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Oscillatory and Asymptotic Behavior of Differential Equations with Deviating Arguments(*)

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1. Introduction

In this paper we are concerned with the oscillatory and asymptotic behavior of the n-th order ($n>1$) nonlinear differential equation with deviating arguments of the form

$$(*)\qquad x^{(n)}(t)+\left\{\prod_{j=1}^{m_0}|x[\tau_{0j}(t)]|^{\rho_j}\right\}F(t;\ [x]^2<\tau_0(t)>,\ [x']^2<\tau_1(t)>,\dots$$

$$\dots,[x^{(n-1)}]^2<\tau_{n-1}(t)>)\prod_{j=1}^{2\lambda-1}\operatorname{sgn}x[\tau_{0j}(t)]=0,\qquad t\geqq t_0,$$

where λ is a positive integer so that $2\lambda-1\leqq m_0$ and:

$$(\forall j=1,2,\dots,m_0)\rho_j\geqq 0,$$

$$\sum_{j=1}^{m_0}\rho_j=1,$$

$$\tau_i(t)=(\tau_{i1}(t),\ \tau_{i2}(t),\dots,\ \tau_{im_i}(t)),$$

$$h<\sigma(t)>\ =(h[\sigma_1(t)],\ h[\sigma_2(t)],\dots,\ h[\sigma_m(t)]),\ \sigma=(\sigma_1,\sigma_2,\dots,\sigma_m).$$

In the particular case, where

$$(\forall i,j)\tau_{ij}(t)\equiv t$$

the above differential equation (*) becomes an ordinary differential equation.

For the real valued functions τ_{ij} ($j=1,2,\dots,m_i$, $i=0,1,\dots,n-1$) and F we suppose that:

(i) The functions τ_{ij} are continuous on the half-line $[t_0,\infty)$ and

$$\lim_{t\to\infty}\tau_{ij}(t)=\infty.$$

(ii) F is non-negative on $[t_0,\infty)\times E_0$ and $\left(\prod_{j=1}^{m_0}y_{0j}^{e_j/2}\right)F(t;y_0,y_1,\dots,y_{n-1})$ is continuous on the same set, where $E_0=[0,\infty)^{m_0}\times[0,\infty)^{m_1}\times\cdots\times[0,\infty)^{m_{n-1}}$.

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Our results generalize and extend recent ones due to C. V. Coffman and J. S. W. Wong [1], T. Kusano and H. Onose [3], Z. Nehari [5], H. Onose [6], V. M. Ševelo and N. V. Vareh [13] and Y. G. Sficas and V. A. Staikos [7].

In order to obtain sufficient conditions for the oscillation of solutions of the differential equation (*) we make use of the *comparison principle* introduced by V. A. Staikos and Y. G. Sficas in [8], [10] and [11]. We exploit this principle by considering the simple differential equation

$$(**)\qquad y^{(n)}(t)+g(t)\left\{\prod_{j=1}^{m_0}|y[\tau_{0j}(t)]|^{\alpha_j}\right\}\prod_{j=1}^{2\lambda-1}\operatorname{sgn} y[\tau_{0j}(t)]=0,\qquad t\geqq t_0,$$

the oscillatory and asymptotic behavior of which is studied here. We suppose that g is a continuous and non-negative function on the half-line $[t_0, \infty)$ and α_j are such that:

$$(\forall j=1,2,\dots,m_0)\alpha_j\geqq 0,$$

$$\sum_{j=1}^{m_0}\alpha_j=\alpha>0.$$

The differential equation (**) is obviously a generalization of the well-known Emden-Fowler differential equation.

In the particular case $m_0=1$, the study of the oscillatory and asymptotic behavior of equation (*) is faced by introducing the concepts of sublinear and superlinear differential equations (cf. [1], [5] and [6]). Here we extend these concepts, at first, for the differential equation (**) and then for differential equation (*).

DEFINITION 1. *The differential equation* (**) *is called*:

(a) *τ_0-distorted sublinear, if*

$$\sum_{j=1}^{m_0}\alpha_j=\alpha\leqq 1,$$

(b) *τ_0-distorted strongly sublinear, if*

$$\sum_{j=1}^{m_0}\alpha_j=\alpha<1,$$

(c) *τ_0-distorted superlinear, if*

$$\sum_{j=1}^{m_0}\alpha_j=\alpha\geqq 1.$$

(d) *τ_0-distorted strongly superlinear, if*

$$\sum_{j=1}^{m_0} \alpha_j = \alpha > 1 .$$

DEFINITION 2. *The differential equation* (*) *is called*:

(a) τ_0*-distorted sublinear, if for any* $t \geqq t_0$ *the function* $F(t; y_0, y_1, \dots, y_{n-1})$ *is non-increasing with respect to* $(y_0, y_1, \dots, y_{n-1}) \in E = (0, \infty)^{m_0} \times (0, \infty)^{m_1} \times \cdots \times (0, \infty)^{m_{n-1}}$, *i.e. for any* $t \geqq t_0$ *we have*

$$(\forall i = 0, 1, \dots, n-1) y_i \leqq z_i \Longrightarrow F(t; y_0, y_1, \dots, y_{n-1}) \geqq F(t; z_0, z_1, \dots, z_{n-1}),$$

(b) τ_0*-distorted strongly sublinear, if there exist non-negative numbers* $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m_0}$ *so that* $\sum_{j=1}^{m_0} \varepsilon_j > 0$ *and for any* $t \geqq t_0$ *the function* $\Phi(t; y_0, y_1, \dots, y_{n-1}) = \left(\prod_{j=1}^{m_0} y_{0j}^{\varepsilon_j}\right) F(t; y_0, y_1, \dots, y_{n-1})$ *is non-increasing with respect to* $(y_0, y_1, \dots, y_{n-1}) \in E$, *i.e. for any* $t \geqq t_0$ *we have*

$$(\forall i = 0, 1, \dots, n-1) y_i \leqq z_i \Longrightarrow \Phi(t; y_0, y_1, \dots, y_{n-1}) \geqq \Phi(t; z_0, z_1, \dots, z_{n-1}),$$

(c) τ_0*-distorted superlinear, if for any* $t \geqq t_0$ *the function* $F(t; y_0, y_1, \dots, y_{n-1})$ *is non-decreasing with respect to* $(y_0, y_1, \dots, y_{n-1}) \in E_0$ *i.e. for any* $t \geqq t_0$ *we have*

$$(\forall i = 0, 1, \dots, n-1) y_i \leqq z_i \Longrightarrow F(t; y_0, y_1, \dots, y_{n-1}) \leqq F(t; z_0, z_1, \dots, z_{n-1}),$$

(d) τ_0*-distorted strongly superlinear, if there exist non-negative numbers* $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m_0}$ *so that* $\sum_{j=1}^{m_0} \varepsilon_j > 0$ *and for any* $t \geqq t_0$ *the function* $\Phi(t; y_0, y_1, \dots, y_{n-1}) = \left(\prod_{j=1}^{m_0} y_{0j}^{-\varepsilon_j}\right) F(t; y_0, y_1, \dots, y_{n-1})$ *is non-decreasing with respect to* $(y_0, y_1, \dots, y_{n-1}) \in E_0$, *i.e. for any* $t \geqq t_0$ *we have*

$$(\forall i = 0, 1, \dots, n-1) y_i \leqq z_i \Longrightarrow \Phi(t; y_0, y_1, \dots, y_{n-1}) \leqq \Phi(t; z_0, z_1, \dots, z_{n-1}).$$

REMARK. The order in the euclidean space $\boldsymbol{R}^m$ is considered in the usual sense, i.e.

$$y \leqq z \Longleftrightarrow (\forall i = 0, 1, \dots, m) y_i \leqq z_i.$$

Also the vectors (1, 1,..., 1) and (0, 0,..., 0) of the space $\boldsymbol{R}^m$ are denoted briefly by **1** and **0** respectively, i.e.

$$\mathbf{1} = (1, 1, \dots, 1) \quad \text{and} \quad \mathbf{0} = (0, 0, \dots, 0).$$

In what follows we consider only such solutions of the equation (*) which are defined for all large t. The oscillatory character is considered in the usual sense, i.e. a solution x of the differential equation (*) is called *oscillatory* if it has no last zero, otherwise it is called *nonoscillatory*.

To obtain our results we make use of the following three lemmas, which are adaptations of the lemmas in [8], [2] and [11] respectively.

LEMMA 1 (*Comparison principle*). *Consider the differential equations with deviating arguments*

$$(E) \qquad x^{(n)}(t)+F(t;\, x<\tau_0(t)>,\, x'<\tau_1(t)>,\dots,\, x^{(n-1)}<\tau_{n-1}(t)>)=0$$

and

$$(E_g) \qquad y^{(n)}(t)+g(t)G(t;\, y<\sigma_0(t)>,\, y'<\sigma_1(t)>,\dots,\, y^{(n-1)}<\sigma_{n-1}(t)>) = 0,$$

where g belongs to a certain function class $\mathscr{G}$, and let g_z denoted the function defined by

$$g_z(t)=\frac{F(t;\, z<\tau_0(t)>,\, z'<\tau_1(t)>,\dots,\, z^{(n-1)}<\tau_{n-1}(t)>)}{G(t;\, z<\sigma_0(t)>,\, z'<\sigma_1(t)>,\dots,\, z^{(n-1)}<\sigma_{n-1}(t)>)}$$

If P is a propositional function with domain a function class $\mathscr{E}$ and

$$\mathscr{S} = \{x\in\mathscr{E}:\ x \text{ is a solution of } (E)\}$$

$$\mathscr{S}_g = \{x\in\mathscr{E}:\ x \text{ is a solution of } (E_g)\}$$

then

$$(\forall g\in\mathscr{G})(\forall y\in\mathscr{S}_g)P(y)$$

and

$$(\forall x\in\mathscr{S}) \sim P(x) \Longrightarrow g_x\in\mathscr{G}$$

imply

$$(\forall x\in\mathscr{S})P(x).$$

LEMMA 2. *Let u be an n-times differentiable function on the interval $[a, \infty)$ with $u^{(k)}$ $(k=0, 1,\dots, n-1)$ absolutely continuous on $[a, \infty)$. If $u^{(n)}(t)$ is not identically zero for all large t and*

$$u(t) \neq 0,\ u(t)u^{(n)}(t) \leqq 0 \quad \text{for every } t\in[a, \infty),$$

then there exists an integer l with $0\leqq l<n$, $n+l$ odd, such that

$$u(t)u^{(k)}(t) \geqq 0 \qquad \text{for every} \quad t\in[a, \infty)\ (k = 0, 1,\dots, l),$$

$(-1)^{n+k-1}u(t)u^{(k)}(t)\geqq 0$ *for every* $t\in[a, \infty)$ $(k=l+1,\ l+2,\dots, n)$, *and*

$$|u(t)| \geqq \frac{(t-a)^{n-1}|u^{(n-1)}2^{n-l-1}t)|}{(n-1)(n-2)\cdots(n-l)} \text{ for every } t\in[a, \infty).$$

LEMMA 3. *If* u *is as in* Lemma 2 *and for some* $k=0, 1,\dots, n-2$,

$$\lim_{t\to\infty} u^{(k)}(t) = c, \qquad c\in \boldsymbol{R},$$

then

$$\lim_{t\to\infty} u^{(k+1)}(t) = 0.$$

2. Oscillatory and asymptotic behavior of bounded solutions

In this section we study the oscillatory and asymptotic behavior of the bounded solutions of the differential equation $(*)$ in the case where it is τ_0-distorted sublinear (Theorems 1 and 3) or τ_0-distorted superlinear (Theorems 2 and 4).

THEOREM 1. *Consider the differential equation* $(*)$ *subject to the conditions* (i) *and* (ii). *If the equation* $(*)$ *is* τ_0*-distorted sublinear and for any* μ_1, μ_2 *with* $|\mu_1|>|\mu_2|$

$$(C_1) \qquad \int^{\infty} t^{n-1}F(t;\, \mu_1^2\cdot\mathbf{1},\, \mu_2^2\cdot\mathbf{1},\dots,\, \mu_2^2\cdot\mathbf{1})\,dt = \infty,$$

then for n *even all bounded solutions of the equation* $(*)$ *are oscillatory, while for* n *odd all bounded solutions of the equation* $(*)$ *are either oscillatory or tending monotonically to zero as* $t\to\infty$ *together with their first* $n-1$ *derivatives.*

PROOF. Let x be a bounded non-oscillatory solution of the equation $(*)$ with $\lim_{t\to\infty} x(t)\neq 0$. Since both x and $-x$ are solutions of the differential equation $(*)$, we can assume, without loss of generality, that $x(t)>0$ for every $t\geqq t_0$. Moreover, since $\lim_{t\to\infty}\tau_{0j}(t)=\infty$ $(j=1, 2,\dots, m_0)$, there exists a $t_1\geqq t_0$ so that

$$(1) \qquad \tau_{0j}(t) \geqq t_0 \quad \text{for every} \quad t \geqq t_1 \ (j = 1, 2,\dots, m_0).$$

Thus, from equation $(*)$, by (1) and condition (ii), it follows that

$$(2) \qquad x^{(n)}(t) \leqq 0 \quad \text{for every} \quad t \geqq t_0.$$

We prove now that $x^{(n)}(t)$ is not identically zero for all large t. To do this we suppose the opposite and remark that in this case the solution $x(t)$ coincides with a polynomial for all large t. Thus, since x is bounded, it must be constant, i.e. $x(t)=\mu_0>0$ for all large t (by hypothesis $\lim_{t\to\infty} x(t)\neq 0$). Hence

$$(3) \qquad -x^{(n)}(t) = \left\{\prod_{j=1}^{m_0} (x[\tau_{0j}(t)])^{\rho_j}\right\} F(t;\, [x]^2<\tau_0(t)>,\, [x']^2<\tau(t)>,\dots$$

$$\ldots, [x^{(n-1)}]^2 < \tau_{n-1}(t) >)$$

$$= \mu_0 F(t;\ \mu_0^2 \cdot \mathbf{1},\ \mathbf{0}, \ldots, \mathbf{0}) \qquad \text{for all large } t,$$

and therefore

$$-x^{(n)}(t) = \mu_0 F(t;\ \mu_0^2 \cdot \mathbf{1},\ \mathbf{0}, \ldots, \mathbf{0})$$

$$\geqq \mu_0 F\left(t;\ \mu_0^2 \cdot \mathbf{1},\ \frac{\mu_0^2}{4} \cdot \mathbf{1}, \ldots, \frac{\mu_0^2}{4} \cdot \mathbf{1}\right) \qquad \text{for all large } t.$$

Since, by (C_1), $F\left(t;\ \mu_0^2 \cdot \mathbf{1}, \frac{\mu_0^2}{4} \cdot \mathbf{1}, \ldots, \frac{\mu_0^2}{4} \cdot \mathbf{1}\right)$ is not identically zero for all large t, the same holds for the function $x^{(n)}(t)$, which contradicts our assumption.

Now, by Lemma 2, we conclude that there exists an integer l with $0 \leqq l < n$, $n + l$ odd, so that for every $t \geqq t_1$

$$x^{(k)}(t) \geqq 0 \qquad (k = 0, 1, \ldots, l)$$

and

$$(-1)^{n+k-1} x^{(k)}(t) \geqq 0 \qquad (k = l+1,\ l+2, \ldots, n).$$

We shall show that $l = 0$ or $l = 1$. Indeed, if $l > 1$, then by Taylor's formula we have

$$x(t) \geqq x(T) + \frac{x'(T)}{1!}(t - T) + \cdots + \frac{x^{(l-1)}(T)}{(l-1)!}(t - T)^{l-1}$$

for every $t \geqq T$, where T is chosen so that $x^{(l-1)}(T) > 0$. (A such choice of T is possible, since as we have proved $x^{(n)}(t)$ is not identically zero for all large t.) This inequality is obviously a contradiction to our assumption that x is bounded.

Since $n + l$ is odd, for n even we have $l = 1$ and

$$x'(t) \geqq 0 \qquad \text{for all} \quad t \geqq t_1,$$

while for n odd we have $l = 0$ and

$$x'(t) \leqq 0 \qquad \text{for all} \quad t \geqq t_1.$$

If we put $c = \lim_{t \to \infty} x(t)$ in the case where x is non-decreasing (i.e. n is even) and $c = 2 \lim_{t \to \infty} x(t)$ in case where x is non-increasing (i.e. n is odd), then we can easily derive that for a $t_2 \geqq t_1$ and for every $t \geqq t_2$

$$\text{(4)} \qquad \frac{c}{2} \leqq x[\tau_{0_j}(t)] \leqq c \qquad (j = 1, 2, \ldots, m_0).$$

Since $\lim_{t \to \infty} x(t)$ exists in R, by Lemma 3, we conclude that

$$\lim_{t\to\infty} x^{(k)}(t) = 0 \qquad (k = 1, 2,\dots, n-1)$$

and consequently there exists a $t_3 \geqq t_2$ so that for every $t \geqq t_3$

(5) $$|x^{(k)}[\tau_{ij}(t)]| \leqq \frac{c}{2} \quad (k = 1, 2,\dots, n-1\,;\, j = 1, 2,\dots, m_i\,;\, i = 0, 1,\dots, n-1)\,.$$

Now, consider the differential equation (∗) and

(6) $$y^{(n)}(t) + g(t)y^{\alpha}(t) = 0,$$

where $\alpha = \frac{\mu}{\nu} > 1$ and μ, ν are odd integers, in the place of the equations (E) and (E_g) of Lemma 1 respectively.

Let $\mathscr{E}$ be the class of all continuous and bounded functions w, which are defined on an interval of the form $[t_w, \infty)$ and P the propositional function:

$$P(w):\quad w \text{ is oscillatory or } \lim_{t\to\infty} w(t) = 0$$

Furthermore, let $\mathscr{G}$ be the class of all non-negative functions g defined on an interval of the form $[t_g, \infty)$, which satisfy the condition

(7) $$\int^{\infty} t^{n-1} g(t)dt = \infty$$

It is well-known (cf. [4]) that, under condition (7), all solutions of the differential equation (6) are oscillatory or tending to zero as $t\to\infty$. That is

$$(\forall g \in \mathscr{G})(\forall y \in \mathscr{S}_g)P(y)\,.$$

We remark that for any bounded solution x of the equation (∗) for which $\sim P(x)$ is satisfied, i.e. x is non-oscillatory and $\lim_{t\to\infty} x(t) \neq 0$, the function g_x associated to x belongs to the function class $\mathscr{G}$. In fact, using the sublinearity of the differential equation (∗) and taking into account (4) and (5), we obtain

(8) $$g_x(t) = x^{-\alpha}(t)\left\{\prod_{j=1}^{m_0} (x[\tau_{0j}(t)])^{\rho_j}\right\} F(t;\, [x]^2 <\tau_0(t)>,\, [x']^2 <\tau_1(t)>,\dots$$
$$\dots,\, [x^{(n-1)}]^2 <\tau_{n-1}(t)>)$$
$$\geqq F\left(t;\, c^2\cdot\mathbf{1}, \frac{c^2}{4}\cdot\mathbf{1},\dots, \frac{c^2}{4}\cdot\mathbf{1}\right) \quad \text{for every } t \geqq t_3$$

and consequently, by (C_1),

$$\int^{\infty} t^{n-1} g_x(t)dt = \infty,$$

i.e.

$$(\forall x \in \mathscr{S}) \sim P(x) \Longrightarrow g_x \in \mathscr{G}.$$

Next, applying Lemma 1, we have

$$(\forall x \in \mathscr{S}) P(x)$$

i.e. every bounded solution x of the differential equation $(*)$ is oscillatory or $\lim_{t\to\infty} x(t)=0$. By Lemma 2, if x is a bounded non-oscillatory solution of the equation $(*)$ with $\lim_{t\to\infty} x(t)=0$, then $x'\leqq 0$ and, therefore, n is an odd integer. Moreover, by Lemma 3, in this case we also have

$$\lim_{t\to\infty} x^{(k)}(t) = 0 \qquad (k = 1, 2, \dots, n-1).$$

THEOREM 2. *Consider the differential equation* $(*)$ *subject to the conditions* (i) *and* (ii). *If the equation* $(*)$ *is* τ_0*-distorted superlinear and for any* $\mu\neq 0$

$$\text{(C}_2\text{)} \qquad \int^{\infty} t^{n-1} F(t;\, \mu^2 \cdot \mathbf{1},\, \mathbf{0}, \dots, \mathbf{0}) dt = \infty$$

then for n even all bounded solutions of the equation $(*)$ *are oscillatory, while for n odd all bounded solutions of the equation* $(*)$ *are either oscillatory or tending monotonically to zero as* $t\to\infty$ *together with their first* $n-1$ *derivatives.*

PROOF. Let x be a bounded non-oscillatory solution of the differential equation $(*)$ with $\lim_{t\to\infty} x(t)\neq 0$. As in the proof of Theorem 1, we can assume, without loss of generality, that $x(t)>0$ for every $t\geqq t_0$ and by choosing $t_1\geqq t_0$ as in (1) we derive (2). Then, we prove (3), which implies that $x^{(n)}(t)$ is not identically zero for all large t, since, by (C$_2$), this holds for $F(t;\, \mu_0^2\cdot \mathbf{1},\, \mathbf{0}, \dots, \mathbf{0})$.

Finally we remark that the proof of the theorem follows exactly the same way as that of the theorem 1, by using in place of (8) the inequality

$$g_x(t) = x^{-\alpha}(t)\left\{\prod_{j=1}^{m_0} (x[\tau_{0j}(t)])^{\rho_j}\right\} F(t;\, [x]^2 < \tau_0(t) >,\, [x']^2 < \tau_1(t) >, \dots$$
$$\dots,\, [x^{(n-1)}]^2 < \tau_{n-1}(t) >)$$
$$\geqq \frac{c^{1-\alpha}}{2} F\left(t;\, \frac{c^2}{4}\cdot \mathbf{1},\, \mathbf{0}, \dots, \mathbf{0}\right).$$

THEOREM 3. *Consider the differential equation* $(*)$ *subject to the conditions* (i), (ii) *and*

(I) $\tau_{ij}(t) \leqq t$ *for every* $t \geqq t_0$ $(j = 1, 2, \dots, m_i;\ i = 0, 1, \dots, n-1)$. *If the equation* $(*)$ *is* τ_0*-distorted sublinear and there exists* $\mu\neq 0$ *so that*

$$(C_3) \qquad \int^{\infty} t^{n-1}F(t;\, \mu^2\cdot\mathbf{1},\, \mathbf{0},\ldots,\, \mathbf{0})dt < \infty\,,$$

then for n even the differential equation (∗) *has a bounded non-oscillatory solution, while for n odd the differential equation* (∗) *has a bounded non-oscillatory solution x with*

$$\lim_{t\to\infty} x(t) \neq 0.$$

PROOF. It is enough to show that there exists a solution x of the equation (∗) with $\lim_{t\to\infty} x(t) = c_0$, where $c_0 \neq 0$ is chosen properly.

The proof of this is based on the arguments developed by V. A. Staikos and Y. G. Sficas [9] and needs the application of following fixed point theorem, which is a special case of Tychonoff's fixed point theorem (See [12]):

FIXED POINT THEOREM. *Let Y be a Fréchet space and X a convex and closed subset of Y. If S is a continuous mapping of X into itself and the closure $\overline{SX}$ is a compact subset of X, then there exists at least one fixed point $x \in X$ of S (i.e. a point $x \in X$ so that $x = Sx$).*

Without loss of generality we suppose that μ in the condition (C_3) is positive and choose a c_0 with $0 < \mu < c_0$. Put $\delta \equiv c_0 - \mu$ and, by (C_3), consider a $T \geqq \max\{t_0, 0\}$ so that for every $k = 0, 1,\ldots, n-1$

$$(9) \qquad (2c_0 - \mu)\int_T^{\infty} (s-T)^{n-i-k}F(s;\, \mu^2\cdot\mathbf{1},\, \mathbf{0},\ldots,\, \mathbf{0})ds \leqq \delta$$

For $T_0 = \min_{i,j}(\min_{t\geqq T} \tau_{ij}(t))$ let Y be the vector space of all continuous real valued functions which are constant on the interval $[T_0, T]$ and $n-1$ times continuously differentiable on the interval $[T, \infty)$.

Consider now in the space Y the sequence of seminorms (p_ν):

$$p_\nu(y) = \sup_{t\in[T, T+\nu]} |y^{(n-1)}(t)| + \sum_{k=0}^{n-2} |y^{(k)}(T)| \qquad (\nu = 1, 2,\ldots)$$

and introduce by it a total paranorm, defined by the formula (See [14])

$$p(x) = \sum_{\nu=1}^{\infty} \frac{1}{2^\nu}\frac{p_\nu(x)}{1+p_\nu(x)} \qquad \text{(Fréchet's combination)}.$$

The space Y endowed with the topology introduced by p becomes a Fréchet space.

Let X be the set of all $x \in Y$ with:

$$(A) \qquad |x(t) - c_0| \leqq \delta, \qquad \text{if} \quad t \geqq T_0$$

and

(B) $$|x^{(k)}(t)| \leqq \delta, \quad \text{if} \quad t \geqq T (k = 1, 2,\dots, n-1).$$

Obviously, X is *nonempty* and it is easy to see that X is also a *convex* set. Moreover, X is *closed*. To prove this, we consider a sequence (y_ν) in X with p-lim $y_\nu = x$. Then for any non-negative integer μ we have

(10) $$|y_\mu(t) - c_0| \leqq \delta, \quad \text{if} \quad t \geqq T_0$$

and

(11) $$|y_\mu^{(k)}(t)| \leqq \delta, \quad \text{if} \quad t \geqq T (k = 1, 2,\dots, n-1).$$

We remark now that, by the definition of p, the sequence $(y_\mu^{(n-1)})$ converges uniformly to the function $x^{(n-1)}$ on any interval $[T, T+\nu]$. Since, moreover, each sequence $(y_\mu^{(k)}(T))$ converges to $x^{(k)}(T)$ $(k=0, 1,\dots, n-2)$, it is easy to see that the sequence $(y_\mu^{(k)})$ converges uniformly to the function $x^{(k)}$ on the interval $[T, T+\nu]$ for every $k=0, 1,\dots, n-1$. But, since ν is an arbitrary natural number, this implies the pointwise convergence

$$\lim y_\mu^{(k)}(t) = x^{(k)}(t)$$

for every $t \geqq T$. By (10) and (11), we obviously have that for the limit function x the conditions (A) and (B) are both satisfied, i.e. $x \in X$.

Now, since, by (A) and the choice of δ, the elements of X are positive functions, we can define the mapping $S\colon X \to Y$ by the following formula:

(12) $$y(t) = (Sx)(t) = \begin{cases} c_0 + \dfrac{(-1)^{n-1}}{(n-1)!} \displaystyle\int_t^\infty (s-t)^{n-1} \left\{ \prod_{j=1}^{m_0} (x[\hat{\tau}_{0_j}(s)])^{\rho_j} \right\} \cdot \\ \cdot F(s; [x]^2 <\hat{\tau}_0(s)>,\dots, [x^{(n-1)}]^2 <\hat{\tau}_{n-1}(s)>)ds \\ \qquad\qquad\qquad\qquad \text{if} \quad t \geqq T, \\ c_0 + \dfrac{(-1)^{n-1}}{(n-1)!} \displaystyle\int_T^\infty (s-T)^{n-1} \left\{ \prod_{j=1}^{m_0} (x[\hat{\tau}_{0_j}(s)])^{\rho_j} \right\} \cdot \\ \cdot F(s; [x]^2 <\hat{\tau}_0(s)>,\dots, [x^{(n-1)}]^2 <\hat{\tau}_{n-1}(s)>)ds \\ \qquad\qquad\qquad\qquad \text{if} \quad T_0 \leqq t < T, \end{cases}$$

where

$$\hat{\tau}_i = (\tau_{i1}, \tau_{i2},\dots, \tau_{im_i})$$

and

$$\hat{\tau}_{ij}(t) = \begin{cases} \tau_{ij}(t), & \text{if} \quad \tau_{ij}(t) \geqq T \\ T, & \text{if} \quad \tau_{ij}(t) < T \end{cases} \quad (j = 1, 2,\dots, m_i;\ i = 0, 1,\dots, n-1).$$

Since for every $t \geqq T$

$$(13) \qquad \left|\left\{\prod_{j=1}^{m_0} (x[\hat{\tau}_{0j}(t)])^{\rho_j}\right\} F(t;\ [x]^2 < \hat{\tau}_0(t) >, \dots,\ [x^{(n-1)}]^2 < \hat{\tau}_{n-1}(t) >)\right|$$

$$\leqq (c_0+\delta) F(t;\ (c_0-\delta)^2 \cdot \mathbf{1},\ \mathbf{0}, \dots,\ \mathbf{0})$$

$$= (2c_0-\mu) F(t;\ \mu^2 \cdot \mathbf{1},\ \mathbf{0}, \dots,\ \mathbf{0})$$

by (9), it follows that the mapping S is defined on the whole set X, i.e. $S: X \mapsto Y$. Moreover, the mapping S has the properties as it is required in the fixed point theorem:

(a) $SX \subseteq X$.

In fact, for $y = Sx$ with $x \in X$ and for every $t \geqq T$, by (9) and (13), we have

$$|y(t) - c_0| \leqq \frac{1}{(n-1)!} \int_t^\infty (s-t)^{n-1} \left\{\prod_{j=1}^{m_0} (x[\hat{\tau}_{0j}(s)])^{\rho_j}\right\} F(s;\ [x]^2 < \hat{\tau}_0(s) >, \dots$$

$$\dots,\ [x^{(n-1)}]^2 < \hat{\tau}_{n-1}(s) >) ds$$

$$\leqq \frac{2c_0-\mu}{(n-1)!} \int_t^\infty (s-t)^{n-1} F(s;\ \mu^2 \cdot \mathbf{1},\ \mathbf{0}, \dots,\ \mathbf{0}) ds$$

$$\leqq \frac{2c_0-\mu}{(n-1)!} \int_T^\infty (s-T)^{n-1} F(s;\ \mu^2 \cdot \mathbf{1},\ \mathbf{0}, \dots,\ \mathbf{0}) ds \leqq \delta$$

and

$$|y^{(k)}(t)| \leqq \frac{(n-1)(n-2)\cdots(n-k)}{(n-1)!} \int_t^\infty (s-t)^{n-1-k} \left\{\prod_{j=1}^{m_0} (x[\hat{\tau}_{0j}(s)])^{\rho_j}\right\} \cdot$$

$$\cdot F(s;\ [x]^2 < \hat{\tau}_0(s) >, \dots,\ [x^{(n-1)}]^2 < \hat{\tau}_{n-1}(s) >) ds$$

$$\leqq \frac{2c_0-\mu}{(n-1-k)!} \int_t^\infty (s-T)^{n-1-k} F(s;\ \mu^2 \cdot \mathbf{1},\ \mathbf{0}, \dots,\ \mathbf{0}) ds$$

$$\leqq \frac{2c_0-\mu}{(n-1-k)!} \int_T^\infty (s-T)^{n-1-k} F(s;\ \mu^2 \cdot \mathbf{1},\ \mathbf{0}, \dots,\ \mathbf{0}) ds \leqq \delta .$$

(b) $\overline{SX}$ *is a compact subset of* X.

Let $y \in SX$ and t_1, t_2 in $[T, \infty)$. Consider an $x \in X$ with $y = Sx$. Then we have

$$|y^{(n-1)}(t_1) - y^{(n-1)}(t_2)| \leqq \left|\int_{t_1}^{t_2} \left\{\prod_{j=1}^{m_0} (x[\hat{\tau}_{0j}(s)])^{\rho_j}\right\} F(s;\ [x]^2 < \hat{\tau}_0(s) >, \dots\right.$$

$$\left.\dots,\ [x^{(n-1)}]^2 < \hat{\tau}_{n-1}(s) >) ds\right|$$

$$\leqq (2c_0-\mu)\left|\int_{t_1}^{t_2} F(s;\, \mu^2\cdot\mathbf{1},\, \mathbf{0},\ldots,\, \mathbf{0})\, ds\right|$$

and consequently the $(n-1)$-th derivatives of the function $y\in SX$ are equicontinuous at each point of the interval $[T, \infty)$. Since, moreover, by (B), the $(n-1)$-th derivatives of the functions of X are uniformly bounded, by Ascoli's theorem, for any sequence (y_ν) in SX there exists a subsequence (z_ν) of (y_ν) so that the sequence $(z_\nu^{(n-1)})$ converges uniformly on every compact subinterval of $[T, \infty)$. Also, by (A) and (B), the sequences $(z_\nu^{(k)}(T))$ $(k=0, 1,\ldots, n-2)$ are bounded and hence there exists a subsequence (w_ν) of (z_ν) so that each sequence $(w_\nu^{(k)}(T))$ $(k=0, 1,\ldots, n-2)$ is convergent. Thus, we easily conclude that the sequence (w_ν) is p-fundamental and then, since the space Y (as Fréchet space) is complete, there exists a $u\in Y$ so that

$$p-\lim w_\nu = u.$$

Hence, we have proved that each sequence in $\overline{SX}$ has a subsequence which is convergent in $\overline{SX}$ and then from this fact we easily conclude that the set $\overline{SX}$ has the property of Bolzano-Weierstrass.

(c) *The mapping S is continuous.*
Let $x\in X$ and (u_ν) be an arbitrary sequence in X with

$$p-\lim u_\nu = x.$$

If we put $y=Sx$ and $v_\nu=Su_\nu$, then for every $t\geqq T$

$$y(t) = \frac{(-1)^{n-1}}{(n-1)!}\int_t^\infty (s-t)^{n-1}\left\{\prod_{j=1}^{m_0} (x[\hat{\tau}_{0_j}(s)])^{\rho_j}\right\}F(s\colon [x]^2<\hat{\tau}_0(s)>,\ldots$$
$$\ldots, [x^{(n-1)}]^2<\hat{\tau}_{n-1}(s)>)ds$$

and

$$v_\nu(t) = \frac{(-1)^{n-1}}{(n-1)!}\int (s-t)^{n-1}\left\{\prod_{j=1}^{m_0} (u_\nu[\hat{\tau}_{0_j}(s)])^{\rho_j}\right\}F(s;\, [u_\nu]^2<\hat{\tau}_0(s)>,\ldots$$
$$\ldots, [u_\nu^{(n-1)}]^2<\hat{\tau}_{n-1}(s)>)ds.$$

For the function u_ν $(\nu=1, 2,\ldots)$, by (13), we have

$$|(s-t)^{n-1}\left\{\prod_{j=1}^{m_0} (u_\nu[\hat{\tau}_{0_j}(s)])^{\rho_j}\right\}F(s;\, [u_\nu]^2<\hat{\tau}_0(s)>,\ldots,\, [u_\nu^{(n-1)}]^2<\hat{\tau}_{n-1}(s)>)|$$
$$\leqq (2c_0-\mu)(s-t)^{n-1}F(s;\, \mu^2\cdot\mathbf{1},\, \mathbf{0},\ldots,\, \mathbf{0}).$$

Hence, by (9), we remark that

$$(2c_0-\mu)\int_t^\infty (s-t)^{n-1}F(s;\, \mu^2\cdot \mathbf{1},\, \mathbf{0},\dots,\, \mathbf{0})ds$$

$$\leqq (2c_0-\mu)\int_t^\infty (s-T)^{n-1}F(s;\, \mu^2\cdot \mathbf{1},\, \mathbf{0},\dots,\, \mathbf{0})ds$$

$$\leqq (2c_0-\mu)\int_T^\infty (s-T)^{n-1}F(s;\, \mu^2\cdot \mathbf{1},\, \mathbf{0},\dots,\, \mathbf{0})ds \leqq \delta < \infty.$$

We can now apply the Lebesgue dominated convergence theorem to obtain

$$\lim_\nu v_\nu(t) = \frac{(-1)^{n-1}}{(n-1)!}$$

$$\lim_\nu \int_t^\infty (s-t)^{n-1}\left\{\prod_{j=1}^{m_0} (u_\nu[\hat{\tau}_{0_j}(s)])^{\rho_j}\right\}F(s;\, [u_\nu]^2 <\hat{\tau}_0(s)>, \dots$$

$$\dots, [u_\nu^{(n-1)}]^2 <\hat{\tau}_{n-1}(s)>)ds$$

$$= \frac{(-1)^{n-1}}{(n-1)!}\int_t^\infty (s-t)^{n-1}\left\{\prod_{j=1}^{m_0} (x[\hat{\tau}_{0_j}(s)])^{\rho_j}\right\}F(s;\, [x]^2 <\hat{\tau}_0(s)>, \dots$$

$$\dots,\, [x^{(n-1)}]^2 <\hat{\tau}_{n-1}(t)>)ds\,.$$

Thus, for every $t \geqq T$ we have the pointwise convergence

$$\lim_\nu v_\nu(t) = y(t)\,.$$

To complete this proof, consider an arbitrary subsequence $(z_\mu)_{\mu\in M}$ of (v_ν). Since the set $\overline{SX}$ is compact, there exist a subsequence $(w_\lambda)_{\lambda\in\Lambda}$ and $\psi \in SX$ so that

$$p-\lim_{\lambda\in\Lambda} w_\lambda = \psi.$$

But, as we have shown in (b), the p-convergence implies the pointwise convergence and, hence, it is easy to see that

$$\psi = y.$$

Therefore

$$p-\lim_{\lambda\in\Lambda} w_\lambda = y$$

and consequently

$$p-\lim v_\nu = y.$$

Now, we can apply the fixed point theorem to conclude that there exists an $x\in X$ with $x=Sx$, which is the desired solution of the differential equation

(*), since, by (13) and (9),

$$|x(t)-c_0| = \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\left\{\prod_{j=1}^{m_0} (x[\hat{\tau}_{0_j}(s)])^{p_j}\right\}F(s;[x]^2<\hat{\tau}_0(s)>,\ldots$$

$$\ldots,[x^{(n-1)}]^2<\hat{\tau}_{n-1}(s)>)ds$$

$$\leqq \frac{2c_0-\mu}{(n-1)!}\int_t^\infty (s-T)^{n-1}F(s;\mu^2\cdot\mathbf{1},\mathbf{0},\ldots,\mathbf{0})ds \to 0 \text{ as } t\to\infty .$$

THEOREM 4. *Consider the differential equation* (*) *subject to the conditions* (i), (ii) *and* (I). *If the equation* (*) *is* τ_0*-distorted superlinear and there exist* $\mu_1\neq 0$, $\mu_2\neq 0$ *so that*

$$\text{(C}_4\text{)}\qquad \int^\infty t^{n-1}F(t;\mu_1^2\cdot\mathbf{1},\mu_2^2\cdot\mathbf{1},\ldots,\mu_2^2\cdot\mathbf{1})\,dt<\infty$$

then for n even the differential equation (*) *has a bounded non-oscillatory solution, while for n odd the differential equation* (*) *has a bounded non-oscillatory solution x with*

$$\lim_{t\to\infty} x(t)\neq 0 .$$

PROOF. The proof of this theorem is similar to the proof of Theorem 3. Without loss of generality, we suppose that μ_1, μ_2 in the condition (C_4) are positive and we choose a c_0 with $\max\left\{\frac{\mu_1}{2},\ \mu_1-\mu_2\right\}\leqq c_0<\mu_1$. We put $\delta=\mu_1-c_0$ and, by (C_4), choose a $T\geqq\max\{t_0, 0\}$ so that

$$\text{(14)}\qquad \mu_1\int_T^\infty (s-T)^{n-1-k}F(s;\mu_1^2\cdot\mathbf{1},\mu_2^2\cdot\mathbf{1},\ldots,\mu_2^2\cdot\mathbf{1})ds\leqq\delta$$

Furthermore, as in the proof of Theorem 3, we consider the corresponding space Y and its nonempty and convex subset X. It is easy to see that the elements of X are positive functions and therefore we can define the mapping S by formula (12).

By τ_0-distorted superlinearity of the differential equation (*) and by (C_4), we obtain that for every $t\geqq T$ and $x\in X$

$$\left|\left\{\prod_{j=1}^{m_0} (x[\hat{\tau}_{0_j}(t)])^{p_j}\right\}F(t;[x]^2<\hat{\tau}_0(t)>,\ldots,[x^{(n-1)}]^2<\hat{\tau}_{n-1}(t)>)\right|$$

$$\leqq (c_0+\delta)F(t;(c_0+\delta)^2\cdot\mathbf{1},\delta^2\cdot\mathbf{1},\ldots,\delta^2\cdot\mathbf{1})$$

$$\leqq \mu_1F(t;\mu_1^2\cdot\mathbf{1},\mu_2^2\cdot\mathbf{1},\ldots,\mu_2^2\cdot\mathbf{1}),$$

because $\delta=\mu_1-c_0\leqq\mu_1-(\mu_1-\mu_2)=\mu_2$. Using this inequality instead of (13), and applying step by step the proof of Theorem 3, we conclude the existence of a solution x of the differential equation (*) with

$$\lim_{t\to\infty} x(t) = c_0 \neq 0.$$

Consider the particular case of function F when

$$F(t; y_0, y_1,\dots, y_{n-1}) \equiv f(t; y_0)\varphi(y_0, y_1,\dots, y_{n-1})$$

where, in addition to (i) and (I), we suppose that following conditions are satisfied:

(II_1) f is non-negative on $[t_0, \infty)\times[0, \infty)^{m_0}$ and $\left(\prod_{j=1}^{m_0} y_{0j}^{\varrho_j/2}\right) f(t; y_0)$ is continuous on the same set.

(II_2) The function φ is positive and continuous on the set E_0.

In this case the differential equation (*) has the following form

$$(15)\qquad x^{(n)}(t)+\left\{\prod_{j=1}^{m_0} |x[\tau_{0j}(t)]|^{\rho_j}\right\} f(t;\ [x]^2<\tau_0(t)>)\varphi([x]^2<\tau_0(t)>,\dots$$

$$\dots, [x^{(n-1)}]^2<\tau_{n-1}(t)>)\prod_{j=1}^{2\lambda-1} \operatorname{sgn} x[\tau_{0j}(t)] = 0,\quad t \geqq t_0$$

Under the above assumptions from Theorems 1–4 we obtain the following.

COROLLARY. *If equation* (15) *is either τ_0-distorted sublinear or τ_0-distorted superlinear, then the condition*

$$(\mathrm{C}_5)\qquad \int^{\infty} t^{n-1} f(t;\ \mu^2\cdot\mathbf{1})dt = \infty \qquad \text{for every} \quad \mu \neq 0$$

is a necessary and sufficient condition in order that:

(α) *for n even all bounded solutions of* (15) *are oscillatory,*

(β) *for n odd all bounded solutions of* (15) *are oscillatory or tending monotonically to zero as $t\to\infty$ together with their first $n-1$ derivatives.*

3. Oscillatory and asymptotic behavor of all solutions

In this section we study the oscillatory and asymptotic behavior of all solutions of the differential equation (*), when it is τ_0-distorted strongly sublinear (Theorem 5) or τ_0-distorted strongly superlinear (Theorem 6). The proof of these theorems are based on the comparison principle, which is applied for the differential equations (*) and (**). For this we apply the fundamental conclusions for the more simple differential equation (**) and namely the following Propositions 1 and 2.

PROPOSITION 1. *Consider the differential equation* (**) *subject to the conditions* (i) *and*

(III) $\tau_{0j}(t) \leqq t$ *for every* $t \geqq t_0$ $(j=1, 2,..., m_0)$.

If the equation (**) *is* τ_0*-distorted strongly sublinear and*

$$(C_6) \qquad \int^{\infty} \left\{ \prod_{j=1}^{m_0} [\tau_{0j}(t)]^{(n-1)\alpha_j} \right\} g(t)dt = \infty,$$

then for n even all solutions of the equation (**) *are oscillatory, while for n odd all solutions of the equation* (**) *are either oscillatory or tending monotonically to zero as* $t \to \infty$ *together with their first* $n-1$ *derivatives.*

PROOF. Let y be a non-oscillatory solution of the differential equation (**) with $\lim_{t\to\infty} y(t) \neq 0$. Since both y and $-y$ are solutions of the equation (**), we can assume, without loss of generality, that $y(t)>0$ for every $t \geqq t_0$. Moreover, since $\lim_{t\to\infty} \tau_{0j}(t) = \infty$ $(j=1, 2,..., m_0)$, there exists a $t_1 \geqq t_0$ so that for every $t \geqq t_1$

$$(16) \qquad \tau_{0j}(t) \geqq \max\{t_0, 0\} \qquad (j = 1, 2,..., m_0).$$

From the equation (**), by (16), it follows that

$$y^{(n)}(t) \leqq 0 \qquad \text{for every} \quad t \geqq t_1$$

and consequently, by Lemma 2,

$$y^{(n-1)}(t) \geqq 0 \qquad \text{for every} \quad t \geqq t_1.$$

Now, we shall show that there exist $c>0$ and $t_2 \geqq t_1$ so that

$$(17) \qquad y(t) \geqq c[\tau_{0i}(t)]^{n-1} y^{(n-1)}(t) \qquad \text{for every} \quad t \geqq t_2 \ (j = 1, 2,..., m_0).$$

For this we consider the integer l, as it is defined by Lemma 2, for solution y and remark that for every $t \geqq t_2 = 2^{n-1-l} t_1$ the inequality

$$(18) \qquad y(2^{l-n+1}t) \geqq \frac{2^{(l-n+1)(n-1)}}{(n-1)(n-2)\cdots(n-l)} (t-t_1)^{n-1} y^{(n-1)}(t)$$

holds.

If y is non-decreasing, then for every $t \geqq t_2$

$$y(t) \geqq y(2^{l-n+1}t) \geqq c[\tau_{0j}(t)]^{n-1} y^{(n-1)}(t) \quad (j = 1, 2,..., m_0),$$

where

$$c = \frac{2^{(l-n+1)(n-1)}}{(n-1)(n-1)\cdots(n-l)}$$

If y is non-increasing, we remark that

$$\lim_{t\to\infty} \frac{y(t)}{y(2^{l-n+1}t)} = 1$$

and it is easy to see that

$$\inf_{t\geqq t_2} \frac{y(t)}{y(2^{l-n+1}t)} > 0$$

Consequently, by (18), for every $t \geqq t_2$ we obtain

$$y(t) = \frac{y(t)}{y(2^{l-n+1}t)} \cdot y(2^{l-n+1}t) \geqq ct^{n-1}y^{(n-1)}(t) \geqq c[\tau_{0_j}(t)]^{n-1}y^{(n-1)}(t)$$

$$(j = 1, 2,\ldots, m_0)$$

where

$$c = \frac{2^{(l-n+1)(n-1)}}{(n-1)(n-2)\cdots(n-l)} \inf_{t\geqq t_2} \frac{y(t)}{y(2^{l-n+1}t)} \,.$$

The equation (**), in view of (17), for every $t \geqq t_2$ yields

$$y^{(n)}(t) + c^\alpha g(t)\,[y^{(n-1)}(t)]^\alpha \prod_{j=1}^{m_0} [\tau_{0_j}(t)]^{(n-1)\alpha_j} \leqq 0\,. \tag{19}$$

Dividing (19) by $[y^{(n-1)}(t)]^\alpha$ and integrating from t_2 to t, we obtain

$$\int_t^{t_2} \frac{y^{(n)}(s)}{[y^{(n-1)}(s)]^\alpha} ds + c^\alpha \int_{t_2}^t g(s)\left\{\prod_{j=1}^{m_0} [\tau_{0_j}(s)]^{(n-1)\alpha_j}\right\} ds \leqq 0\,. \tag{20}$$

Hence, since $\alpha < 1$, we obtain

$$\begin{aligned}\int_{t_2}^t \left\{\prod_{j=1}^{m_0} [\tau_{0_j}(s)]^{(n-1)\alpha_j}\right\} g(s)ds &\leqq \frac{1}{c^\alpha}\int_t^{t_2} \frac{y^{(n)}(s)}{[y^{(n-1)}(s)]^\alpha} ds \\ &\leqq \frac{1}{c^\alpha}\int_{y^{(n-1)}(t)}^{y^{(n-1)}(t_2)} \frac{dz}{z^\alpha} \\ &\leqq \frac{1}{c^\alpha}\int_0^{y^{(n-1)}(t_2)} \frac{dz}{z^\alpha} < \infty\end{aligned}$$

which contradicts (C_6).

The conclusion of the proposition follows now immediately from Lemmas 2 and 3.

PROPOSITION 2. *Consider the differential equation* (**) *subject to the* *conditions* (i) *and* (III). *If the equation* (**) *is* τ_0*-distorted strongly superlinear and*

$$(C_7)\qquad \int^{\infty} \frac{\sum_{j=1}^{m_0} [\tau_{0_j}(t)]^{n-1}}{\prod_{j=1}^{m_0} [\tau_{0_j}(t)]^{(n-1)(\alpha-\alpha_j)}} g(t)dt = \infty$$

then for n even all solutions of the equation (**) *are oscillatory, while for n odd all solutions of the equation* (**) *are oscillatory or tending monotonically to zero as* $t\to\infty$ *together with their first* $n-1$ *derivatives.*

PROOF. The proof follows immediately as an application of the theorem in [7] in the case, where

$$\varphi(y_1, y_2,\dots, y_{m_0}) = \prod_{j=1}^{m_0} |y_j|^{\alpha_j} \prod_{j=1}^{2\lambda-1} \operatorname{sgn} y_j$$

and

$$\rho(y) = |y|^{\alpha-1}\operatorname{sgn} y.$$

THEOREM 5. *Consider the differential equation* (*) *subject to the conditions* (i), (ii) *and* (I). *If the equation* (*) *is* τ_0*-distorted strongly sublinear and for any* $\mu\neq 0$

$$(C_8)\qquad \int^{\infty}\left\{\prod_{j=1}^{m_0} [\tau_{0j}(t)]^{(n-1)\rho_j}\right\}F(t;\, \mu^2\cdot\boldsymbol{\tau}_0^{2(n-1)}(t),\, \mu^2\cdot\boldsymbol{\tau}_1^{2(n-2)}(t),\dots,\, \mu^2\cdot\mathbf{1})dt = \infty,$$

where

$$\boldsymbol{\tau}_i^{2(n-1-i)}(t) = (\tau_{i1}^{2(n-1-i)}(t),\, \tau_{i2}^{2(n-1-i)}(t),\dots,\, \tau_{im_i}^{2(n-1-i)}(t))$$
$$(i = 0, 1,\dots, n-1),$$

then for n even all solutions of the equation (*) *are oscillatory, while for n odd all solutions of the equation* (*) *are either oscillatory or tending monotonically to zero as* $t\to\infty$ *together with their first* $n-1$ *derivatives.*

PROOF. Let x be a non-oscillatory solution of the differential equation (*) with $\lim_{t\to\infty} x(t)\neq 0$. As in the proof of Theorem 1, without loss of generality, we assume that $x(t)>0$ for every $t\geqq t_0$ and by choosing $t_1\geqq t_0$ as in (1) we derive (2). By Lemma 2, for every $t\geqq t_1$ we obtain

$$x^{(n-1)}(t) \geqq 0$$

for any l. Hence, by Taylor's formula, for every $t\geqq t_1$ we have

$$x^{(k)}(t) \leqq x^{(k)}(t_1) + \frac{x^{(k+1)}(t_1)}{1!}(t-t_1) + \cdots + \frac{x^{(n-1)}(t_1)}{(n-1-k)!}(t-t_1)^{n-1-k}$$

where $0 \leqq k \leqq l$. Thus, for any k, $0 \leqq k \leqq l$, the function $\frac{x^{(k)}(t)}{t^{n-1-k}}$ is eventually bounded and consequently there exist constants μ_k $(k=0, 1,\dots, l)$ so that

(21) $\quad x^{(k)}[\tau_{ij}(t)] \leqq \mu_k[\tau_{ij}(t)]^{n-1-k} \quad (j = 1, 2,\dots, m_i;\ i = 0, 1,\dots, n-1)$

for every $t \geqq t_2$, where t_2 is chosen so that

$$\tau_{ij}(t) \geqq t_2.$$

Moreover, since the functions $|x^{(k)}|$ for $k=l+1, l+2,\dots, n-1$ are all bounded, obviously there exist $\mu_k>0$ $(k=l+1, l+2,\dots, n-1)$ such that

(22) $\quad |x^{(k)}[\tau_{ij}(t)]| \leqq \mu_k[\tau_{ij}(t)]^{n-1-k} \qquad (k = l+1, l+2,\dots, n-1)$

for every $t \geqq t_3$, where $t_3 \geqq t_2$ is chosen properly. For $\mu = \max_k \mu_k$ and for every $t \geqq t_4 = \max\{t_2, t_3\}$ from inequalities (21) and (22) we obtain

(23) $\quad |x^{(k)}[\tau_{ij}(t)]| \leqq \mu[\tau_{ij}(t)]^{n-1-k} \qquad (k = 0, 1,\dots, n-1).$

Consider now the exponents ε_j $(j=1, 2,\dots, m_0)$, which correspond to function F, by the definition of τ_0-distorted strongly sublinearity, and moreover arbitrary numbers η_j $(j=1, 2,\dots, m_0)$ with

$$0 \leqq \eta_j \leqq \varepsilon_j.$$

Then for any $(t; y_0, y_1,\dots, y_{n-1})$, $(t; z_0, z_1,\dots, z_{n-1})$ with $t \geqq t_0$ and $0 \leqq y \leqq z$ we obtain

(24) $\quad \left(\prod_{j=1}^{m_0} y_{0_j}^{\eta_j}\right) F(t; y_0, y_1,\dots, y_{n-1}) \geqq \left(\prod_{j=1}^{m_0} z_{0_j}^{\eta_j}\right) F(t; z_0, z_1,\dots, z_{n-1}).$

In fact, we first remark that

$$\prod_{j=1}^{m_0} y_{0_j}^{\eta_j-\varepsilon_j} \geqq \prod_{j=1}^{m_0} z_{0_j}^{\eta_j-\varepsilon_j} \geqq 0$$

and then, by sublinearity of the equation $(*)$, we have

$$\left(\prod_{j=1}^{m_0} y_{0_j}^{\varepsilon_j}\right) F(t; y_0, y_1,\dots, y_{n-1}) \geqq \left(\prod_{j=1}^{m_0} z_{0_j}^{\varepsilon_j}\right) F(t; z_0, z_1,\dots, z_{n-1}) \geqq 0$$

and consequently multiplying these inequalities we get (24).

Therefore we choose the η_j $(j=1, 2,\dots, m_0)$ so that

$$0<\eta_j<\rho_j \qquad (j = 1, 2,\dots, m_0).$$

Hence, if we put

$$\alpha_j = \rho_j - \eta_j \qquad (j = 1, 2, \ldots, m_0)$$

then we have $\alpha_j > 0$ and

$$\alpha = \sum_{j=1}^{m_0} \alpha_j = \sum_{j=1}^{m_0} \rho_j - \sum_{j=1}^{m_0} \eta_j < \sum_{j=1}^{m_0} \rho_j = 1$$

i.e. the differential equation (**) is τ_0-distorted strongly sublinear.

Consider now the equations (*) and (**) in the place of the equations (E) and (E_g) of Lemma 1 respectively.

Let $\mathscr{E}$ be the class of all continuous functions x, which are defined on an interval of the form $[t_x, \infty)$ and P the propositional function:

$$P(x): \quad x \text{ is oscillatory or } \lim_{t\to\infty} x(t) = 0$$

Furthermore, let $\mathscr{G}$ be the class of all continuous and non-negative functions g defined on an interval of the form $[t_g, \infty)$, which satisfy the condition (C_6). By Proposition 1, under the condition (C_6), all solutions of the equation (*) are oscillatory or tending to zero as $t\to\infty$, i.e.

$$(\forall g \in \mathscr{G})(\forall y \in \mathscr{S}_g)P(y).$$

We remark that for any solution x of equation (*) for which $\sim P(x)$ is satisfied, i.e. x is non-oscillatory and $\lim_{t\to\infty} x(t) \neq 0$, the function g_x associated to x belongs to the function class $\mathscr{G}$. In fact, using the strongly sublinearity of the equation (*) and taking into account (23) and (24), we conclude that

$$\left\{\prod_{j=1}^{m_0} [\tau_{0j}(t)]^{(n-1)\alpha_j}\right\} g_x(t) = \left\{\prod_{j=1}^{m_0} [\tau_{0j}(t)]^{(n-1)\alpha_j}\right\} \cdot$$

$$\cdot \frac{\left\{\prod_{j=1}^{m_0} (x[\tau_{0_j}(t)])^{\rho_j}\right\} F(t;\, [x]^2 < \tau_0(t)>,\, [x']^2 < \tau_1(t)>, \ldots, [x^{(n-1)}]^2 < \tau_{n-1}(t)>)}{\prod_{j=1}^{m_0} (x[\tau_{0_j}(t)])^{\alpha_j}} \geqq$$

$$\geqq \left\{\prod_{j=1}^{m_0} [\tau_{0j}(t)]^{(n-1)\alpha_j}\right\} \left\{\prod_{j=1}^{m_0} (x[\tau_{0j}(t)])^{\eta_j}\right\} F(t;\, [x]^2 < \tau_0(t)>,\, [x']^2 < \tau_1(t)>, \ldots$$

$$\ldots, [x^{(n-1)}]^2 < \tau_{n-1}(t)>)$$

$$\geqq \left\{\prod_{j=1}^{m_0} [\tau_{0j}(t)]^{(n-1)\alpha_j}\right\} \left\{\mu^\eta \prod_{j=1}^{m_0} [\tau_{0j}(t)]^{(n-1)\eta_j}\right\} F(t;\, \mu^2 \cdot \boldsymbol{\tau}_0^{2(n-1)}(t),\, \mu^2 \cdot \boldsymbol{\tau}_1^{2(n-2)}(t),$$

$$\ldots, \mu^2 \cdot \mathbf{1})$$

$$= \mu^{1-\alpha}\left\{\prod_{j=1}^{m_0} [\tau_{0j}(t)]^{(n-1)\rho_j}\right\} F(t;\, \mu^2 \cdot \boldsymbol{\tau}_0^{2(n-1)}(t),\, \mu^2 \cdot \boldsymbol{\tau}_1^{2(n-2)}(t), \ldots,\, \mu^2 \cdot \mathbf{1})$$

where $\eta = \sum_{j=1}^{m_0} \eta_j$, and consequently, by ($C_6$),

$$\int^{\infty} \left\{\prod_{j=1}^{m_0} [\tau_{0j}(t)]^{(n-1)\alpha_j}\right\} g_x(t)dt = \infty$$

i.e.

$$(\forall x \in \mathscr{E}) \sim P(x) \Longrightarrow g_x \in \mathscr{G}.$$

Now, applying Lemma 1, we obtain

$$(\forall x \in \mathscr{S}) P(x).$$

This proves the theorem.

THEOREM 6. *Consider the differential equation* (*) *subject to the conditions* (i), (ii), (I) *and*

(IV) *The functions* τ_{0j} $(j=1, 2,\ldots, m_0)$ *are differentiable on* $[t_0, \infty)$ *and*

$$\tau_{0_j}'(t) \geqq 0 \qquad \textit{for every} \quad t \geqq t_0.$$

If equation (*) *is* τ_0*-distorted strongly superlinear and for each* $\mu \neq 0$

$$(C_9) \qquad \int^{\infty} \frac{\sum_{j=1}^{m_0} [\tau_{0_j}(t)]^{n-1}}{\prod_{j=1}^{m_0} [\tau_{0_j}(t)]^{(n-1)(\alpha-\alpha_j)}} F(t;\, \mu^2 \cdot \mathbf{1}, \mathbf{0}, \ldots, \mathbf{0})dt = \infty$$

where $\alpha_j = \rho_j + \varepsilon_j$ $(j=1, 2,\ldots, m_0)$, $\alpha = \sum_{j=1}^{m_0} \alpha_j$ *and* ε_j $(j=1, 2,\ldots, m_0)$ *are the exponents which correspond to the function F, by the definition of* τ_0*-distorted strongly superlinearity, then for n even all solutions of the equation* (*) *are oscillatory, while for n odd all solutions of the equation* (*) *are either oscillatory or tending monotonically to zero as* $t \to \infty$ *together with their first* $n-1$ *derivatives.*

PROOF. Let x be a non-oscillatory solution of the equation (*) with $\lim_{t\to\infty} x(t) \neq 0$. As in the proof of Theorem 1, without loss of generality, we suppose that $x(t) > 0$ for every $t \geqq t_0$ and by choosing a $t_1 \geqq t_0$ as in (1) we derive (2). Hence, by Lemma 2, all derivatives of arbitrary order are of constant sign on the interval $[t_1, \infty)$. Therefore, the function x is monotonous and consequently there exists $\mu_0 > 0$ so that

$$|x(t)| \geqq \mu_0 \qquad \textit{for every} \quad t \geqq t_1.$$

Moreover, since $\lim_{t\to\infty} \tau_{0j}(t) = \infty$ $(j=1, 2,\dots, m_0)$, there exists a $t_2 \geqq t_1$ so that for every $t \geqq t_2$

$$\tau_{0j}(t) \geqq t_1 \qquad (j=1, 2,\dots, m_0)$$

and consequently

$$(25) \qquad |x[\tau_{0j}(t)]| \geqq \mu_0 \qquad \text{for every} \quad t \geqq t_2 \qquad (j = 1, 2,\dots, m_0).$$

Consider now in the place of the equations (E) and (E_g) of Lemma 1 respectively the equations $(*)$ and $(**)$, where $\alpha_j = \rho_j + \varepsilon_j$ $(j=1, 2,\dots, m_0)$ and $\sum_{j=1}^{m_0} \alpha_j = \alpha > 1$, since $\sum_{j=1}^{m_0} \varepsilon_j > 0$.

We therefore define the class $\mathscr{E}$ and the propositional function P exactly as in the proof of Theorem 5, while as $\mathscr{G}$ we define the class of all continuous and non-negative functions g defined on an interval of the form $[t_g, \infty)$, which satisfy the condition (C_7). By Proposition 2, under the condition (C_7), all solutions of the equation $(**)$ are oscillatory or tending to zero as $t \to \infty$, i.e.

$$(\forall g \in \mathscr{G})(\forall y \in \mathscr{S}_g)P(y).$$

If x is a solution of the equation $(*)$ for which $\sim P(x)$ is satisfied, i.e. x is non-oscillatory and $\lim_{t\to\infty} x(t) \neq 0$, then the function g_x associated to x belongs to the function class $\mathscr{G}$. In fact, since equation $(*)$ is τ_0-distorted strongly superlinear, by (25) and Lemma 3, it follows that for every $t \geqq t_2$

$$g_x(t) = \frac{\left\{\prod_{j=1}^{m_0} (x[\tau_{0_j}(t)])^{\rho_j}\right\} F(t; [x]^2 < \tau_0(t) >,\dots, [x^{(n-1)}]^2 < \tau_{n-1}(t) >)}{\prod_{j=1}^{m_0} (x[\tau_{0_j}(t)])^{\alpha_j}}$$

$$= \left\{\prod_{j=1}^{m_0} (x[\tau_{0_j}(t)])^{-\varepsilon_j}\right\} F(t; [x]^2 < \tau_0(t) >,\dots, [x^{(n-1)}]^2 < \tau_{n-1}(t) >)$$

$$\geqq \mu^{1-\alpha} F(t; \mu^2 \cdot \mathbf{1}, \mathbf{0},\dots, \mathbf{0})$$

and consequently, by (C_7),

$$\int^{\infty} \frac{\sum_{j=1}^{m_0} [\tau_{0_j}(t)]^{n-1}}{\prod_{j=1}^{m_0} [\tau_{0_j}(t)]^{(n-1)(\alpha-\alpha_j)}} \, g_x(t)dt = \infty$$

i.e.

$$(\forall x \in \mathscr{S}) \sim P(x) \Longrightarrow g_x \in \mathscr{G}.$$

Applying Lemma 1, we obtain

$$(\forall x \in \mathscr{S}) P(x)$$

which proves the theorem.

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