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Remarks on the Multiplicative Products of Distributions

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Many attempts have been made for defining the multiplication between distributions. Y. Hirata and H. Ogata [3] have defined a product of distributions and J. Mikusiński [9] has also defined the same product in a different fashion. In [5], we have considered the multiplication invariant under diffeomorphism which covers the multiplication in the above sense. If S, $T\in\mathscr{D}'(R^N)$ and if $\alpha S*\check{T}$ has the value $(\alpha S*\check{T})(0)$ at 0 in the sense of S. Łojasiewicz [8] for any $\alpha\in\mathscr{D}(R^N)$, then there exists a unique distribution W such that $<W,\alpha>=(\alpha S*\check{T})(0)$. In [10], R. Shiraishi has defined a restricted δ-sequence $\{\rho_n\}$ as a sequence of non-negative functions $\rho_n\in\mathscr{D}(R^N)$ such that

(i) $\operatorname{supp}\rho_n$ converges to $\{0\}$ as $n\to\infty$

(ii) $\int\rho_n(x)dx$ converges to 1 as $n\to\infty$;

(iii) $\int|x|^{|p|}|D^p\rho_n(x)|dx\leqq M_p$ (M_p being independent of n),

where the integral is extended over the whole N-dimensional space, and he has shown that the existence of the product $W=S\circ T$ of S and T is equivalent to each of the following conditions:

(1) The distributional limit $\lim_{n\to\infty}(S*\rho_n)(T*\tilde{\rho}_n)$ exists for every restricted δ-sequences $\{\rho_n\}$ and $\{\tilde{\rho}_n\}$:

(2) The distributional limit $\lim_{n\to\infty}(S*\rho_n)T$ exists for every restricted δ-sequence $\{\rho_n\}$.

(3) The distributional limit $\lim_{n\to\infty}S(T*\rho_n)$ exists for every restricted δ-sequence $\{\rho_n\}$

And if one of these conditions is satisfied, the limit equals W.

On the other hand, we may define the multiplicative product $S\vartriangle T$ as the distributional limit $\lim_{n\to\infty}(S*\rho_n)(T*\rho_n)$, if it exists for every restricted δ-sequence $\{\rho_n\}$ ([10, p. 97]). The purpose of this paper is to investigate this multiplication Δ by making a comparison with the multiplication $\circ$.

By the definition stated above we see that if $S\circ T$ exists, then $S\vartriangle T$ exists and is equal to $S\circ T$. However the converse does not hold ([10, p. 97]).

LEMMA 1. *$\delta \triangle \mathrm{Pf}.\frac{1}{x}=-\frac{1}{2}\delta'$ but $\delta\circ \mathrm{Pf}.\frac{1}{x}$ does not exist.*

PROOF. Since the distribution $\mathrm{Pf}.\frac{1}{x}$ has not a value at 0, the product $\delta\circ\mathrm{Pf}.\frac{1}{x}$ does not exist. For any restricted δ-sequence $\{\rho_n\}$ and any $\varphi\in\mathscr{D}(R)$ we can write

$$<\rho_n\left(\frac{1}{x}*\rho\right)\ \phi> \;=\; <\frac{1}{x}, \check{\rho}_n*\phi\rho_n>$$

If we write $\phi(x)=\phi(0)+\phi'(0)x+x^2\psi(x)$, then $\psi(0)=\lim_{x\to 0}\psi(x)=2^{-1}\phi''(0)$, $\lim_{x\to 0}x^{-1}(\psi(x)-\psi(0))=(3!)^{-1}\phi'''(0)$ and so on. Since $\rho_n*\rho_n$ is an even function, $<\frac{1}{x}, \rho_n*\rho_n>$ vanishes. Put $\alpha_n=\rho_n*x\rho_n$. Then $\alpha_n=\rho_n*(-x)\check{\rho}_n=(-x)(\rho_n*\check{\rho}_n)$ $+(x\rho_n)*\check{\rho}_n, \alpha_n-\check{\alpha}_n=x(\rho_n*\check{\rho}_n)$ and therefore

$$<\frac{1}{x}, \alpha_n> \;=\int_0^\infty(\rho_n*\check{\rho}_n)\,dx=\frac{1}{2}\int_{-\infty}^\infty(\rho_n*\check{\rho}_n)\,dx=\frac{1}{2}.$$

On the other hand, if we put $\beta_n=\check{\rho}_n*x^2\psi\rho_n$, then

$$\beta_n''=\check{\rho}_n*x^2\psi\rho_n''+2\check{\rho}_n*x(2\psi+x\psi')\rho_n'+\check{\rho}_n*(x^2\psi)''\rho_n.$$

By the property (iii) of $\{\rho_n\}$ we see that $\beta_n'(x)=-\int_x^\infty\beta_n''(x)dx$ is bounded and therefore $x^{-1}(\beta_n-\check{\beta}_n)=x^{-1}(\beta_n(x)-\beta_n(0))+x^{-1}(\beta_n(0)-\check{\beta}_n(x))$ is bounded. Since $\mathrm{supp}\{x^{-1}(\beta_n-\check{\beta}_n)\}$ tends to $\{0\}$ as $n\to\infty$, we see that $\lim_{n\to\infty}\int_{-\infty}^\infty x^{-1}(\beta_n-\check{\beta}_n)dx=0$. Thus the product $\delta\circ\mathrm{Pf}.\frac{1}{x}$ exists and is equal to $-2^{-1}\delta'$.

PROPOSITION 1. *For any non-negative integer k, the product $\delta^{(k)}\triangle\mathrm{Pf}.\frac{1}{x^{k+1}}$ exists and*

$$\delta^{(k)}\triangle\mathrm{Pf}.\frac{1}{x^{k+1}}=\frac{(-1)^{k+1}k!}{2\,(2k+1)!}\delta^{(2k+1)}$$

but the product $\delta^{(k)}\circ\mathrm{Pf}.\frac{1}{x^{k+1}}$ does not exist.

PROOF. Given $S, T\in\mathscr{D}'(R)$, the existence of the product $S\circ T'$ implies the existence of $S'\circ T$ and $S\circ T$ ([5, p. 162]). Thus it follows from Lemma 1 that the product $\delta^{(k)}\circ\mathrm{Pf}.\frac{1}{x^{k+1}}$ does not exist.

For any restricted δ-sequence $\{\rho_n\}$ and any $\varphi\in\mathscr{D}(R)$, we have

$$<(\delta^{(k)}*\rho_n)\left(\frac{1}{x^{k+1}}*\rho_n\right),\ \phi> = \frac{(-1)^k}{k!}<\frac{1}{x},\ (\rho_n^{(k)})^{\vee}*\phi\rho_n^{(k)}>,$$

where we write

$$\phi(x) = \phi(0)+\phi'(0)x+\cdots+\frac{1}{(2k+1)!}\phi^{(2k+1)}(0)x^{2k+1}+x^{2k+2}\psi(x).$$

$(\rho_n^{(k)})^{\vee}*\rho_n^{(k)}$ is an even function and

$$(\rho_n^{(k)})^{\vee}*x\rho_n^{(k)} = x((\rho_n^{(k)})^{\vee}*\rho_n^{(k)})-x(\rho_n^{(k)})^{\vee}*\rho_n^{(k)}$$

and therefore we have

$$<\frac{1}{x},\ (\rho_n^{(k)})^{\vee}*\rho_n^{(k)}> = 0$$

and

$$<\frac{1}{x},\ (\rho_n^{(k)})^{\vee}*x\rho_n^{(k)}> = \int_0^{\infty}(\rho_n^{(k)})^{\vee}*\rho_n^{(k)}dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}(\rho_n^{(k)})^{\vee}*\rho_n^{(k)}\,dx = \begin{cases} \frac{1}{2} & (k=0), \\ 0 & (k\geqq 1). \end{cases}$$

Let $k\geqq 1$ and put $\beta_{l,n}(x)=(\rho_n^{(k)})^{\vee}*x^l\rho_n^{(k)}$ for $0\leqq l\leqq 2k+1$. For $l=2p+1$, $p=1, 2,\ldots, k$, we can write

$$\beta_{l,n}(x) = x\{\sum_{i=0}^{2p}(-x)^i(\rho_n^{(k)})^{\vee}*x^{2p-i}\rho_n^{(k)}\}-x^{2p+1}(\rho_n^{(k)})^{\vee}*\rho_n^{(k)}$$

and hence

$$\beta_{l,n}(x)-\beta_{l,n}(-x) = x\{x\sum_{i=0}^{2p}(-x)^i(\rho_n^{(k)})^{\vee}*x^{2p-i}\rho_n^{(k)}\}$$

If $p<k$, then $i+(2p-i)=2p<2k$ and therefore either i or $2p-i$ is less than k. Moreover the function $\sum_{i=0}^{2p}(-x)^i(\rho_n^{(k)})^{\vee}*x^{2p-i}\rho_n^{(k)}$ is an even function with compact support. Thus we have

$$<\frac{1}{x},\ \beta_{l,n}> = \frac{1}{2}\sum_{i=0}^{2p}\int_{-\infty}^{\infty}\{(-x)^i(\rho_n^{(k)})^{\vee}*x^{2p-i}\rho_n^{(k)}\}dx=0.$$

In the case where $p=k$, we have

$$<\frac{1}{x},\ \beta_{2k+1,n}> = \frac{1}{2}\sum_{i=0}^{2k}\int_{-\infty}^{\infty}\{(-x)^i(\rho_n^{(k)})^{\vee}*x^{2k-i}\rho_n^{(k)}\}dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \{(-x)^k (\rho_n^{(k)})^{\vee} * x^k \rho_n^{(k)}\} dx$$

$$= \frac{1}{2}\left\{\int_{-\infty}^{\infty} (x^k \rho_n^{(k)}) dx\right\}^2 = \frac{1}{2}(k!)^2.$$

For $l=2p, p=1, 2,\dots,$ fe, we can write

$$\beta_{l,n}(x) - \beta_{l,n}(-x) = x\{\sum_{i=0}^{2p-1}(-x)^i (\rho_n^{(k)})^{\vee} * x^{2p-i-1}\rho_n^{(k)}\},$$

where $\sum_{i=0}^{2P-1}(-x)^i(\rho_n^{(k)})^{\vee} * x^{2p-i-1}\rho_n^{(k)}$ is an even function and either i or $2p-i-1$ is less than fc. Thus we have

$$\int_0^{\infty} \frac{1}{x}(\beta_n(x) - \beta_n(-x)) dx = 0.$$

For $\beta_n(x) = (\rho_n^{(k)})^{\vee} * x^{2k+2}\psi\rho_n^{(k)}$ we have

$$\beta_n'' = (-1)^k \check{\rho}_n * (x^{2k+2}\psi\rho_n^{(k)})^{(k+2)}$$

$$= (-1)^k \sum_{j=0}^{k+2}\binom{k+2}{j}(\check{\rho}_n * (x^{2k+2}\psi)^{(j)}\rho_n^{(2k+2-j)}).$$

By the property (iii) of $\{\rho_n\}$ we see that $\beta_n' = -\int_x^{\infty}\beta_n''(x)dx$ is bounded and so is $x^{-1}(\beta_n(x) - \beta_n(-x))$. Moreover supp $\{x^{-1}(\beta_n(x) - \beta_n(-x))\}$ tends to $\{0\}$ as $n\to\infty$. Thus we have $\lim_{n\to\infty} x^{-1}(\beta_n(x) - \beta_n(-x)) = 0$.

Consequently the product $\delta^{(k)} \Delta \mathrm{Pf}.\frac{1}{x^{k+1}}$ exists and is equal to $\frac{(-1)^{k+1}k!}{2\,(2k+1)!}\cdot\delta^{(2k+1)}$.

In [1], B. Fisher has introduced the product of two distributions on the open interval (a, b), $-\infty \leqq a < b \leqq \infty$, as the distributional limit of $(S*\delta_n)(T*\delta_n)$, where $\delta_n(x) = n\rho(nx)$ and p is a fixed C^{∞} function having the following properties: (1) $\rho(x)=0$ for $|x|\geqq 1$, (2) $\rho(x)\geqq 0$, (3) $\rho(x)=\rho(-x)$, (4) $\int_{-1}^{1}\rho(x)dx=1$ and (5) $\rho^{(r)}(x)$ has only r changes of sign for $r=1, 2,\dots$. And he has shown that the product of the distributions $\delta^{(k)}$ and Pf. $\frac{1}{x^{k+1}}$ in his sense exists and equals $\frac{(-1)^{k+1}k!}{2\,(2k+1)!}\delta^{(2k+1)}$ for any non-negative integer fc.

In our previous paper [6] with collaboration of S. Hatano, we have shown that the product of the distributions $\delta^{(k)}$ and Pf. $\frac{1}{x^{k+1}}$ in the sense of H. G. Tillmann exists and the same result as above holds true.

In the proof of Proposition 2 below we shall need the following lemma, which we have shown in [4, p. 71].

LEMMA 2. *Let E and F be spaces of type* (F). *Let G be a locally convex space. If a family of separately continuous bilinear maps* u_α, $\alpha \in A$, *of* $E \times F$ *into G is bounded at each point of* $E \times F$, *then* $\{u_\alpha\}_{\alpha \in A}$ *is equicontinuous.*

PROPOSITION 2. *Let* $S, T \in \mathscr{D}'(R^N)$. *If the multiplicative product* $(S*(\alpha T)^{\vee})\vartriangle\delta$ *exists for any* $\alpha \in \mathscr{D}(R^N)$, *then the product* $S \vartriangle T$ *exists.*

PROOF. Let K be any compact subset of R^N and let φ e $\mathscr{D}_K$. There exist two compact subsets K' and K'' of R^N such that $K \subset K''$ and $K'' \subset \mathring{K}'$. Let $\alpha_1 \in \mathscr{D}(R^N)$ such that $\alpha_1 = 1$ on K' and

$$\alpha_1 S*(\psi T)^{\vee} = S*(\psi T)^{\vee}$$

in a 0-neighbourhood for any ψ e $\mathscr{D}_{K''}$. Then we have

$$(\alpha_1 S*(\psi T)^{\vee})\vartriangle\delta = (S*(\psi T)^{\vee})\vartriangle\delta, \qquad \psi \in \mathscr{D}_{K''}.$$

For $\alpha_2 \in \mathscr{D}_{K''}$ sucn that $\alpha_2 = 1$ in a small neighbourhood of K we have

$$\lim_{n\to\infty} <(S*\rho_n)(T*\rho_n), \phi> = \lim_{n\to\infty} <(\alpha_1 S*\rho_n)(\alpha_2 T*\rho_n), \phi>,$$

where $\{\rho_n\}$ is any restricted δ-sequence.

To estimate the right hand side of the equation we may assume that φ is a periodic function with period $2l$ for each coordinate where / is taken large enough. Thus we can write

$$\phi = \sum c_m e^{i\frac{\pi}{l}<m,x>},$$

where $\Sigma|c_m|(1+|m|)^k < \infty$ for any positive integer fc. Writing $e(m) = e^{i\frac{\pi}{l}<m,x>}$, we have

$$\begin{aligned} <(\alpha_1 S*\rho_n)(\alpha_2 T*\rho_n), \varphi> &= \Sigma c_m <(\alpha_1 S*\rho_n)(\alpha_2 T*\rho_n), e(m)> \\ &= \Sigma c_m <\alpha_1 S*\rho_n, e(m)\alpha_2 T*\rho_n e(m)> \\ &= \Sigma c_m <\alpha_1 S*(e(m)\alpha_2 T)^{\vee}, \check{\rho}_n*\rho_n e(m)>. \end{aligned}$$

Since the product $(\alpha_1 S*(\psi T)^{\vee})\vartriangle\delta$ exists for any $\psi \in \mathscr{D}_{K''}$, we can write for any $\chi \in \mathscr{B}$

$$\begin{aligned} <(\alpha_1 S*(\psi T)^{\vee})\vartriangle\delta, \chi> &= \lim_{n\to\infty} <(\alpha_1 S*(\psi T)^{\vee}*\rho_n)\rho_n, \chi> \\ &= \lim_{n\to\infty} <\alpha_1 S*(\psi T)^{\vee}, \check{\rho}_n*\rho_n\chi>. \end{aligned}$$

Here the map

$$(\psi, \chi) \longrightarrow <\alpha_1 S*(\psi T)^{\vee}, \check{\rho}_n*\rho_n\chi>$$

of $\mathscr{D}_{K''} \times \mathscr{B}$ into the complex number field is separately continuous for any ρ_n and the family of maps is bounded at each point of $\mathscr{D}_{K''} \times \mathscr{B}$. By virtue of Lemma 2 this family of maps is equicontinuous. Thus there exist a constant M and a positive constant k such that

$$|<\alpha_1 S*(\psi T)^{\vee}, \check{\rho}_n*\rho_n\chi>| \leqq M \sup_{|p|\leq k} |D^p\psi| \sup_{|p|\leq k} |D^p\chi|$$

and consequently there exists a constant M' such that

$$|<\alpha_1 S*(e(m)\alpha_2 T)^{\vee},\ \check{\rho}_n*\rho_n e(m)>| \leqq M'(1+|m|)^{2k}.$$

From $\Sigma|c_m|(1+|m|)^{2k}<\infty$ it follows that $\Sigma c_m<\alpha_1 S*(e(m)\alpha_2 T)^{\vee},\ \check{\rho}_n*\rho_n e(m)>$ is normally convergent. Furthermore each term has a limit as $n\to\infty$. Thus the multiplicative product $S\vartriangle T$ exists.

PROPOSITION 3. *Let $S, T\in\mathscr{D}'(R^N)$. If the multiplicative product $S\vartriangle\beta T$ exists for any $\beta\in\mathscr{E}(R^N)$, then the product $(S*(\alpha T)^{\vee})\vartriangle\delta$ exists for any $\alpha\in\mathscr{D}(R^N)$.*

PROOF. Let K be any compact subset of R^N and take a compact subset K_1 with $K\subset\mathring{K}_1$. If we take $\beta\in\mathscr{D}(R^N)$ such that $\beta=1$ on K_1, then $\beta S*(\alpha T)^{\vee}=S*(\alpha T)^{\vee}$ in a 0-neighbourhood for any $\alpha\in\mathscr{D}_K$. From the fact that

$$(1-\beta)S\vartriangle(\alpha T)^{\vee}=\lim_{n\to\infty}((1-\beta)S*\rho_n)((\alpha T)^{\vee}*\rho_n)=0,$$

it follows that the multiplicative product $\beta S\vartriangle\alpha T$ exists for any $\alpha\in\mathscr{D}_K$ and equals $S\vartriangle\alpha T$. Thus we have for any $\varphi\in\mathscr{D}(R^N)$

$$<(S*(\alpha T)^{\vee})\vartriangle\delta,\ \varphi> = \lim_{n\to\infty} <(\beta S*(\alpha T)^{\vee}*\rho_n)\rho_n,\varphi>$$

if the right hand side exists. In the same way as in the proof of Proposition 2 we can write

$$\varphi = \Sigma c_m e^{i\frac{\pi}{l}<x,m>} = \Sigma c_m e(m),$$

where $\Sigma|c_m|(1+|m|)^k<\infty$ for any positive integer k and l is taken sufficiently large. Then we have

$$\begin{aligned}<(\beta S*(\alpha T)^{\vee}*\rho_n)\rho_n,\ \phi> &= \Sigma c_m<\beta S*(\alpha T)^{\vee},\check{\rho}_n*\rho_n e(m)>\\ &= \Sigma c_m<\beta S,\ (\alpha T)*\check{\rho}_n*\rho_n e(m)>\\ &= \Sigma c_m<(\beta S*\rho_n)(e(-m)\alpha T*\rho_n),\ e(m)>.\end{aligned}$$

By the existence of the product $\beta S\vartriangle\chi T$ for any $\chi\in\mathscr{D}_K$ there exist a constant M and a positive integer k such that

$$|<(\beta S*\rho_n)(\chi T*\rho_n)\psi>| \leqq M \sup_{|p|\leq k} |D^p\chi| \sup_{|p|\leq k} |D^p\psi|$$

for any $\psi \in \mathscr{B}$ and therefore we have the inequality

$$|<(\beta S*\rho_n)(e(-m)\alpha T*\rho_n), e(m)>| \leqq M_1(1+|m|)^{2k}$$

with a constant M_1. Thus the sequence $\Sigma c_m<(\beta S*\rho_n)(e(-m)\alpha T*\rho_n), e(m)>$ is normally convergent and we have

$$\lim_{n\to\infty} <(\beta S*\rho_n)(e(-m)\alpha T*\rho_n), e(m)> = <\beta S*e(-m)\alpha T, e(m)>.$$

Consequently we see that the limit of $<(\beta S*(\alpha T)^{\vee}*\rho_n)\rho_n\phi>$ exists as $n\to\infty$, which means that the product $(\beta S*(\alpha T)^{\vee})\Delta\delta = (S*(\alpha T)^{\vee})\Delta\delta$ exists.

Now recall the definition of the multiplicative product ST of $S\in\mathscr{D}'(R^N)$ and $T\in\mathscr{D}'(R^N)$in the sense of Y. Hirata-H. Ogata and J. Mikusiński. The product ST is defined as one of the limits of the sequences in the following equivalent conditions ([11]):

(1) The distributional limit $\lim_{n\to\infty}(S*\rho_n)(T*\tilde{\rho}_n)$ exists for every δ-sequences $\{\rho_n\}$ and $\{\tilde{\rho}_n\}$:

(2) The distributional limit $\lim_{n\to\infty}(S*\rho_n)T$exists for every δ-sequence $\{\rho_n\}$:

(3) The distributional limit $\lim_{n\to\infty} S(T*\rho_n)$ exists for every δ-sequence $\{\rho_n\}$.

Here a δ-sequence $\{\rho_n\}$ is a sequence of non-negative functions $\rho_n\in\mathscr{D}(R^N)$ with the following properties:

(i) $\operatorname{supp}\rho_n$ converges to $\{0\}$ as $n\to\infty$;

(ii) $\int\rho_n(x)dx=1$, the integral being extended to the whole N-dimensional space.

In [11, p. 229] we showed that ST exists if and only if, for any $\alpha\in\mathscr{D}(R^N)$, there exists a 0-neighbourhood in which $\alpha S*\check{T}$ is a bounded function continuous at 0 and that $<ST, \alpha> = (\alpha S*\check{T})(0)$ in this case.

We may define the multiplicative product $S\cdot T$ as the distributional limit $\lim_{n\to\infty}(S*\rho_n)(T*\rho_n)$, if it exists for every δ-sequence $\{\rho_n\}$. Since the property (iii) of a restricted δ-sequence $\{\rho_n\}$ does not play any role in the proofs of the above two propositions, we have the analogues of Propositions 2 and 3 for such multiplication.

PROPOSITION 4. *Let $S, T\in\mathscr{D}'(R^N)$. If the product $(S*(\alpha T)^{\vee})\ \delta$ exists for any $\alpha\in\mathscr{D}(R^N)$, then the product $S\ \ T$ exists.*

PROPOSITION 5. *Let 5, $T\in\mathscr{D}'(R^N)$. If the product $S\ \beta T$ exists for any β e $\mathscr{E}(R^N)$, then the product $(S*(\alpha T)^{\vee})\ \delta$ exists for any $\alpha\in\mathscr{D}(R^N)$.*

REMARK. Let $\{\rho_n\}$ be any fixed δ-sequence. We may define another multiplicative product of $S \in \mathscr{D}'(R^N)$ and $T \in \mathscr{D}'(R^N)$ as the distributional limit $\lim_{n\to\infty} (S*\rho_n)(T*\rho_n)$, if it exists. For such multiplication we have also the analogues of Propositions 4 and 5.

We have shown in [5, p. 162] that if the products $S\circ\frac{\partial T}{\partial x_j}$ exist for $j=1, 2,\dots, N$, then the products $S\circ T$ and $\frac{\partial S}{\partial x_j}\circ T$ exist for $j=1, 2,\dots, N$ and $\frac{\partial}{\partial x_j}(S\circ T) =\frac{\partial S}{\partial x_j}\circ T+ S\circ\frac{\partial T}{\partial x_j}$ holds. The same property holds also true of the multiplicative product ST. But the statement is not true in general for the multiplication Δ. Let $N=1$. The product $\delta'\circ\text{Pf.}\frac{1}{x^2}$ exists but the product $\delta'\Delta\text{Pf.}\frac{1}{x}$ does not exist. In fact, let $\rho\in\mathscr{D}(R)$ such that $\rho\geqq 0$ and $\int\rho(x)dx=1$, and put $\rho_n(x) =n\rho(nx)$. Then $\{\rho_n\}$ is a restricted δ-sequence and we have

$$<\left(\frac{1}{x}*\rho_n\right)\rho'_n, \varphi> = <\frac{1}{x}, \rho'_n\phi*\check{\rho}_n>$$

for any $\varphi \in \mathscr{D}(R)$. If we take $\varphi \in \mathscr{D}(R)$ such that $\varphi=1$ in a 0-neighbourhood, then for a sufficiently large n we have

$$<\frac{1}{x}, \rho'_n*\check{\rho}_n> = n^2<\frac{1}{x}, \rho'*\check{\rho}>$$

$$= 2n^2\int_0^\infty x^{-2}(\rho*\check{\rho}(x)-\rho*\check{\rho}(0))dx,$$

where $\rho*\check{\rho}\geqq 0$ and $\rho*\check{\rho}(0)=\int_{-\infty}^{\infty}\rho^2(x)dx$. In the relations

$$(\rho*\check{\rho})^2 = \left(\int\rho(x-t)\rho(-t)dt\right)^2$$

$$\leqq \left(\int\rho^2(x-t)dt\right)\left(\int\rho^2(-t)dt\right) = (\rho*\check{\rho}(0))^2,$$

the equality does not hold and therefore

$$\int_0^\infty x^{-2}(\rho*\check{\rho}(x)-\rho*\check{\rho}(0))dx < 0.$$

Thus $<\left(\frac{1}{x}*\rho_n\right)\rho'_n, \Phi>$ does not converge. This means the product $\delta'\Delta\text{Pf.}\frac{1}{x}$ does not exist.

Moreover the product $\delta''\Delta\text{Pf.}\frac{1}{x}$ does not exist. In fact, for any restricted δ-sequence $\{\rho_n\}$ and any $\varphi \in \mathscr{D}(R)$ we have

$$<\rho''_n\Big(\frac{1}{x}*\rho_n\Big),\ \phi> = <\frac{1}{x},\ \rho''_n\phi*\check{\rho}_n>$$

$$= <\frac{1}{x},\ (\rho'_n\phi)'*\check{\rho}_n - \rho'_n\phi'*\check{\rho}_n>$$

$$= <\frac{1}{x^2},\ \rho'_n\phi*\check{\rho}_n> - <\rho'_n\Big(\frac{1}{x}*\rho_n\Big),\ \phi'> .$$

From the facts that the product $\delta'\triangle\mathrm{Pf.}\frac{1}{x^2}$ exists but the product $\delta'\triangle\mathrm{Pf.}\frac{1}{x}$ does not exist it follows that $<\frac{1}{x^2},\ \rho'_n\phi*\check{\rho}_n>$ converges to $<\delta'\triangle\mathrm{Pf.}\frac{1}{x^2},\ \phi>$ but $<\rho'_n\Big(\frac{1}{x}*\rho_n\Big),\ \phi'>$ does not converge as $n\to\infty$ and therefore $<\rho''_n\Big(\frac{1}{x}*\rho_n\Big),\ \phi>$ does not converge.

It is easily shown that if the products $S\triangle T$ and $S\triangle\frac{\partial T}{\partial x_j}$ exist for $j = 1, 2,\dots, N$, then the product $\frac{\partial S}{\partial x_j}\triangle T$ exists and $\frac{\partial}{\partial x_j}(S\triangle T) = \frac{\partial S}{\partial x_j}\triangle T + S\triangle\frac{\partial T}{\partial x_j}$ holds. From the facts that $\delta\triangle\mathrm{Pf.}\frac{1}{x}$ exists but $\delta'\triangle\mathrm{Pf.}\frac{1}{x}$ does not exist it follows that $\delta\triangle\mathrm{Pf.}\frac{1}{x^2}$ does not exist.

We have shown in [5, p. 162] that if the product $S\circ T$ exists, then $(\alpha S)\circ T$ and $S\circ(\alpha T)$ exists for any $\alpha\in\mathscr{E}(R^N)$ and $(\alpha S)\circ T = \alpha(S\circ T) = S\circ(\alpha T)$. The same property holds also true of the multiplicative product ST. But the statement is not true in general for $S\triangle T$. In fact, let $N=1$ and take $S=\delta'$ and $T=\mathrm{Pf.}\frac{1}{x^2}$. Then the product $\delta'\triangle\mathrm{Pf.}\frac{1}{x^2}$ exists but $\delta'\triangle\mathrm{Pf.}\frac{1}{x}$ and $\delta\triangle\mathrm{Pf.}\frac{1}{x^2}$ do not exist. On the other hand, if we take $S=\delta$ and $T=\mathrm{Pf.}\frac{1}{x}$, then $(\alpha S)\triangle T = -\frac{\alpha(0)}{2}\delta'$ and $\alpha(S\triangle T) = \frac{1}{2}\alpha'(0)\delta - \frac{\alpha(0)}{2}\delta'$ for any $\alpha\in\mathscr{E}(R)$ and therefore $\alpha(S\triangle T)$ is not equal to $(\alpha S)\triangle T$ in general.

Let 5, T be tempered distributions on R^N and suppose S and T are $\mathscr{S}'$-composable, that is, $(S_x\otimes T_y)\phi(\hat{x}+\hat{y})\in(\mathscr{D}'_{L^1})$ for any $\varphi\in\mathscr{S}(R^N)$. Then the product Sf exists and $(S*T)^\wedge = \hat{S}\hat{T}$ ([3, p. 151]). Furthermore $(S*\rho_n)(f*\tilde{\rho}_n)$ converges in $\mathscr{S}'(R^N)$ to $\hat{S}\hat{T}$ as $n\to\infty$ for any δ-sequences $\{\rho_n\}$ and $\{\tilde{\rho}_n\}$ ([11, p. 233]). Thus, if 5 and T are $\mathscr{S}'$-composable, then $\hat{S}\triangle\hat{T}$ exists, $\hat{S}\triangle\hat{T} = (S*T)^\wedge$ and $(\hat{S}*\rho_n)(\hat{T}*\rho_n)$ converges to $\hat{S}\triangle\hat{T}$ in $\mathscr{S}'(R^N)$ for any restricted δ-sequence $\{\rho_n\}$

PROPOSITION 6. *Let* $S\in\mathscr{D}'(R^N)$. *Then the following conditions are equivalent to each other:*

(1) $S\in\mathscr{E}(R^N)$.

(2) $S\triangle T$ *exists for any* $T\in\mathscr{D}'(R^N)$.

(3) *$S \triangle T$ exists for any $T \in \mathscr{E}'(R^N)$.*

PROOF. It suffices to prove the implications (1)⇒(2) and (3)⇒(1).

(1)⇒(2). Let $S \in \mathscr{E}(R^N)$. Then the product ST exists for any $T \in \mathscr{D}'(R^N)$, and a fortiori $S \triangle T$ exists.

(3)⇒(1). Suppose $S \triangle T$ exists for any $T \in \mathscr{E}'(R^N)$. For any restricted δ-sequence $\{\rho_n\}$, the map

$$T \longrightarrow (S*\rho_n)(T*\rho_n)$$

of $\mathscr{E}'(R^N)$ into $\mathscr{D}'(R^N)$ is continuous and $\mathscr{E}'(R^N)$ is a barrelled space. By the Banach-Steinhaus theorem the map $\mathscr{E}'(R^N) \ni T \to \lim_{n\to\infty} (S*\rho_n)(T*\rho_n) = S \triangle T \in \mathscr{D}'(R^N)$ is continuous, and therefore for any ϕ e $\mathscr{D}(R^N)$ there exists an element $\phi(S)$ e $\mathscr{E}(R^N)$ such that

$$<S\triangle T, \varphi> = <\phi(S), T> .$$

If we take $\Gamma = \alpha$ E $\mathscr{D}(R^N)$, then $< S\triangle\alpha, \phi > = < \alpha S, \varphi > = < \phi S, \alpha >$. Thus $\varphi S = \phi(S)$e $\mathscr{E}(R^N)$, which implies $S \in \mathscr{E}(R^N)$.

Let ξ be a C^∞ map of a non-empty open subset $\Omega \subset R^N$ into another open subset $\Omega' \subset R^n$. If the map ξ^*: $\mathscr{D}(\Omega') \ni \alpha \to \alpha \circ \xi \in \mathscr{D}'(\Omega)$ is continuously extended to the map of $\mathscr{D}'(\Omega)$ (or equivalently of $\mathscr{E}'(\Omega)$) into $\mathscr{D}'(\Omega)$, then the map ξ is said to be admissible ([7, p. 76]) and $\xi^* S$ is said to be the transposed image of 5 e $\mathscr{D}'(\Omega')$. Then we see that $n \leqq N$ ([7, p. 77]).

Let ξ and η be C^∞ maps of a non-empty open subset $\Omega \subset R^N$ into another open subsets $\Omega_1 \subset R^p$ and $\Omega_2 \subset R^q$ respectively, and assume the map $\chi = (\xi, \eta)$ of Ω into Ω_1 x Ω_2 has no critical point. By the facts that the multiplication o is invariant under the diffeomorphism and has the local property ([5, pp. 162–165]) we conclude that the multiplicative product $(\xi^* S)\triangle(\eta^* T)$ exists for every S e $\mathscr{D}'(\Omega_1)$ and $T \in \mathscr{D}'(\Omega_2)$.

PROPOSITION 7. *Let ξ be an admissible map of $\Omega \subset R^N$ into $\Omega_1 \subset R^p$ and η be an admissible map of Ω into $\Omega_2 \subset R^q$. If the multiplicative product $(\xi^* S)\triangle$ $(\eta^* T)$ exists for every $S \in \mathscr{D}'(\Omega_1)$ and $T \in \mathscr{D}'(\Omega_2)$, then the map $\chi = (\xi, \eta)$ is admissible.*

PROOF. Let $\{\rho_n\}$ be any restricted δ-sequence defined in Ω and consider the map

$$(S, T) \longrightarrow ((\xi^* S)*\rho_n)((\eta^* T)*\rho_n)$$

of $\mathscr{E}'(\Omega_1) \times \mathscr{E}'(\Omega_2)$ into $\mathscr{D}'(\Omega)$ for a large n. It is a separately continuous bilinear map for each ρ_n and $((\xi^* S)*\rho_n)((\eta^* T)*\rho_n)$ converges in $\mathscr{D}'(\Omega)$ to $(\xi^* S)\triangle$

$(\eta*T)$ as $n\to\infty$. Since $\mathscr{E}'(\Omega_1)$ and $\mathscr{E}'(\Omega_2)$ are barrelled, the bilinear map

$$(S, T) \longrightarrow (\xi*S)\triangle(\eta*T)$$

of $\mathscr{E}'(\Omega_1)\times\mathscr{E}'(\Omega_2)$ into $\mathscr{D}'(\Omega)$ is hypocontinuous. Owing to the theorem of Grothendieck ([2, p. 66]), since $\mathscr{E}'(\Omega_1)$ and $\mathscr{E}'(\Omega_2)$ are (DF)-spaces, the map is continuous and therefore it can be continuously extended to the map of $\mathscr{E}'(\Omega_1)\hat{\otimes}_\pi\mathscr{E}'(\Omega_2)=\mathscr{E}'(\Omega_1\times\Omega_2)$ into $\mathscr{D}'(\Omega)$; this means that the map $\chi=(\xi, \eta)$ is admissible.

Since the property (iii) of a restricted δ-sequence $\{\rho_n\}$ does not play any role in the proofs of the above two propositions, the analogues of Propositions 6 and 7 remain valid for the multiplication.

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