defined by (21) is continuous and $\|f\|\leq\|\{x_i'\}\|_{p',q'}$. Indeed, from Lemma 3 we have for any $\{x_i\}\in l_{p,q}\{E\}$

$$\begin{aligned}|f(\{x_i\})| &\leq \sum_{i=1}^{\infty} |<x_i, x_i'>| \\ &\leq \|\{x_i\}\|_{p,q}\|\{x_i'\}\|_{p',q'},\end{aligned}$$

whence we have $f\in l_{p,q}\{E\}'$ and $\|f\|\leq\|\{x_i'\}\|_{p',q'}$ simultaneously with convergence of the series in (21).

Conversely, let $f\in l_{p,q}\{E\}'$. If we define $x_i'\in E'$ $(i\in I)$ by

$$<x, x_i'> = f((0,\dots, 0, \overset{i}{\smile}{x}, 0,\dots)) \qquad \text{for each } x\in E,$$

then for any $\{x_i\}\in l_{p,q}\{E\}$ the series $\sum_{i=1}^{\infty}<x_i, x_i'>$ converges unconditionally by (22). Hence, if for any $\{x_i\}\in l_{p,q}\{E\}$ we put $\boldsymbol{u}_n$ as in (7), then by Proposition 3 we have

$$\begin{aligned}f(\{x_i\}) &= f(\lim_{n\to\infty} \boldsymbol{u}_n) \\ &= \lim_{n\to\infty} f(\boldsymbol{u}_n) \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} <x_{\phi(i)}, x'_{\phi(i)}> \\ &= \sum_{i=1}^{\infty} <x_i, x_i'>,\end{aligned}$$

where ϕ is the permutation such that $\{\|x_{\phi(i)}\|\}$ is the non-increasing rearrangement of $\{\|x_i\|\}$. Since $f \in l_p\{E\}'$ from Corollary to Proposition 2 and $l_p\{E\}' = l_{p'}\{E'\}$, $\{x'_i\} \in l_{p'}\{E'\}$. Therefore, $\lim_{i\to\infty} x'_i = 0$ and we can take the non-increasing rearrangement $\{\|x'_{\psi(i)}\|\}$ of $\{\|x'_i\|\}$. Let n be an arbitrary positive integer. Then for any $\varepsilon > 0$ there exist $x_i \in E$, $1 \le i \le n$, such that $\|x_i\| = 1$ and

$$<x_i, x'_{\psi(i)}> \ \ge \ \|x'_{\psi(i)}\| - \varepsilon_i,$$

where

$$\varepsilon_i = \frac{\varepsilon \|x'_{\psi(i)}\|}{\left(\sum_{i=1}^{n} i^{q'/p'-1} \|x'_{\psi(i)}\|^{q'}\right)^{1/q'}}.$$

Put

$$\boldsymbol{t}_n = \sum_{i=1}^{n} i^{q'/p'q-1/p} \|x'_{\psi(i)}\|^{q'/q} (0,\dots, 0, \overset{\psi(i)}{\overbrace{x_i}}, 0,\dots).$$

Since the non-increasing rearrangement of $\{i^{q'/p'q-1/p}\|x'_{\psi(i)}\|^{q'/q}\}_{1\le i\le n}$ is invariant because of $q'/p'q - 1/p = q'/p' - 1 < 0$, we have

$$\begin{aligned}
(24) \qquad |f(\boldsymbol{t}_n)| &\le \|f\| \, \|\boldsymbol{t}_n\|_{p,q} \\
&= \|f\| \left(\sum_{i=1}^{n} i^{q/p-1+q'/p'-q/p} \|x'_{\psi(i)}\|^{q'}\right)^{1/q} \\
&= \|f\| \left(\sum_{i=1}^{n} i^{q'/p'-1} \|x'_{\psi(i)}\|^{q'}\right)^{1/q}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(25) \qquad |f(\boldsymbol{t}_n)| &= \sum_{i=1}^{n} i^{q'/p'q-1/p} \|x'_{\psi(i)}\|^{q'/q} <x_i, x'_{\psi(i)}> \\
&\ge \sum_{i=1}^{n} i^{q'/p'-1} \|x'_{\psi(i)}\|^{q'/q} (\|x'_{\psi(i)}\| - \varepsilon_i) \\
&= \sum_{i=1}^{n} i^{q'/p'-1} \|x'_{\psi(i)}\|^{q'} - \varepsilon \left(\sum_{i=1}^{n} i^{q'/p'-1} \|x'_{\psi(i)}\|^{q'}\right)^{1/q}.
\end{aligned}$$

By (24) and (25) we have

$$\left(\sum_{i=1}^{n} i^{q'/p'-1} \|x'_{\psi(i)}\|^{q'}\right)^{1/q'} \le \|f\| + \varepsilon.$$

Since ε and n are arbitrary, this shows that $\{x'_i\} \in l_{p',q'}\{E'\}$ and $\|\{x'_i\}\|_{p',q'} \le \|f\|$. We have thus proved the theorem for $1 < p < q < \infty$.

(ii) Let $1 < q < p \le \infty$ and suppose that E is reflexive. Since $l_{p,q}\{E\}'$ is isometrically isomorphic to $l^0_{p',q'}\{E'\}$ by Proposition 5, we have only to prove that $l_{p',q'}\{E'\}$ and $l^0_{p',q'}\{E'\}$ are isomorphic. Here one should note that the former is a quasi-normed space and the latter is a normed space. Let $\{x'_i\} \in l_{p',q'}\{E'\}$. Then, by Lemma 3 for any $\{x_i\} \in l_{p,q}\{E\}$

$$\sum_{i=1}^{\infty} |<x_i, x'_i>| \leq \|\{x_i\}\|_{p,q} \|\{x'_i\}\|_{p',q'} < \infty,$$

from which it follows that $\{x'_i\} \in l^0_{p',q'}\{E'\}$ and $\|\{x'_i\}\|^0_{p',q'} \leq \|\{x'_i\}\|_{p',q'}$. Thus we have

$$l_{p',q'}\{E'\} \subset l^0_{p',q'}\{E'\}, \quad \|\cdot\|^0_{p',q'} \leq \|\cdot\|_{p',q'}. \tag{26}$$

Let i be the canonical injection of $l_{p',q'}\{E'\}$ into $l^0_{p',q'}\{E'\}$; $i(\{x'_i\}) = \{x'_i\}$ for each $\{x'_i\} \in l_{p',q'}\{E'\}$. Then the image of i is dense in $l^0_{p',q'}\{E'\}$, since $\mathscr{F}(E') \subset l_{p',q'}\{E'\} \subset l^0_{p',q'}\{E'\}$ and $\mathscr{F}(E')$ is dense in $l^0_{p',q'}\{E'\}$ by Proposition 4. Therefore ${}^t i$: $l^0_{p',q'}\{E'\}' \to l_{p',q'}\{E'\}'$, the transpose of i, is also a continuous injection. Since $l^0_{p',q'}\{E'\}' = l_{p,q}\{E\}''$ by Proposition 5 and since $l_{p',q'}\{E'\}' = l_{p,q}\{E''\} = l_{p,q}\{E\}$ by (i) and our assumption, we can regard ${}^t i$ as an injection of $l_{p,q}\{E\}''$ into $l_{p,q}\{E\}$. Furthermore, ${}^t i$ maps $l_{p,q}\{E\}$ onto $l_{p,q}\{E\}$ identically by the definition of ${}^t i$. Consequently, $l_{p,q}\{E\}'' = l_{p,q}\{E\}$ and ${}^t i$ must be an isometric isomorphism. Hence

$${}^t({}^t i): l_{p',q'}\{E'\}'' \to l^0_{p',q'}\{E'\}''$$

is also an isometric isomorphism, from which it follows that the quasi-norm $\|\cdot\|_{p',q'}$ and the norm $\|\cdot\|^0_{p',q'}$ are equivalent on $l_{p',q'}\{E'\}$ by Banach's homomorphism theorem (cf. [4], p. 294). Therefore $l_{p',q'}\{E'\}$ is complete for the norm $\|\cdot\|^0_{p',q'}$ since it is complete for its own quasi-norm by Theorem 1. This, combined with the fact that $l_{p',q'}\{E'\}$ is dense in $l^0_{p',q'}\{E'\}$, shows that $l_{p',q'}\{E'\} = l^0_{p',q'}\{E'\}$. Thus $l_{p',q'}\{E'\}$ and $l^0_{p',q'}\{E'\}$ are isomorphic.

(iii) Let $1<q<p\leq\infty$ and suppose that E is an arbitrary Banach space. Let $\{x'_i\} \in l^0_{p',q'}\{E'\}$. Put $e'_i = x'_i/\|x'_i\|$ (resp. $e'_i=0$) if $x'_i \neq 0$ (resp. $x'_i=0$) and $\alpha_i = \|x'_i\|$. Then, $x'_i = \alpha_i e'_i$ for each $i \in I$. For any $\varepsilon>0$, if $e'_i \neq 0$, there exists an $e_i \in E$ such that $\|e_i\|=1$ and $<e_i, e'_i> > 1-\varepsilon$. If $e'_i=0$, we put $e_i=0$. Then, for any $\{\xi_i\} \in l_{p,q}$ with $\|\{\xi_i\}\|_{p,q} \leq 1$ we have

$$\begin{aligned}\sum_{i=1}^{\infty} |\xi_i \alpha_i| &\leq \frac{1}{1-\varepsilon} \sum_{i=1}^{\infty} |<\xi_i e_i, \alpha_i e'_i>| \\ &\leq \frac{1}{1-\varepsilon} \sup_{\|\{x_i\}\|_{p,q} \leq 1} \sum_{i=1}^{\infty} |<x_i, x'_i>| \\ &= \frac{1}{1-\varepsilon} \|\{x'_i\}\|^0_{p',q'}.\end{aligned}$$

Letting $\varepsilon \to 0$, we have $\sum_{i=1}^{\infty} |\xi_i \alpha_i| \leq \|\{x_i'\}\|_{p',q'}^0$, which implies

$$\{\alpha_i\} \in l_{p',q'}^0 \quad \text{and} \quad \|\{\alpha_i\}\|_{p',q'}^0 \leq \|\{x_i'\}\|_{p',q'}^0.$$

Since $l_{p',q'}^0$ is isomorphic to $l_{p',q'}$ by (ii), $\{\alpha_i\} \in l_{p',q'}$. Let $M_{p,q}$ be a positive number such that $M_{p,q} > 1$ and

$$\|\{\beta_i\}\|_{p',q'}^0 \leq \|\{\beta_i\}\|_{p',q'} \leq M_{p,q} \|\{\beta_i\}\|_{p',q'}^0 \qquad \text{for every } \{\beta_i\} \in l_{p',q'}^0.$$

Then we have

$$\begin{aligned} \|\{x_i'\}\|_{p',q'} = \|\{\alpha_i\}\|_{p',q'} &\leq M_{p,q} \|\{\alpha_i\}\|_{p',q'}^0 \\ &\leq M_{p,q} \|\{x_i'\}\|_{p',q'}^0, \end{aligned}$$

whence $\{x_i'\} \in l_{p',q'}\{E'\}$. Thus we have

$$l_{p',q'}^0\{E'\} \subset l_{p',q'}\{E'\}, \quad \|\cdot\|_{p',q'} \leq M_{p,q} \|\cdot\|_{p',q'}^0. \tag{27}$$

It follows from (26) and (27) that $l_{p',q'}\{E'\}$ is isomorphic to $l_{p',q'}^0\{E'\}$, which is isometrically isomorphic to $l_{p,q}\{E\}'$ and (23) holds for this $M_{p,q}$. This completes the proof of the theorem.

COROLLARY. *Let $1 < p < q < \infty$. $l_{p,q}\{E\}$ is a Banach space with the following norm $\|\cdot\|_{p,q}^0$ which is equivalent to $\|\cdot\|_{p,q}$:*

$$\|\{x_i\}\|_{p,q}^0 = \sup_{\|\{x_i'\}\|_{p',q'} \leq 1} \left| \sum_{i=1}^{\infty} \langle x_i, x_i' \rangle \right| \qquad \textit{for each } \{x_i\} \in l_{p,q}\{E\}.$$

PROOF. Since $l_{p,q}\{E\}''$ and $l_{p,q}\{E''\}$ are isomorphic by Theorem 2, it is easily seen that $\|\cdot\|_{p,q}^0$ is a norm on $l_{p,q}\{E\}$ which is equivalent to the quasi-norm $\|\cdot\|_{p,q}$.

As a consequence of Theorem 2 we have the following

THEOREM 3. *Let $1 < p \leq \infty$, $1 < q < \infty$. If E is reflexive, then $l_{p,q}\{E\}$ is reflexive.*

The assertion for $p < q$ means $l_{p,q}\{E\}$ is reflexive as a topological vector space (cf. [7]). However, it is also reflexive as the Banach space defined in the preceding corollary.

§4. Conjugates of $(p, q; r)$-absolutely summing operators

In this section, as an application of the result obtained in the preceding section, we shall characterize the conjugates of $(p, q; r)$-absolutely summing operators ([6], [9]).

We first recall the definitions of the space of weakly p-summable sequences $l_p(E)$ and the space of strongly p-summable sequences $l_p\langle E\rangle$ ([2]).

For $1\leq p\leq\infty$ $l_p(E)$ is the normed space consisting of all E-valued sequences $\{x_i\}$ such that for any $x'\in E'$ the sequence $\{<x_i, x'>\}$ belongs to l_p, where the norm is given by

$$\varepsilon_p(\{x_i\}) = \begin{cases} \sup\limits_{\|x'\|\leq 1}\left(\sum\limits_{i=1}^{\infty}|<x_i, x'>|^p\right)^{1/p} & \text{if } 1\leq p<\infty, \\ \sup\limits_i \|x_i\| & \text{if } p=\infty. \end{cases}$$

For $1\leq p\leq\infty$ $l_p\langle E\rangle$ is the normed space consisting of all E-valued sequences $\{x_i\}$ such that for each $\{x_i'\}\in l_{p'}(E')$ the series $\sum_{i=1}^{\infty}<x_i, x_i'>$ converges, where the norm is given by

$$\sigma_p(\{x_i\}) = \sup_{\varepsilon_{p'}(\{x_i'\})\leq 1}\left|\sum_{i=1}^{\infty}<x_i, x_i'>\right|.$$

We now recall the following

Definition 3. *For $1\leq p, q, r\leq\infty$ an operator $T: E\to F$ is called $(p, q; r)$-absolutely summing* ([9]) *provided there exists a constant $c\geq 0$ such that*

$$\|\{Tx_i\}\|_{p,q}\leq c\,\varepsilon_r(\{x_i\}) \qquad \text{for every } \{x_i\}\in\mathscr{F}(E) \tag{28}$$

and $(r; p, q)$-strongly summing ([6]) *provided there exists a constant $c\geq 0$ such that*

$$\sigma_r(\{Tx_i\})\leq c\|\{x_i\}\|_{p,q} \qquad \text{for every } \{x_i\}\in\mathscr{F}(E). \tag{29}$$

The smallest number c for which (28) *(resp.* (29)*) holds is denoted by $\Pi_{p,q;r}(T)$ (resp. $D_{r;p,q}(T)$).*

$\Pi_{p,q;r}$ (resp. $D_{r;p,q}$) is a norm for $p\geq q$ and a quasi-norm for $p<q$ on the space of $(p, q; r)$-absolutely summing (resp. $(r; p, q)$-strongly summing) operators.

$(p, p; r)$-absolutely summing operators are exactly (p, r)-absolutely summing operators (B. Mitjagin and A. Pełczyński [8]), $(p, p; p)$-absolutely summing operators coincide with absolutely p-summing operators (A. Pietsch [11]) and $(p; p, p)$-strongly summing operators coincide with strongly p-summing operators (J. S. Cohen [2]).

Theorem 4. *Let $1<p\leq\infty$, $1<q<\infty$, $1\leq r\leq\infty$. An operator $T: E\to F$ is $(p, q; r)$-absolutely summing if and only if its conjugate operator $T': F'\to E'$ is $(r'; p', q')$-strongly summing. In this case,*

$$D_{r';p',q'}(T') \leq 2^{2/p}\Pi_{p,q;r}(T) \leq 2^{2/p}M_{p,q}D_{r';p',q'}(T') \qquad \text{for} \quad p<q,$$

$$D_{r';p',q'}(T') \leq \Pi_{p,q;r}(T) \leq M_{p,q}D_{r';p',q'}(T') \qquad \text{for} \quad p \geq q,$$

where $M_{p,q}$ *is as in* Theorem 2.

PROOF. By Proposition 1 we can improve the estimate in Theorem 2 of [6] as follows: For every $(p, q; r)$-absolutely summing operator T, its conjugate operator T' is $(r'; p', q')$-strongly summing and we have

$$D_{r';p',q'}(T') \leq 2^{2/p}\Pi_{p,q;r}(T) \qquad \text{for} \quad p < q,$$

$$D_{r';p',q'}(T') \leq \Pi_{p,q;r}(T) \qquad \text{for} \quad p \geq q.$$

On the other hand, since $l_{p,q}\{E\}'$ and $l_{p',q'}\{E'\}$ are isomorphic by Theorem 2, we have by Remark 1 in [6] that if T' is $(r'; p', q')$-strongly summing, then T is $(p, q; r)$-absolutely summing and $\Pi_{p,q;r}(T) \leq D_{r';p',q'}(T')$.

Similarly, by Theorem 4 and Remark 2 in [6], combined with Theorem 2, we have the following characterization of an operator whose conjugate is $(p, q; r)$-absolutely summing.

THEOREM 5. *Let* $1<p\leq\infty$, $1<q<\infty$, $1\leq r\leq\infty$. *For an operator* $T: E\to F$, *the conjugate operator* $T': F'\to E'$ *is* $(p, q; r)$*-absolutely summing if and only if* T *is* $(r'; p', q')$*-strongly summing. In this case,*

$$D_{r';p',q'}(T) \leq \Pi_{p,q;r}(T') \leq M_{p',q'}D_{r';p',q'}(T)$$

for a certain number $M_{p',q'}>1$.

References

[1] P. L. Butzer and H. Berens, *Semi-groups of operators and approximation,* Springer, Berlin, 1967.

[2] J. Cohen, *Absolutely p-summing, p-nuclear operators and their conjugates,* Dissertation, Univ. of Maryland, 1970.

[3] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities,* Cambridge Univ. Press, 1967.

[4] J. Horváth, *Topological vector spaces and distributions* I, Addison-Wesley, Reading, Mass., 1966.

[5] R. A. Hunt, *On L(p, q) spaces,* Enseignement Math. **12** (1966), 249–276.

[6] M. Kato, *Conjugates of (p, q; r)-absolutely summing operators,* Hiroshima Math. J. **5** (1975), 127–134.

[7] G. Köthe, *Topologische lineare Räume,* Springer, Berlin, 1960.

[8] B. Mitjagin and A. Pełczyński, *Nuclear operators and approximative dimension,* Proc. I. C. M. Moscow, 1966, 366–372.

[9] K. Miyazaki, *(p, q; r)-absolutely summing operators,* J. Math. Soc. Japan **24** (1972),

341–354.

[10] K. Miyazaki, *(p, q)-nuclear and (p, q)-integral operators,* Hiroshima Math. J. **4** (1974), 99–132.

[11] A. Pietsch, *Absolut p-summierende Abbildungen in normierten Räumen*, Studia Math. **28** (1967), 333–353.

[12] K. Yoshinaga, *Lorentz spaces and the Calderón-Zygmund theorem*, Bull. Kyushu Inst. Tech. **16** (1969), 1–38.

Department of Mathematics,
Faculty of Science,
Hiroshima University