HIROSHIMA MATH. J.
6 (1976), 353-357

On the Limits of p-Precise Functions along Lines Parallel to the Coordinate Axes of R^n

Yoshihiro MIZUTA

(Received January 16, 1976)

1. Introduction and statement of the main result

Recently, C. Fefferman [2] proved the following result: Let $1<p<n$ and let u be a C^1-function on $R^n=R\times R^{n-1}$ $(n\geqq 2)$ such that $\int_{R^n}|\operatorname{grad} u|^p dx<\infty$. Then there is a constant c such that $\lim_{x_1\to\infty}u(x_1,x')=c$ for almost all $x'\in R^{n-1}$.

In the present note, we shall give an improvement of this result by using the capacity $C_{1,p}$:

$$C_{1,p}(E) = \inf\|f\|_p^p \qquad \text{for} \quad E\subset R^n,$$

where the infimum is taken over all non-negative functions f in $L^p(R^n)$ such that $\int|x-y|^{1-n}f(y)dy\geqq 1$ for all $x\in E$. This capacity is a special case of the capacity $C_{k;\mu;p}$ introduced by N. G. Meyers [4]. We shall show

THEOREM 1. *Let* $1<p<n$ *and let u be a p-precise function on* $R^n=R\times R^{n-1}$. *Then there are a constant c and a Borel set* E' *in* R^{n-1} *with* $C_{1,p}(\{0\}\times E')=0$ *such that*

$$\lim_{x_1\to\infty} u(x_1, x') = c \qquad \text{for all} \quad x'\in R^{n-1}-E'.$$

For p-precise functions, see [6; Chap. IV] (also cf. [3; Chap. III, §2], in which they are called Beppo Levi functions of order p). Note that for a p-precise function u on R^n, $\operatorname{grad} u$ is defined almost everywhere and $\int_{R^n}|\operatorname{grad} u|^p dx<\infty$. Also note that if $C_{1,p}(\{0\}\times E')=0$, then the $(n-1)$-dimensional Lebesgue measure of E' is zero (see [3; Theorem A], [1; Theorem 1 in §IV] and our Lemma 2).

The proof of this theorem is based on the following proposition, which is a special case of Theorem 1 on account of [6; Theorem 9.6] (also cf. [5; Theorem 5.1]).

PROPOSITION 1. *Let* $1<p<n$ *and let* $f\in L^p(R^n)$. *Then there is a Borel set* $E'\subset R^{n-1}$ *with* $C_{1,p}(\{0\}\times E')=0$ *such that*

$$\lim_{x_1\to\infty}\int_{R^n}|x-y|^{1-n}f(y)dy \neq 0 \qquad \textit{for all} \quad x'\in R^{n-1}-E',$$

where $x=(x_1, x')$.

We shall see that Proposition 1 is the best possible as to the size of the exceptional set (Remark 2).

2. Proof of Proposition 1

We may assume that $f\geqq 0$. Let r be a positive number and j a positive integer. If $|x|>2r$, then we have by Holder's inequality

$$(1) \qquad \int_{|y|\leqq r}|x-y|^{1-n}f(y)dy \leqq \|f\|_p\left\{\int_{|y|\leqq r}|x-y|^{p'(1-n)}dy\right\}^{1/p'}$$

$$\leqq \|f\|_p\left\{\int_{|x-y|\geqq r}|x-y|^{p'(1-n)}dy\right\}^{1/p'} = M\|f\|_p r^{1-n/p},$$

where $1/p+1/p'=1$ and M is a constant independent of r. On the other hand, from the definition of $C_{1,p}$ it follows that

$$(2) \qquad C_{1,p}\left(\left\{x;\int_{|y|>r}|x-y|^{1-n}f(y)dy>\frac{1}{2j}\right\}\right)\leqq (2j)^p\int_{|y|>r}f(y)^p dy.$$

If r is sufficiently large, say $r\geqq r_j$, then the right-hand sides of (1) and (2) are smaller than $(2j)^{-1}$ and 2^{-j} respectively. Set

$$\omega_j=\left\{x;\ |x|>2r_j, \int|x-y|^{1-n}f(y)dy>1/j\right\}.$$

Then

$$C_{1,p}(\omega_j)\leqq C_{1,p}\left(\left\{x;\int_{|y|>r_j}|x-y|^{1-n}f(y)dy>\frac{1}{2j}\right\}\right)<2^{-j}.$$

Set $E_k=\cup_{j=k}^{\infty}\omega_j$ and $E=\cap_{k=1}^{\infty}E_k^*$, where E_k^* is the projection of E_k to the hyperplane $R_0^n=\{(0, x');\ x'\in R^{n-1}\}$. It is easy to see that $\lim_{x_1\to\infty}\int\{(x_1-y_1)^2+|x'-y'|^2\}^{(1-n)/2}f(y)dy\neq 0$ if $(0, x')$ does not belong to E. If we show that $C_{1,p}(E_k^*)\leqq C_{1,p}(E_k)$ for each fc, then we have $C_{1,p}(E)=0$, and hence the proposition. Thus it remains to show

LEMMA 1 *(cf.* [6; *Theorem* 8.1]). *Let* $1<p<\infty$. *For any set* $E\subset R^n$ *denote by* E^* *the projection of* E *to* R_0^n. *Then we have*

$$C_{1,p}(E^*)\leqq C_{1,p}(E).$$

3. Proof of Lemma 1

To prove Lemma 1, we consider the symmetrization of functions with respect to R_0^n. First, let $\varphi: R^1 \to R^1$ be a non-negative measurable function. The symmetrization φ^* of φ is defined by

$$\varphi^*(t) = \inf\left\{r \geqq 0; \int_{\varphi(s) \geqq r} ds \leqq 2|t|\right\}.$$

For a non-negative measurable function f on R^n, we define its symmetrization f^* (with respect to R_0^n) by $f^*(x_1, x') = \varphi_{x'}^*(x_1)$, where $\varphi_{x'}(x_1) = f(x_1, x')$, for $x' \in R^{n-1}$ such that $\varphi_{x'}$ is measurable. We see that f^* is a non-negative measurable function defined a.e. on R^n and has the following properties:

(a)
$$\int_{R^n} f^*(x)^p dx = \int_{R^n} f(x)^p dx;$$

(b)
$$\int_{R^n} f^*(x) g^*(x) dx \geqq \int_{R^n} f(x) g(x) dx$$

for any non-negative measurable function g on R^n.

Now, let f be a non-negative function in $L^p(R^n)$ such that $\int_{R^n} |x-y|^{1-n} f(y) dy \geqq 1$ for all $x \in E$. Let $x = (x_1, x') \in E$ and put $x^* = (0, x')$. Since the symmetrization of the function $|x-y|^{1-n}$ as a function in y is $|x^*-y|^{1-n}$, we have by property (b)

$$\int_{R^n} |x^*-y|^{1-n} f^*(y) dy \geqq \int_{R^n} |x-y|^{1-n} f(y) dy \geqq 1.$$

Hence, in view of (a), we obtain Lemma 1.

4. Proof of Theorem 1

First, we remark the following lemma (cf. [4; Theorem 3]):

LEMMA 2. *Let* $1 < p < n$ *and* $E \subset R^n$. *Then* $C_{1,p}(E) = 0$ *if and only if there is a non-negative function* f *in* $L^p(R^n)$ *such that* $\int_{R^n} |x-y|^{1-n} f(y) dy = \infty$ *for every* $x \in E$.

In view of this lemma, [6; Theorem 9.11 and its remark, Theorem 9.3] or [5; Theorems 4.1 and 3.2] implies that a p-precise function u on R^n has the following integral representation:

$$u(x)=c_1\sum_{i=1}^{n}\int\frac{x_i-y_i}{|x-y|^n}\frac{\partial u}{\partial y_i}(y)dy+c_2$$

except for x in a Borel set E_1 with $C_{1,p}(E_1)=0$, where c_1 and c_2 are constants. Let E_1^* be the projection of E_1 to R_0^n. By Proposition 1, there is a Borel set $E_2 \subset R_0^n$ such that $C_{1,p}(E_2)=0$ and

$$\lim_{x_1\to\infty}\int|x-y|^{1-n}|\mathrm{grad}\, u|dy=0$$

for all $(0,x')\in R_0^n-E_2$, where $x=(x_1,x')$. Obviously, $C_{1,p}(E_1^*\cup E_2)=0$ (cf. [4; Theorem 1]) and $\lim_{x_1\to\infty}u(x_1,x')=c_2$ if $(0,x')\notin E_1^*\cup E_2$. Thus Theorem 1 is proved.

5. Remarks

REMARK 1. If we combine our theorem with a result of B. Fuglede [3; Theorem A] and the above Lemma 2, we have

THEOREM 2. *Let u be a p-precise function on R^n $(1<p<n)$. Then there is a constant c such that $\lim_{x_1\to\infty}u(x_1,x')=c$ if $(0,x')\notin E$, where E is a Borel set in R_0^n such that $C_p(E)=0$ if $p\leqq 2$ and $C_{p-\varepsilon}(E)=0$ for any ε with $0<\varepsilon<p$ if $p>2$.*

REMARK 2. Proposition 1 is the best possible as to the size of the exceptional set: Given a set $E\subset R_0^n$ with $C_{1,p}(E)=0$, we set $\tilde{E}=\{x+(j,0);\ x\in E$ and j is an integer$\}$. Then $C_{1,p}(\tilde{E})=0$. By Lemma 2 there is a non-negative function f in $L^p(R^n)$ such that $\int|x-y|^{1-n}f(y)dy=\infty$ for every $x\in\tilde{E}$. We see that $\limsup_{x_1\to\infty}\int|x-y|^{1-n}f(y)dy=\infty$ if $(0,x')\in E$, where $x=(x_1,x')$.

REMARK 3. In connection with Proposition 1, we may be concerned with functions of the following form:

$$u(x)=\int_{R^n}|x-y|^{1-n}f(y)\omega(y)dy,$$

where ω is a positive continuous function on R^n and $f\in L^p(R^n)$. The next two propositions show that it is of little value to consider a weight function ω.

PROPOSITION 2. *Let $1<p<\infty$. If $\omega(y)=\omega(y_1,y')\to+\infty$ as $|y_1|\to\infty$, then there exists a non-negative function $f\in L^p(R^n)$ such that*

$$\limsup_{x_1\to\infty}\int_{R^n}|x-y|^{1-n}f(y)\omega(y)dy=+\infty$$

for every $x' \in R^{n-1}$, *where* $x=(x_1, x')$.

PROOF. Let ε be a positive number and set $g(y)=1$ if $|y|<1$ and $=|y|^{-n/p-\varepsilon}$ if $|y| \geqq 1$. Then $g \in L^p(R^n)$. Set $a_r = \inf\left\{\omega(y_1, y');\ |y_1| > \frac{r}{2}\right\}$ for r>0 and set $g_r(y) = a_r^{-1/2} g(y - re_1)$, where $e_1 = (1, 0, \ldots, 0) \in R^n$. We have a sequence $\{r_j\}$, $r_j > 2$, such that $\sum_{j=1}^{\infty} a_{r_j}^{-1/2} < \infty$ Let $x^* = (0, x') \in R^n$ and $x^{(j)} = x^* + r_j e_1$. Setting $f = \sum_{j=1}^{\infty} g_{r_j}$ and $u(x) = \int |x-y|^{1-n} f(y) \omega(y) dy$, we note

$$u(x^{(j)}) \geqq \int |x^{(j)} - y|^{1-n} g_{r_j}(y) \omega(y) dy$$

$$\geqq a_{r_j}^{1/2} \int_{|x^*-z|<1} |x^* - z|^{1-n} g(z) dz \longrightarrow \infty$$

as $j \to \infty$, which implies that f is the required function.

Proposition 3. *Let* $1<p<n$ *and suppose* $\omega(y_1, y') \to 0$ *as* $|y_1| \to \infty$. *Then, Proposition* 1 *and Remark2 remain valid for the function* $\int_{R^n} |x-y|^{1-n} f(y) \omega(y) dy$.

PROOF. This is seen from the fact that $\int_{R^n} |x-y|^{1-n} f(y) \omega(y) dy \neq \infty$ if and only if $\int_{R^n} |x-y|^{1-n} f(y) dy \neq \infty$ for a non-negative function $f \in L^p(R^n)$.

References

[1] L. Carleson, Selected problems on exceptional sets, Van Nostrand, Princeton, 1967.

[2] C. Fefferman, Convergence on almost every line for functions with gradient in $L^p(R^n)$, Ann. Inst. Fourier **24** (1974), 159–164.

[3] B. Fuglede, Extremal length and functional completion, Acta Math. **98** (1957), 171-219.

[4] N. G. Meyers, A theory of capacities for potentials of functions in Lebesgue classes, Math. Scand. 26 (1970), 255-292.

[5] Y. Mizuta, Integral representations of Beppo Levi functions of higher order, Hiroshima Math. J. 4 (1974), 375-396.

[6] M. Ohtsuka, Extremal length and precise functions in 3-space, Lecture notes, Hiroshima Univ., 1973.

Department of Mathematics,
Faculty of Science,
Hiroshima University