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Note on J-groups of Lens Spaces

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§1. Introduction

Let $J(X)$ be the J-group of a CW-complex X of finite dimension. Then by J. F. Adams [2] and D. Quillen [9], it is shown that

(1.1) $$J(X) = KO(X)/\operatorname{Ker} J, \qquad \operatorname{Ker} J = \textstyle\sum_k(\cap_e k^e(\Psi^k - 1)KO(X)),$$

where $KO(X)$ is the KO-group of real vector bundles over X, $J\colon KO(X)\to J(X)$ is the natural epimorphism and Ψ^k is the Adams operation.

In this note, we consider the standard lens space

$$L^n(m) = S^{2n+1}/Z_m, \qquad Z_m = \{\exp(2\pi li/m)\colon 0 \leqq l < m\},$$

which has the CW-decomposition $L^n(m) = \cup_{j=0}^{2n+1} e^j$ and its $2n$-skeleton

$$L_0^n(m) = \{[z_0,\dots, z_n] \in L^n(m)\colon z_n \text{ is real} \geqq 0\}.$$

Let η be the canonical complex line bundle over $L^n(m)$ or $L_0^n(m)$. Then

(1.2) (N. Mahammed [7]) *the K-rings of complex vector bundles over these spaces are given by*

$$K(L^n(m)) = K(L_0^n(m)) = Z[\eta]/ < \eta^m - 1, (\eta - 1)^{n+1} >,$$

and the reduced group $\tilde{K}(L^n(m)) = \tilde{K}(L_0^n(m))$ *is of order* m^n,
where $Z[\eta]$ is the integral polynomial ring with one variable η and the denominator is the ideal generated by $\eta^m - 1$ and $(\eta - 1)^{n+1}$.

Now, we consider the case $m = p^r$, where p is an *odd prime* and $r \geqq 1$. Then by (1.2) and (1.1), we have the following

PROPOSITION 1.3. (i) $$J(L^n(p^r)) = \begin{cases} J(L_0^n(p^r)) \oplus Z_2 & \textit{if } \ n \equiv 0 \bmod 4, \\ J(L_0^n(p^r)) & \textit{otherwise.} \end{cases}$$

(ii) $$Jr\colon K(L_0^n(p^r)) \xrightarrow{r} KO(L_0^n(p^r)) \xrightarrow{J} J(L_0^n(p^r))$$

(r is the real restriction) is epimorphic, and Ker Jr *is the subgroup of* $K(L_0^n(p^r))$ *of* (1.2) *for* $m = p^r$ *generated additively by the elements*

$$(1+\sigma)\sigma^d\sigma(s), \qquad 0 \leqq s < r, \quad 0 \leqq d < p^s(p-1)-1,$$

where σ and $\sigma(s)$ are the elements given by

(1.4) $$\sigma = \eta - 1 = \sigma(0), \quad \sigma(s) = \eta^{p^s} - 1 = (1+\sigma)^{p^s} - 1 \qquad (0 \leqq s \leqq r).$$

The purpose of this note is to give some generators of the abelian group $\tilde{K}(L^n(p^r))$ more explicitly, and to study the J-group $J(L^n(p^r))$ by the above proposition.

Consider the following integers and elements of $\tilde{K}(L^n(p^r))=\tilde{K}(L_0^n(p^r))$ for $0 \leqq s<r$, $0 \leqq d<p^s(p-1)$ and $d+p^s \leqq n$:

(1.5) $$n-p^s+1 = a_s p^s(p-1)+b_s, \; 0 \leqq b_s < p^s(p-1),$$
$$t(d+p^s) = p^{r-s-1+\bar{a}_s}, \; \bar{a}_s = \begin{cases} a_s+1 & \text{if } d < b_s, \\ a_s & \text{if } d \geqq b_s; \end{cases}$$

(1.6) $$\sigma(s, d) = \begin{cases} \sum_{t=0}^{s} p^{(p^t-1)\bar{a}_s}\sigma^d\sigma(s-t)^{p^t} \\ \qquad \text{if } b_s \leqq d < b_s+p^s-1 \text{ or } d < b_s-p^s(p-2)-1, \\ \sigma^d\sigma(s) \quad \text{otherwise.} \end{cases}$$

THEOREM 1.7 (cf. [8, Th. 3]). *Let p be an odd prime. Then the reduced K-groups of the lens space $L^n(p^r)$ and its subspace $L_0^n(p^r)$ are given by*

$$\tilde{K}(L^n(p^r)) = \tilde{K}(L_0^n(p^r)) = \textstyle\sum_{i=1}^{N} Z_{t(i)} \; (\textit{direct sum}), \quad N = \min\{n, p^r-1\},$$

where the summand $Z_{t(i)}$ is the cyclic subgroup of order $t(i)$ of (1.5) *generated by the element $\sigma(s, d)$, $i=d+p^s$, of* (1.6).

Consider the following elements of the reduced J-group $\tilde{J}(L_0^n(p^r))$:

(1.8) $$\alpha_s = Jr(\sigma(s)) = Jr(\eta^{p^s})-2, \qquad 0 \leqq s < r.$$

Then we have the following theorem, where a cyclic group Z_t of order t is denoted by $Z(t)$.

THEOREM 1.9. *Let p be an odd prime, and assume that the integers a_s in* (1.5) *satisfy the condition*

(1.10) $$a_s \not\equiv 0 \bmod p \qquad \textit{for} \quad 0 \leqq s < R(n) = \min\{r-1, r(n)\},$$

where $p^{r(n)} \leqq n+1 < p^{r(n)+1}$. Then the reduced J-group of $L_0^n(p^r)$ is the direct sum

$$\tilde{J}(L_0^n(p^r)) = \textstyle\sum_{s=0}^{R(n)} Z(p^{a_s}),$$

where the s-th cyclic subgroup $Z(p^{a_s})$ *of order* p^{a_s} *is generated by the elements* α_s *of* (1.8), *and* $\alpha_s=0$ *if* $s\geqq r(n)$.

As a corollary, we have the following result for the case $r=1$:

COROLLARY 1.11 ([4, Prop. 1]). $\tilde{J}(L_0^n(p))=Z(p^{a_0})$ *and it is generated by* α_0, *where* $a_0=[n/(p-1)]$.

We notice that (1.10) is equivalent to the following condition (cf. Lemma 5.10):

$$0 \leqq b_s \leqq p^s(p-2) \quad \text{for} \quad 0 < s \leqq R(n). \tag{1.12}$$

For the case that the assumption (1.10) or (1.12) does not hold, we take the integer ρ, $0<\rho\leqq R(n)$, such that

$$b_s \leqq p^s(p-2) \quad \text{for} \quad \rho < s \leqq R(n), \quad b_\rho > p^\rho(p-2), \tag{1.13}$$

and consider the integers $\tilde{n}$ and $\tilde{a}_s$ given as follows:

$$\tilde{n} = a_\rho p^\rho(p-1)+p^{\rho+1}-1 = n+p^\rho(p-1)-b_\rho;$$
$$\tilde{a}_s = \begin{cases} a_s & \text{if} \quad \rho < s \leqq R(n), \\ a_\rho p^{\rho-s}+(p^{\rho-s+1}-1)/(p-1) & \text{if} \quad 0 \leqq s \leqq \rho. \end{cases} \tag{1.14}$$

PROPOSITION 1.15. *Under the above situations,* $\tilde{J}(L_0^n(p^r))$ *is the abelian group of order* p^ν, $\nu=\sum_{s=0}^{r-1}[n/p^s(p-1)]$, *generated by the elements* α_s, $0\leqq s\leqq R(n)$, *of* (1.8), *with the relations*

$$\begin{aligned} &p^{\tilde{a}_s}\alpha_s = 0 \quad \textit{for} \quad 0 \leqq s \leqq R(n), \\ &\textstyle\sum_{s=0}^{\rho}\theta(a(p-1);\, s+1)\alpha_s = 0 \quad \textit{for} \quad a_0 < a \leqq \tilde{a}_0, \end{aligned} \tag{1.16}$$

where $\theta(a(p-1);\, s+1)=\sum_{i=0}^{\infty}(-1)^i\binom{a(p-1)}{ip^{s+1}}$.

By this proposition, we obtain the results for the special case that $r=2$, 3 or 4 in Proposition 6.13, 7.5 or 7.7.

§2. Proof of Proposition 1.3

LEMMA 2.1. *Let m be an odd integer. Then*

(i) $J(L^n(m)) = J(L_0^n(m))\oplus Z_2$ *if* $n\equiv 0 \bmod 4$, $\quad = J(L_0^n(m))$ *otherwise.*

(ii) *The real restriction* $r\colon K(L_0^n(m))\to KO(L_0^n(m))$ *is epimorphic, and* $\operatorname{Ker} r$ *is generated additively by the elements* $\eta^j-\eta^{m-j}$, $0<j<m$.

PROOF. (i) From the exact sequence of $(L^n(m), L_0^n(m))$, we have the split exact sequence

$$0 \longrightarrow \widetilde{KO}(S^{2n+1}) \longrightarrow KO(L^n(m)) \longrightarrow KO(L_0^n(m)) \longrightarrow 0,$$

where $\widetilde{KO}(S^{2n+1})=Z_2$ if $n\equiv 0 \bmod 4$, $=0$ otherwise, and the reduced group $\widetilde{KO}(L_0^n(m))$ is of order $m^{[n/2]}$, (cf. [5, Lemmas 2.4, 2.3]). Hence the operator Ψ^k-1 splits by the naturality, and we have the desired result by (1.1) and the fact that $\tilde{J}(S^{2n+1})=\widetilde{KO}(S^{2n+1})$ [2, II, (3.5)].

(ii) The first half is proved in the proof of [5, Prop. 2.11]. Let $t\colon K\to K$ be the conjugation. Then $t(\eta^j)=\eta^{-j}=\eta^{m-j}$ by (1.2), and so $r(\eta^j-\eta^{m-j})=r(\eta^j-t(\eta^j))=0$ since $rt=r$. Conversely, assume $x\in \operatorname{Ker} r$. Then $x\in\tilde{K}(L_0^m(m))$, which is of odd order by (1.2), and hence $y=x/2$ exists and $ry=0$. Thus we have $y+ty=cry=0$, where $c\colon KO\to K$ is the complexification. Therefore $x=y+y=y-ty$ and is a linear combination of $\eta^j-\eta^{m-j}$ since y is a linear combination of η^j by (1.2). q.e.d.

Proof of Proposition 1.3. By the above lemma for $m=p^r$, it is sufficient to determine $\operatorname{Ker} Jr$. Since r is epimorphic and $r\Psi^k=\Psi^k r$ [3, Lemma A2], (1.1) shows that

$$\operatorname{Ker} Jr = \operatorname{Ker} r+\textstyle\sum_k K_k, \quad K_k = \bigcap_e k^e(\Psi^k-1)K(L_0^n(m)).$$

Since $\Psi^k\eta^j=\eta^{kj}$ by [1, Th. 5.1], (1.2) shows that $K_k=0$ if $k\equiv 0 \bmod p$ and K_k is generated by $\{\eta^{kj}-\eta^j\}$ otherwise. By these facts, (ii) of the above lemma and the relation $\eta^{p^r}=1$ of (1.2), we see that $\operatorname{Ker} Jr$ is generated by the elements

$$(*) \qquad \alpha(s, k) = \eta^{kp^s}-\eta^{p^s}, \quad 0\leqq s<r,\ 1\leqq k<p^{r-s},\ (k, p) = 1.$$

Since $\alpha(t, 1)=0$ and $\alpha(t, k+p^{s-t})-\alpha(t, k)=\eta^{kp^t}\sigma(s)$ for $0\leqq t\leqq s$, the elements

$$\eta^j\sigma(s), \quad 0\leqq s<r,\ 1\leqq j<p^s(p-1),$$

are linear combinations of the elements of $(*)$ and the converse is also true. Further it is easy to see that the elements of (ii) of the proposition are linear combinations of these elements and the converse is true. q.e.d.

§3. Proof of Theorem 1.7

We prove Theorem 1.7 quite arithmetically by starting (1.2).

Let p be a prime, and consider the integer

$$(3.1) \qquad h(a; s) = [(n+p^s-1-a)/p^s(p-1)].$$

Then, we have the following relations in $\tilde{K}(L^n(p^r))=\tilde{K}(L_0^n(p^r))$.

Proposition 3.2. *For any given sequence* $(k_0,\dots, k_{s-1})$, $0\leqq s<r$, *we set*

$$\alpha = \prod_{t=0}^{s-1} \sigma(t)^{k_t}, \qquad d = \sum_{t=0}^{s-1} k_t p^t,$$

where $\sigma(t) \in \tilde{K}(L^n(p^r))$ *is the element of* (1.4).

(i) *If* $\bar{k}_s$ *is a non-negative integer with* $n \leqq d + \bar{k}_s p^s < n + p^s$, *then*

$$p^{r-s-l}\alpha\sigma(s)^{\bar{k}_s+l} = 0 \qquad \text{for} \quad 0 < l \leqq r-s.$$

(ii) $p^{r-s+h}\alpha\sigma(s)^k = 0,\ h = h(d+kp^s;\ s),\ if\ k > 0.$

PROOF. The relation $\sigma^{n+1}=0$ of (1.2) implies (i) for $s=0$. By assuming inductively (i) for $s(\geqq 0)$, we prove (ii) for s and (i) for $s+1$.

(i) *implies* (ii). We denote simply by $h(k) = h(d+kp^s;\ s)$ for $k>0$. Then we see easily by (3.1) that

$$(*) \qquad h(k) = h \quad \text{if and only if} \quad \bar{k}_s-(p-1)(h+1) < k \leqq \bar{k}_s-(p-1)h.$$

This shows that $k>\bar{k}_s$ and $h(k) \geqq \bar{k}_s - k$ if $h(k)<0$. Hence (i) implies (ii) for k such that $h(k)<0$.

Now, assume inductively (ii) for k such that $h(k)<h$ $(h \geqq 0)$, and let $h(k)=h$. By the relation $(1+\sigma)^{p^r}-1=0$ of (1.2) and (1.4), we see that $(1+\sigma(s))^{p^{r-s}}-1=0$. Multiplying this relation by $p^h\alpha\sigma(s)^{k-1}$, we have

$$(**) \qquad \sum_{i=1}^{p^{r-s}} \binom{p^{r-s}}{i} p^h\alpha\sigma(s)^{k+i-1} = 0.$$

If $i=jp^v \geqq 1$ and $(p, j)=1$, then we see easily that $i>v(p-1)$ and so $h(k+i-1) \leqq h-v$ by $(*)$. This fact and the inductive assumption show that the i-th term of $(**)$ is 0 if $h(k+i-1)<h$. Therefore, by $(**)$ and $(*)$, we have (ii) for $k=k'$ $(=\bar{k}_s-(p-1)h), \dots, k'-p+2$, successively, i.e., for k such that $h(k)=h$. These show (ii) by the induction on h.

(ii) *for* s *implies* (i) *for* $s+1$. By (1.4), we have

$$(3.3) \qquad \sigma(s+1) = (1+\sigma(s))^p - 1 = pA(s)\sigma(s) + \sigma(s)^p,$$

$$A(s) = \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i}\sigma(s)^{i-1}.$$

For any sequence $(k_0, \dots, k_s, k)$, this implies

$$p^{r-s-1-l}\alpha\sigma(s)^{k_s}\sigma(s+1)^k = \sum_{i=0}^{k}\binom{k}{i} A(s)^i p^{r-s-1-l+i}\alpha\sigma(s)^{k_s+pk-i(p-1)}.$$

If $\bar{k}_{s+1}$ is a non-negative integer with $n \leqq d + k_s p^s + \bar{k}_{s+1}p^{s+1} = d'$, and $k=\bar{k}_{s+1}+l$, $l>0$, then we see easily that

$$h(k_s+pk-i(p-1)) = h(d'+lp^s-(i-l)p^s(p-1);\ s) \leqq i-l-1,$$

by (3.1), and hence each term of the above summation is 0 as desired by (ii). q. e. d.

LEMMA 3.4. *Under the notations of the above lemma, we have*

$$p^{r-s-1+h'}\alpha\sigma(s)^k = (-1)^l p^{r-s-1+h'+l}\alpha A(s)^l\sigma(s)^{k-l(p-1)}$$

if $k-l(p-1)>0$, *where* $h'=h(d+kp^s;\ s+1)$ *and* $A(s)$ *is the element in* (3.3).

PROOF. It is sufficient to show that

$$p^{r-s-1+h'+l'}\alpha(pA(s)\sigma(s)^{k-(l'+1)(p-1)}+\sigma(s)^{k-l'(p-1)})=0$$

for $0\leqq l'<l$. The left hand side is equal to $p^{r-s-1+h'+l'}\alpha\sigma(s)^{k-l'(p-1)-p}\sigma(s+1)$ by (3.3), and this is 0 as desired by (ii) of the above proposition, since $h(d+kp^s-l'p^s(p-1);\ s+1)\leqq h'+l'$ by (3.1). q.e.d.

LEMMA 3.5. *If* $n\leqq d+kp^s<n+p^s$, $l>0$ *and* $m=k-l(p-1)>0$, *then*

$$p^{r-s-1}\sigma^d\sigma(s)^k = (-1)^l p^{r-s-1+l}\sigma^d\sigma(s)^m.$$

PROOF. Since $h(d+kp^s;\ s+1)=0$ by the assumption, the left hand side is equal to $(-1)^l p^{r-s-1+l}\sigma^d A(s)^l\sigma(s)^m$ by the above lemma for $\alpha=\sigma^d$. Further this is equal to the right hand side by Proposition 3.2 (ii), since $A(s)$ is a polynomial in $\sigma(s)$ with the constant term 1. q.e.d.

LEMMA 3.6. *Assume that* p *is odd. Then under the assumption of the above lemma, we have the following relation for* $0<t\leqq s$:

$$p^{r-s-1}\sigma^d(\sigma(s-t+1)^{kp^{t-1}}-\sigma(s-t)^{kp^t}) = (-1)^{l+1}p^{r-s-1+lp^t}\sigma^d\sigma(s-t)^{mp^t}.$$

PROOF. Set $u=s-t$. By (3.3), the left hand side is equal to

$$(*)\qquad \sum_{i=1}^{kp^{t-1}}\binom{kp^{t-1}}{i}p^{r-s-1+i}\sigma^d A(u)^i\sigma(u)^{kp^t-i(p-1)}.$$

If $i=jp^v\geqq 1$ and $(p,j)=1$, then we see easily that $p(i-v)>1+i$ by the assumption $p\geqq 3$, and hence that

$$i-v>h(d+(kp^t-i(p-1))p^u;\ u+1)$$

by the assumption $n\leqq d+kp^s$ and (3.1). Since the coefficient of the i-th term in $(*)$ is a multiple of $p^{t-1-v+r-s-1+i}=p^{r-u-2+i-v}$, this inequality and Lemma 3.4 show that $(*)$ is equal to

$$\sum_{i=1}^{kp^{t-1}}(-1)^{lp^t-i}\binom{kp^{t-1}}{i}p^{r-s-1+lp^t}\sigma^d A(u)^{lp^t}\sigma(u)^{kp^t-lp^t(p-1)}$$
$$=(-1)^{l+1}p^{r-s-1+lp^t}\sigma^d A(u)^{lp^t}\sigma(u)^{mp^t}.$$

By setting $A(u)=1+B(u)\sigma(u)$, this is equal to

$$(-1)^{l+1}\sum_{i=0}^{lp^t}\binom{lp^t}{i}p^{r-s-1+lp^t}\sigma^d B(u)^i\sigma(u)^{mp^t+i}.$$

If $i=jp^v\geqq 1$ and $(p, j)=1$, then we see easily that $i>v(p-1)$ and so $lp^t-v>h(d+(mp^t+i)p^u; u)$. Therefore the i-th term in the last summation is 0 for $i\geqq 1$ by Proposition 3.2 (ii), and the lemma is obtained. q.e.d.

PROPOSITION 3.7. *Assume that p is odd, and $n<d+kp^s<n+p^s$, $l>0$ and $m=k-l(p-1)>0$. Then*

$$p^{r-s-1+l}\sum_{t=0}^{s}p^{l(p^t-1)}\sigma^d\sigma(s-t)^{mp^t}=0.$$

PROOF. By the above two lemmas, the left hand side is equal to $(-1)^l p^{r-s-1}\sigma^{d+kp^s}$, which is 0 by $\sigma^{n+1}=0$ of (1.2). q.e.d.

Now, we are ready to prove Theorem 1.7.

LEMMA 3.8. *Assume p is odd. Then*

$$t(d+p^s)\sigma(s, d)=0 \qquad \text{for} \quad 0\leqq d<\min\{p^s(p-1),\ n-p^s+1\},$$

where $t(i)$ is the integer of (1.5) *and $\sigma(s, d)$ is the element of* (1.6).

PROOF. By (1.5), we have

$$\bar{a}_s>0 \qquad \text{for} \quad d<n-p^s+1. \tag{3.9}$$

If $b_s\leqq d<b_s+p^s-1$ or $d<b_s-p^s(p-2)-1$, then we see easily that

$$n<d+(1+\bar{a}_s(p-1))p^s<n+p^s$$

by definition, and so the desired relation for d by (3.9) and the above proposition. Otherwise, the desired relation $t(d+p^s)\sigma^d\sigma(s)=0$ follows from Proposition 3.2 (ii), since $h(d+p^s; s)=a_s+[(b_s+p^s-2-d)/p^s(p-1)]$. q.e.d.

PROOF OF THEOREM 1.7. (1.2) shows that $\tilde{K}(L^n(p^r))$ is generated (additively) by $\{\sigma^i: 1\leqq i\leqq N\}$ and its order is p^{rn}. Hence it is also generated by $\{\sigma(s, d): 0\leqq d<p^s(p-1),\ 1\leqq d+p^s<N\}$, since $\sigma(s, d)=\sum_{j=1}^{i}x_j\sigma^j$, $i=d+p^s$, and $x_i\equiv 1 \bmod p$ by the definition (1.6) and (3.9). Also, we see easily that $\prod_{i=1}^{N}t(i)=p^{rn}$ by (1.5). Therefore we have the theorem by the above lemma. q.e.d.

§4. Some preliminary lemmas for binomial coefficients

In the rest of this note, we assume that p is an odd prime. To study the J-group $J(L_0^n(p^r))$, we prepare some lemmas for the integers

$$\theta(a, b; u, v)=\sum_{i=0}^{\infty}(-1)^i\sum_{c=0}^{b}\binom{a}{ip^v-cp^u}\binom{b}{c}, \tag{4.1}$$

(4.2) $$\theta(a;v)=\theta(a,0;u,v)=\sum_{i=0}^{\infty}(-1)^i\binom{a}{ip^v},$$

where a, b, u, v are non-negative integers.

LEMMA 4.3. (i) $\sum_{j=1}^{p^u}(-1)^j\binom{p^u}{j}\theta(a+j,b;u,v)=-\theta(a,b+1;u,v)$.

(ii) $$\sum_{j=1}^{p^s-1}(-1)^j\binom{p^s}{j}\theta(a+j;v)=\theta(a+p^s;v) \quad \textit{if} \quad s\geqq v.$$

PROOF. We notice that $((x+1)-1)^k(x+1)^a=x^k(x+1)^a$ shows the equality

(*) $$\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{a+j}{l}=(-1)^k\binom{a}{l-k}.$$

(i) By (4.1) and (*), the left hand side is equal to

$$\sum_{i=0}^{\infty}(-1)^i\sum_{c=0}^{d}\sum_{j=1}^{p^u}(-1)^j\binom{p^u}{j}\binom{a+j}{ip^v-cp^u}\binom{b}{c}$$

$$=\sum_{i=0}^{\infty}(-1)^i\sum_{c=0}^{b}\left\{-\binom{a}{ip^v-(c+1)p^u}-\binom{a}{ip^v-cp^u}\right\}\binom{b}{c},$$

and the last is equal to the right hand side, since $\binom{b}{c}+\binom{b}{c-1}=\binom{b+1}{c}$.

(ii) The result follows from (i) for $b=0$ and (ii) of the following lemma. q.e.d.

LEMMA 4.4. (i) *$\theta(a,b;u,v)$ is the constant term q_0 of the right hand side of*

$$(1+x)^a(1+x^{p^u})^b\equiv\sum_{i=0}^{p^v-1}q_ix^i \mod 1+x^{p^v}.$$

(ii) $\theta(a,b;u,v)=0$ *if* $b\geqq 1$, $u\geqq v$.

(iii) $\theta(a,b;u,v)=1$ *if* $a+bp^u<p^v$.

PROOF. (i) follows immediately from the definition (4.1). (ii) and (iii) are seen easily by (i). q.e.d.

LEMMA 4.5. *Assume that* $0\leqq u\leqq v\leqq s$. *Then*

(i) $\theta(a,p^{s-u};u,v+1)=p^k\theta'$ *for some integer* $\theta'=\theta'(a;u,v,s)$, *where* $k=k(a;v,s)=[a/p^v(p-1)]+(p^{s-v}-1)/(p-1)$.

(ii) *If* $a=lp^v(p-1)+m$ *and* $p^v(p-2)\leqq m<p^v(p-1)$, *then*

$$\theta'(a;u,v,s)\equiv(-1)^{k(a;v,s)} \mod p.$$

PROOF. Let Q and Q' be the polynomials in y such that

(4.6) $$(1+y)^p=1+y^p+p(1+y)Q(y),\quad Q(y)=(1+y)Q'(y)-1.$$

Then we see easily that

$$(4.7) \qquad (1+y)^{k(p-1)+1} \equiv p^k(1+y)Q(y)^k \mod 1+y^p.$$

(i) Set $a=lp^v(p-1)+m$, $0\leqq m<p^v(p-1)$. Then

$$\begin{aligned}(1+x)^a(1+x^{p^u})^{p^{s-u}} &= (1+x)^m((1+x)^{p^v})^{l(p-1)}((1+x^{p^u})^{p^{v-u}})^{p^{s-v}}\\ &= (1+x)^m(1+y+pB(x))^{l(p-1)}(1+y+pB'(x))^{p^{s-v}} \qquad (y=x^{p^v})\\ &= (1+x)^m\{(1+y)^{k(p-1)+1}+\textstyle\sum_{i\geqq 1}p^i(1+y)^{k(p-1)+1-i}C_i(x)\} \quad (k=k(a;\, v, s))\\ &\equiv (1+x)^m\{p^k(1+y)Q(y)^k+p^kD(x)\} \mod 1+y^p \qquad (\text{by } (4.7)),\end{aligned}$$

where B, B', C_i and D are some polynomials in x. Therefore we see (i) by (i) of the above lemma.

(ii) Set $m'=m-p^v(p-2)$. Then $0\leqq m'<p^v$ by the assumption. In the same way as the above proof, we see that

$$\begin{aligned}&(1+x)^a(1+x^{p^u})^{p^{s-u}}\\ &= (1+x)^{m'}\{(1+y)^{(k+1)(p-1)}+\textstyle\sum_{i\geqq 1}p^i(1+y)^{(k+1)(p-1)-i}C_i'(x)\}\\ &\equiv (1+x)^{m'}\{p^k(1+y)^{p-1}((1+y)Q'(y)-1)^k+p^{k+1}D_1(x)\} \mod 1+y^p\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{by } (4.7), (4.6))\\ &\equiv (-1)^kp^k(1+x)^{m'}(1+y)^{p-1}+p^{k+1}D_2(x) \mod 1+y^p \qquad (\text{by } (4.7)),\end{aligned}$$

where C_i' and D_i are some polynomials in x. Hence we have (ii) by (i) of the above lemma, since $m'+p^v(p-1)<p^{v+1}$. q.e.d.

LEMMA 4.8. (i) *$\theta(a;\, 0)=0$ if $a>0$; $\quad \theta(a;\, v)=1$ if $a<p^v$.*
(ii) *Let $a=lp^v(p-1)+p^v+m$, $l\geqq 0$ and $0\leqq m<p^v(p-1)$. Then*

$$\theta(a;\, v+1)=\theta(a-p^v,\, p^v;\, 0,\, v+1)=p^l\theta'(a;\, v)$$

for some integer $\theta'(a;\, v)=\theta'(a-p^v;\, 0,\, v,\, v)$, and

$$\theta'(a;\, v)\equiv(-1)^l \mod p \quad \text{if} \quad m\geqq p^v(p-2).$$

PROOF. (i) is clear by the definition (4.2). We see the first equality in (ii) by Lemma 4.4 (i), and so (ii) by the above lemma. q.e.d.

§5. Proof of Theorem 1.9 for $n \geqq p^r-1$

Now, we study the reduced J-group $\tilde{J}(L_0^n(p^r))$.

We have immediately the following lemma from Proposition 1.3 and (1.2).

LEMMA 5.1. (i) $Jr\colon \tilde{K}(L_0^n(p^r))\ (=\tilde{K}(L^n(p^r)))\to \tilde{J}(L_0^n(p^r))$ *is epimorphic.*
(ii) *The abelian group* $\tilde{J}(L_0^n(p^r))$ *is generated by the elements*

$$\alpha_s = Jr(\sigma(s)) = Jr(\eta^{p^s})-2, \qquad 0 \leqq s < r,$$

of (1.8), *and has the relations*

$$(5.2) \qquad Jr(\sigma^d\sigma(s)) = (-1)^d\alpha_s \quad \textit{for} \quad 0\leqq s<r,\ 0\leqq d<p^s(p-1),$$

in addition to the Jr*-images of the relations in* $\tilde{K}(L_0^n(p^r))$.

We notice that (5.2) holds also for $s\geqq r$, since $\sigma(s)=(1+\sigma)^{p^s}-1=\eta^{p^s}-1=0$ by (1.2) and so $\alpha_s=Jr(\sigma(s))=0$ if $s\geqq r$.

LEMMA 5.3. *The following relations hold in* $\tilde{J}(L_0^n(p^r))$:

$$(5.4) \qquad Jr(\sigma^a\sigma(u)^b)$$
$$= (-1)^{a+b}\sum_{v=0}^{r-1}\sum_{i=0}^{\infty}(-1)^i\sum_{c=0}^{b}\binom{a}{ip^{v+1}-cp^u}\binom{b}{c}(\alpha_{v+1}-\alpha_v)$$

for $0\leqq u<r$, $a+bp^u>0$, *where* $\alpha_r=0$ *and the coeffiient of* $\alpha_{v+1}-\alpha_v$ *is the integer* $\theta(a, b; u, v+1)$ *of* (4.1).

PROOF. *The case* $b=0$. It is sufficient to show that

$$(5.5) \quad Jr(\sigma^a) = (-1)^{a+1}\alpha_s+(-1)^a\sum_{v=0}^{s-1}\theta(a;\, v+1)(\alpha_{v+1}-\alpha_v) \qquad \text{for} \quad 0<a<p^{s+1},$$

by Lemma 4.8 (i) and the above notice, where $\theta(a;\, v+1)$ is the integer of (4.2).

We prove (5.5) by the induction on a. If $a<p$, then (5.5) is (5.2) for $s=0$. Assume $p^s\leqq a<p^{s+1}$ and set $d=a-p^s$. By applying Jr to $\sigma^a-\sigma^d\sigma(s)=-\sigma^d((1+\sigma)^{p^s}-1-\sigma^{p^s})$, (5.2) and the inductive assumption show that

$$Jr(\sigma^a)-(-1)^d\alpha_s = -\sum_{j=1}^{p^s-1}\binom{p^s}{j}Jr(\sigma^{d+j}) = (-1)^d\sum_{j=1}^{p^s-1}(-1)^j\binom{p^s}{j}\alpha_s$$
$$-(-1)^d\sum_{v=0}^{s-1}\sum_{j=1}^{p^s-1}(-1)^j\binom{p^s}{j}\theta(d+j;v+1)\,(\alpha_{v+1}-\alpha_v)\,.$$

Clearly the first summation is 0. Hence we have (5.5) by Lemma 4.3 (ii).

The case $b>0$. We can prove (5.4) by the induction on b, using the equality $\sigma^a\sigma(u)^b=\sigma^a((1+\sigma)^{p^u}-1)\sigma(u)^{b-1}$ and Lemma 4.3 (i). q.e.d.

Now, for a given integer n, we consider the integers a_s and b_s of (1.5), and the following assumption:

$$(5.6) \qquad n\geqq p^r-1, \quad 0\leqq b_s\leqq p^s(p-2) \qquad \text{for} \quad 0\leqq s<r,$$

where the condition $b_0 \leqq p-2$ holds always by (1.5).

LEMMA 5.7. *Under the assumption* (5.6), *the following hold in* $\tilde{J}(L_0^n(p^r))$:

(i) $$p^{r-s-1+a_s}\alpha_s = 0 \qquad \text{for} \quad 0 \leqq s < r.$$

(ii) $$p^{r-s-1+a_s} Jr\sigma(s, d) = (-1)^d \sum_{v=0}^{s-1} p^{r-s-1+a_v+k'}\theta'\alpha_v,$$

$$k' = k'(d;\, v,\, s) = [d/p^v(p-1)] - [b_s/p^v(p-1)],$$

for $0 \leqq s < r$ *and* $b_s \leqq d < b_s + p^s - 1$, *where* $\sigma(s, d) \in \tilde{K}(L_0^n(p^r))$ *is the element of* (1.6) *and* $\theta' = \theta'(d;\, v,\, v,\, s)$ *is the integer of Lemma* 4.5 (i).

PROOF. (i) By (1.5), (1.6) and the assumption (5.6), we see easily that

$$\sigma(s, d_0) = \sigma^{d_0}\sigma(s), \; t(d_0 + p^s) = p^{r-s-1+a_s} \qquad \text{for} \quad d_0 = p^s(p-1)-1.$$

Hence we have (i) by Theorem 1.7 and (5.2).

(ii) By (1.6) and (5.4), the left hand side is equal to

$$(-1)^{d+1} \sum_{t=0}^{s} p^{r-s-1+a_s p^t} \sum_{v=0}^{r-1} \theta(d,\, p^t;\, s-t,\, v+1)(\alpha_{v+1} - \alpha_v).$$

In the summation $\sum_{v=0}^{r-1}$, we see that $\sum_{v=0}^{s-t-1} = 0$ by Lemma 4.4 (ii) and $\sum_{v=s}^{r-1} = -\alpha_s$ by Lemma 4.4 (iii), and so $p^{r-s-1+a_s p^t} \sum_{v=s}^{r-1} = 0$ by (i). Hence, by Lemma 4.5 (i), the above one is equal to

$$(*) \qquad (-1)^{d+1} \sum_{v=0}^{s-1} \sum_{t=s-v}^{s} p^{r-s-1+a_s p^t + k(d;v,s)}\, \theta'(d;\, s-t,\, v,\, s)(\alpha_{v+1} - \alpha_v).$$

We notice the following equalities, which are seen easily by the definition (1.5):

$$(5.8) \qquad \begin{aligned} a_u &= a_s p^{s-u} + (p^{s-u}-1)/(p-1) + [b_s/p^u(p-1)], \\ b_s &= [b_s/p^u(p-1)]p^u(p-1) + b_u, \end{aligned} \qquad \text{for} \quad 0 \leqq u \leqq s.$$

By the first equality of (5.8) and the definition of $k(d;\, v,\, s)$ in Lemma 4.5 (i), we see that

$$a_s p^t + k(d;\, v,\, s) = a_v + k'(d;\, v,\, s) + a_s(p^t - p^{s-v}),$$

which is larger than $a_v + s - v$ if $t > s - v$. In the same way, we see $a_s p^t + k(d;\, v,\, s) \geqq a_{v+1} + s - v - 1$. Therefore $(*)$ is equal to the right hand side of the desired relation by (i). q.e.d.

LEMMA 5.9. *Let* $t < r$. *Then, under the assumption* (5.6), *the relations*

$$t(d + p^s) Jr\sigma(s, d) = 0 \qquad \text{for} \quad 0 \leqq s \leqq t, \quad 0 \leqq d < p^s(p-1),$$

which are the Jr-images of the relations in Theorem 1.7, *are reduced to the* *relations*

$$p^{r-t-1+a_s}\alpha_s = 0 \qquad \textit{for} \quad 0 \leqq s \leqq t.$$

PROOF. If $t=0$, then the lemma is clear by (5.2) and Lemma 5.7 (i) for $s=0$.

We prove the lemma by the induction on t. By (1.5), (1.6), (5.2) and the above lemma, it is sufficient to show the last relations of the lemma from the relations

$$(*) \qquad p^{r-t+a_s}\alpha_s = 0 \quad \text{for} \quad 0 \leqq s < t, \quad p^{r-t-1+a_t}\alpha_t = 0,$$

$$(**) \qquad \textstyle\sum_{v=0}^{t-1} p^{r-t-1+a_v+k'(d;v,t)}\,\theta'(d;\, v,\, v,\, t)\alpha_v = 0 \qquad \text{for} \quad b_t \leqq d < b_t + p^t - 1.$$

Assume that the relations $p^{r-t-1+a_s}\alpha_s=0$ for $u<s\leqq t$ are obtained from $(*)$ and $(**)$, where $u<t$. Then $\sum_{v>u}=0$ in $(**)$. We consider the integer

$$d_0 = (l+1)p^u(p-1)-1, \qquad l = [b_t/p^u(p-1)].$$

Since $b_t = lp^u(p-1)+b_u$ by (5.8), we see easily that $b_t \leqq d_0 < b_t + p^t - 1$ and $k'(d_0;\, u,\, t)=0$. Also $k'(d_0;\, v,\, t) \geqq 1$ if $v<u$, since $b_u \leqq p^u(p-2)$ by (5.6). Further, since $\theta'(d_0;\, u,\, u,\, t) \equiv \pm 1 \bmod p$ by Lemma 4.5 (ii), we see that the relation for $d=d_0$ in $(**)$ and the relations $(*)$ imply $p^{r-t-1+a_u}\alpha_u=0$. Thus we have the desired result by the induction on u. q.e.d.

LEMMA 5.10. (i) *If* $b_s \leqq p^s(p-2)$ *and* $s>0$, *then* $a_{s-1} \not\equiv 0 \bmod p$.

(ii) $b_s \leqq p^s(p-2)$ *for* $0<s\leqq t$ *if and only if* $a_s \not\equiv 0 \bmod p$ *for* $0 \leqq s < t$.

(iii) *The conditions* (1.10) *and* (1.12) *are equivalent.*

PROOF. (i) is shown by the equality $a_{s-1} = a_s p + 1 + [b_s/p^{s-1}(p-1)]$ in (5.8).

The necessity of (ii) follows from (i), and (iii) follows from (ii).

The sufficiency of (ii) is proved by the induction on $t>0$ as follows: Assume that $b_s \leqq p^s(p-2)$ for $0<s<t$ and $a_{t-1} \not\equiv 0 \bmod p$. Then we see that $[b_t/p^{t-1}(p-1)] \leqq p-2$ by the above equality, and hence $b_t \leqq (p-2)p^{t-1}(p-1)+b_{t-1} \leqq p^t(p-2)$ by the second equality in (5.8). q.e.d.

By Lemmas 5.1 and 5.9 for $t=r-1$, we have immediately the following lemma, which is Theorem 1.9 for $n \geqq p^r-1$ by (iii) of the above lemma.

LEMMA 5.11. *If* (5.6) *holds, i.e., if* $r(n) \geqq r$ *and* (1.10) *holds, then*

$$\tilde{J}(L_0^n(p^r)) = \textstyle\sum_{s=0}^{r-1} Z(p^{a_s}) \quad \textit{(direct sum)},$$

where the cyclic subgroup $Z(p^{a_s})$ *is generated by* α_s.

§6. Proofs of Theorem 1.9 and Proposition 1.15

To study the case that (5.6) does not hold, we study the relation between $\tilde{J}(L_0^n(p^r))$ and $\tilde{J}(L_0^{n-1}(p^r))$.

By the definition of the lens space $L^n(p^r)$ and its subspace $L_0^n(p^r)$, it is easy to see that we have the cofibering

$$(6.1) \qquad L_0^{n-1}(p^r) \xrightarrow{i} L_0^n(p^r) \xrightarrow{\pi} L_0^n(p^r)/L_0^{n-1}(p^r) = S^{2n-1} \cup_{p^r} e^{2n}.$$

LEMMA 6.2. *The J-group of the mapping cone $X=S^{2n-1}\cup_{p^r}e^{2n}$, of the map $S^{2n-1}\to S^{2n-1}$ of degree p^r, is given by*

$$\tilde{J}(X) = \begin{cases} Z(p^{\min\{r,s+1\}}) & \text{if } n = ap^s(p-1),\ (a, p) = 1,\ s \geqq 0, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Since the sequence $\widetilde{KO}(S^{2n}) \xrightarrow{\times p^r} \widetilde{KO}(S^{2n}) \to \widetilde{KO}(X) \to 0$ is exact, we have the exact sequence

$$\tilde{J}(S^{2n}) \xrightarrow{\times p^r} \tilde{J}(S^{2n}) \longrightarrow \tilde{J}(X) \longrightarrow 0$$

by [2, II, (3.12)] and [9]. Also, by [2, II, (3.5), (3.7)], $\tilde{J}(S^{2n})$ is a finite cyclic group, and the exponent to which the odd prime p occurs in the decomposition of its order into prime powers is equal to $1+s$ if $n=ap^s(p-1)$ and $(a, p)=1$, or 0 otherwise. Hence we have the lemma by the above exact sequence. q.e.d.

PROPOSITION 6.3. (i) *The cofibering* (6.1) *induces the exact sequence*

$$0 \longrightarrow \tilde{J}(S^{2n-1} \cup_{p^r} e^{2n}) \xrightarrow{\pi^*} \tilde{J}(L_0^n(p^r)) \xrightarrow{i^*} \tilde{J}(L_0^{n-1}(p^r)) \longrightarrow 0.$$

(ii) *If $n \not\equiv 0 \mod p-1$, then i^* is isomorphic.*

(iii) *If $n=ap^s(p-1)$ and $(a, p)=1$, then* $\operatorname{Ker} i^*$ *is the cyclic subgroup of $\tilde{J}(L_0^n(p^r))$ of order $p^{\min\{r,s+1\}}$ generated by the element $Jr(\sigma^n)$.*

(iv) *The order of $\tilde{J}(L_0^n(p^r))$ is equal to p^ν, $\nu=\sum_{s=0}^{r-1}[n/p^s(p-1)]$.*

PROOF. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{K}(X) & \xrightarrow{\pi^*} & \tilde{K}(L_0^n(p^r)) & \xrightarrow{i^*} & \tilde{K}(L_0^{n-1}(p^r)) & \longrightarrow & 0 \\
 & & \downarrow r & & \downarrow r & & \downarrow r & & \\
 & & \widetilde{KO}(X) & \xrightarrow{\pi^*} & \widetilde{KO}(L_0^n(p^r)) & \xrightarrow{i^*} & \widetilde{KO}(L_0^{n-1}(p^r)) & \longrightarrow & 0 \\
 & & \downarrow J & & \downarrow J & & \downarrow J & & \\
 & & \tilde{J}(X) & \xrightarrow{\pi^*} & \tilde{J}(L_0^n(p^r)) & \xrightarrow{i^*} & \tilde{J}(L_0^{n-1}(p^r)) & \longrightarrow & 0,
\end{array}$$

where $X=S^{2n-1}\cup_{p^r} e^{2n}$. The upper two sequences are exact and so is the lowest one by [2, II, (3.12)] and [9]. Since Jr is epimorphic and $\pi^*(\tilde{K}(X))$ is generated by σ^n, by the proof of [5, Prop. 2.6], we see that Ker i^* is generated by $Jr(\sigma^n)$ in (iii).

Also, the lowest exact sequence and the above lemma show (ii) and that

(iii)$'$ the order of Ker i^* is a divisor of $p^{\min\{r,s+1\}}$ in (iii).

By considering the epimorphism $i^*\colon \tilde{J}(L_0^n(p^r))\to\tilde{J}(L_0^0(p^r))=0$ induced by the inclusion $i\colon *=L_0^0(p^r)\subset\cdots\subset L_0^{n-1}(p^r)\subset L_0^n(p^r)$, and by using (ii) and (iii)$'$ iteratedly, we see that

(iv)$'$ the order of $\tilde{J}(L_0^n(p^r))$ is a divisor of p^ν in (iv).

On the other hand, (iv) holds for $n=ap^{r-1}(p-1)+p^{r-1}-1$, $a\geqq 1$, by (5.8) and Lemma 5.11. Therefore, we see that the order of Ker i^* is equal to $p^{\min\{r,s+1\}}$ in (iii)$'$, and so (iii), (i) and (iv). q.e.d.

Proof of Theorem 1.9. For the case that $n\geqq p^r-1$, i.e., $r(n)\geqq r$, the theorem is proved in Lemma 5.11.

Assume that $r(n)<r$. Consider the natural projection $\phi\colon (L^n(p^{r(n)}), L_0^n(p^{r(n)}))\to(L^n(p^r), L_0^n(p^r))$, and the induced homomorphism

$$\phi^*\colon \tilde{J}(L_0^n(p^r))\longrightarrow \tilde{J}(L_0^n(p^{r(n)}))\,.$$

Since $\phi^*\eta=\eta$ in $K(\ )$ by definition, we see that $\phi^*\alpha_s=\alpha_s$ $(0\leqq s<r)$ and ϕ^* is epimorphic by Lemma 5.1 (ii). Also, since $n<p^{r(n)}(p-1)$ by the assumption $b_{r(n)}\leqq p^{r(n)}(p-2)$, we see that the orders of these groups coincide by (iv) of the above proposition. Therefore ϕ^* is isomorphic, and we see the theorem by Lemma 5.11 for $L_0^n(p^{r(n)})$. q.e.d.

For the case that the assumption (1.10) or (1.12) does not hold, we take the integer ρ, $0<\rho\leqq R(n)$, of (1.13) satisfying

$$b_\rho > p^\rho(p-2),\quad b_s\leqq p^s(p-2)\quad\text{if}\quad \rho<s\leqq R(n), \tag{6.4}$$

and consider the integer $\tilde{n}$ of (1.14) and the integers $\tilde{a}_s$ and $\tilde{b}_s$ of (1.5) for $\tilde{n}$:

$$\begin{aligned}\tilde{n} &= n+p^\rho(p-1)-b_\rho=(a_\rho+1)p^\rho(p-1)+p^\rho-1,\\ \tilde{n}-p^s+1 &= \tilde{a}_sp^s(p-1)+\tilde{b}_s,\quad 0\leqq\tilde{b}_s<p^s(p-1).\end{aligned} \tag{6.5}$$

Lemma 6.6. (i) *$\tilde{n}=p^{\rho+1}-1$ if $\rho=R(n)<r-1$, and $R(\tilde{n})=R(n)$ otherwise.*
(ii) *If $0\leqq s\leqq\rho$, then $\tilde{b}_s=0$ and*

$$\tilde{a}_s=(a_\rho+1)p^{\rho-s}+(p^{\rho-s}-1)/(p-1)=a_s+p^{\rho-s}-[b_\rho/p^s(p-1)]\,,$$

and hence $\tilde{a}_s-a_s\leqq j$ if and only if $\tilde{a}_0-a_0\leqq jp^s$.
(iii) *$\tilde{a}_s=a_s$, $\tilde{b}_s=b_s+\tilde{n}-n<p^s(p-2)$, if $\rho<s\leqq R(n)$.*

(iv) *If* $n<m\leqq\tilde{n}$ *and* $m\equiv 0 \bmod p-1$, *then there is an integer* i *such that*

$$m = \tilde{n}-i(p-1), \quad 0\leqq i<\tilde{a}_0-a_0 = p^\rho-[b_\rho/(p-1)]\leqq (p^\rho-1)/(p-1).$$

PROOF. (i) and (ii) are shown easily by (6.5), (6.4), (1.5), (1.10) and (5.8).

(iii) Assume that $\rho<s\leqq R(n)$. Then the two equalities

$$b_s = xp^\rho(p-1)+b_\rho \qquad\qquad (x = [b_s/p^\rho(p-1)]),$$

$$p^s(p-2) = yp^\rho(p-1)+p^\rho(p-2) \quad (y = (p-2)(p^{s-\rho}-1)/(p-1)),$$

and (6.4) show that $x<y$ and hence $b_s<p^s(p-2)-p^\rho(p-2)\leqq p^s(p-2)-p^\rho <p^s(p-2)-(\tilde{n}-n)$, and we see (iii).

(iv) The last inequality follows from the assumption $b_\rho>p^\rho(p-2)$ in (6.4). q.e.d.

Now, we see Proposition 1.15 by Proposition 6.3 (iv) and the following

PROPOSITION 6.7. *Under the situations of* (6.4) *and* (6.5), $\tilde{J}(L_0^n(p^r))$ *is the abelian group generated by* α_s, $0\leqq s\leqq R(n)$, *with the relations*

(6.8.1) $$p^{\tilde{a}_s}\alpha_s = 0, \qquad \textit{for} \quad 0\leqq s\leqq R(n),$$

(6.8.2) $$\textstyle\sum_{s=0}^{\rho}\theta(n_i;\, s+1)\alpha_s = 0, \qquad \textit{for} \quad n_i = \tilde{n}-i(p-1), \quad 0\leqq i<\tilde{a}_0-a_0.$$

Here $\theta(n_i;\, s+1)$ *is the integer of* (4.2) *and satisfies*

(6.9) $$\theta(n_i;\, s+1) = p^{\tilde{a}_s-1-i_s}\theta'(n_i;\, s) \qquad \textit{if} \quad i = i_sp^s+j_s, \quad 0\leqq j_s<p^s,$$

where $\theta'(n_i;\, s)$ *is the integer in Lemma* 4.8 (ii).

PROOF. By (i)–(iii) of the above lemma, we see that (1.12) holds for $\tilde{n}$, and hence that $\tilde{J}(L_0^{\tilde{n}}(p^r))$ is the direct sum of the cyclic subgroups $Z(p^{\tilde{a}_s})$ generated by α_s, $0\leqq s\leqq R(n)$, by Theorem 1.9. Also, if $i=i_sp^s+j_s$, $0\leqq j_s<p^s$, then

(6.10) $$n_i = \tilde{n}-i(p-1) = (\tilde{a}_s-1-i_s)p^s(p-1)+p^s+(p^s-j_s)(p-1)-1$$

by (6.5), and (6.9) is seen by Lemma 4.8 (ii). Therefore, by Proposition 6.3 and (iv) of the above lemma, it is sufficient to show the equality

$$Jr(\sigma^{n_i}) = -\textstyle\sum_{s=0}^{\rho}p^{\tilde{a}_s-1-i_s}\theta'(n_i;\, s)\alpha_s, \qquad \text{for} \quad n<n_i\leqq\tilde{n}.$$

Assume that $n<n_i\leqq\tilde{n}$. Then by (5.5) and (6.9), we have

$$Jr(\sigma^{n_i}) = \textstyle\sum_{s=0}^{R(n)}p^{\tilde{a}_s-1-i_s}\theta'(n_i;\, s)(\alpha_{s+1}-\alpha_s).$$

In this equality, we see that $\tilde{a}_s-1-i_s\geqq a_s$ by (6.10) and (1.5) since $n_i>n$. Also

$$a_s = \tilde{a}_s \quad \text{if} \quad s>\rho; \quad a_s\geqq a_{s+1} = \tilde{a}_{s+1} \quad \text{if} \quad s\geqq\rho,$$

$$a_s \geqq a_\rho p^{\rho-s}+(p^{\rho-s}-1)/(p-1) \geqq \tilde{a}_{s+1} \quad \text{if} \quad \rho > s \geqq 0,$$

by (iii) and (ii) of the above lemma and (5.8). Hence we see the desired equality by (6.8.1). q.e.d.

LEMMA 6.11. *In* (6.9),

$$\theta'(n_i;\, s) \equiv (-1)^{\tilde{a}_s-1-i_s} \bmod p, \qquad \text{if} \;\; j_s \leqq (p^s-1)/(p-1).$$

PROOF. We have the lemma by the latter half of Lemma 4.8 (ii) and (6.10). q.e.d.

LEMMA 6.12. *Assume* $\rho=1$ *in* (6.4). *Then* $\tilde{a}_1=a_1+1$, $\tilde{a}_0=a_0+1$, *and the relations* (6.8.1–2) *are reduced to the relations*

$$p^{\tilde{a}_s}\alpha_s = 0 \; (0 \leqq s \leqq R(n)), \qquad p^{a_1}\alpha_1 - p^{a_0}\alpha_0 = 0.$$

PROOF. The first two equalities are seen by Lemma 6.6 (ii) and (iv). Hence (6.8.2) contains only one relation for $i=0$, which is reduced to $p^{a_1}\alpha_1-p^{a_0}\alpha_0=0$ under (6.8.1) by the above lemma. q.e.d.

The following is the result for the case $r=2$.

PROPOSITION 6.13 (cf. [6, Th. 6.9]*). *Let* a_0, a_1 *be the integers in* (1.5), *and consider* $a_1=0$ *if* $a_0=0$. *Then*

$$\tilde{J}(L_0^n(p^2)) = \begin{cases} Z(p^{a_0}) \oplus Z(p^{a_1}) & \text{if} \;\; a_0 \not\equiv 0 \bmod p \;\; \text{or} \;\; a_0 = 0, \\ Z(p^{a_0+1}) \oplus Z(p^{a_1}) & \text{if} \;\; a_0 = (a_1+1)p, \end{cases}$$

and the first summand is generated by α_0 *and the second one by* $\alpha_1-p^{a_0-a_1}\alpha_0$, *which can be replaced by* α_1 *for the upper case.*

PROOF. By Lemma 5.10 (ii), the condition $b_1>p(p-2)$ of (6.4) holds if and only if $a_0\equiv 0 \bmod p$ and $a_0>0$. Therefore, we have the result for the lower case by Proposition 6.7 and the above lemma, and the one for the upper case by Theorem 1.9. q.e.d.

§7. The case that $r\geqq 3$

For the case that $r\geqq 3$, we study the integer θ of (4.2) more precisely.

LEMMA 7.1. *Assume that* $l>0$ *is a multiple of* p^λ, $\lambda\geqq 1$. *Then*

(i) $(1+x)^{lp^\nu(p-1)} \equiv (-1)^l p^{l-1}(p-(1+y^p)/(1+y))+p^{l+\lambda}C(x) \bmod 1+y^p$,

* The proof of this result in [6] is not complete, since the last equality in [6, Lemma 6, 7] is not valid.

where $y=x^{p^v}$ *and* $C(x)$ *is some polynomial in* x.

(ii) *If* $b=a-lp^v(p-1)\geqq 0$ *in addition, then*

$$\theta(a;\ v+1)\equiv(-1)^l p^{l-1}(p\theta(b;\ v+1)-\theta(b;\ v)) \bmod p^{l+\lambda}.$$

PROOF. (i) is shown by the following calculations, where $\equiv$ means $\equiv$ mod $1+y^p$ and B, C' and C'' are some polynomials in x.

$$(1+x)^{lp^v(p-1)}$$

$$=(1+y+pB(x))^{l(p-1)}=\sum_{i=0}^{l(p-1)}\binom{l(p-1)}{i}p^i\,(1+y)^{l(p-1)-i}B(x)^i$$

$$\equiv p^{l-1}(1+y)^{p-1}Q(y)^{l-1}+p^{l+\lambda}C'(x)\quad\text{(by (4.7) and the assumption } p^\lambda|l),$$

$$(1+y)^{p-1}Q(y)^{l-1}$$

$$=(-1)^{l-1}(1+y)^{p-1}+\sum_{i=1}^{l-1}(-1)^{l-1-i}\binom{l-1}{i}(1+y)^{p-1+i}Q'(y)^i\quad\text{(by (4.6))}$$

$$\equiv(-1)^{l-1}(1+y)^{p-1}+pQ(y)(Q(y)^{l-1}-(-1)^{l-1})\quad\text{(by (4.7) and (4.6))}$$

$$\equiv(-1)^l(p-(1+y^p)/(1+y))+p^{\lambda+1}C''(x)\quad\text{(by (4.6) and } p^\lambda|l).$$

(ii) By Lemma 4.4 (i), it is easy to see that $\theta(b;\ v)$ is equal to the constant term q'_0 of the right hand side of

$$(1+x)^b(1+y^p)/(1+y)\equiv\sum_{i=0}^{p^{v+1}-1}q'_i x^i \bmod 1+y^p.$$

Therefore, (ii) is shown by Lemma 4.4 (i) and the equality of (i) multiplied by $(1+x)^b$.

LEMMA 7.2. *Suppose* $p^\lambda|\tilde{a}_s-i_s$, $\lambda\geqq 1$ *and* $j_s\leqq(p^s-1)/(p-1)$. *Then in* (6.9),

$$\theta'(n_i;\ s)\equiv(-1)^{\tilde{a}_s-i_s-1}(1-p) \bmod p^{\lambda+1}.$$

PROOF. By (ii) of the above lemma, (6.10) and the assumption, we see that

$$\theta(n_i;\ s+1)\equiv(-1)^{\tilde{a}_s-i_s}p^{\tilde{a}_s-i_s-1}(p\theta(b;\ s+1)-\theta(b;\ s)) \bmod p^{\tilde{a}_s-i_s+\lambda},$$

where $b=p^s-j_s(p-1)-1$. This implies the desired equality by (6.9) and the second equality of Lemma 4.8 (i), since $b<p^s$. q.e.d.

LEMMA 7.3. *Suppose that* $p^\lambda|\tilde{a}_s-i_s-1$, $\lambda\geqq 1$. *Then the following hold in* (6.9):

(i) $$\theta'(n_i;\, s) \equiv (-1)^{\tilde{a}_s - i_s - 1}(1-\theta(b;\, s)/p) \bmod p^{\lambda},$$
$$b = p^s + (p^s - j_s)(p-1) - 1.$$

(ii) *If* $s=0$ *in addition, then* $\theta'(n_i;\, 0) \equiv (-1)^{\tilde{a}_0 - i - 1} \bmod p^{\lambda}$.

(iii) *Suppose* $s \geqq 1$ *in addition, and set* $j_s = k_s p^{s-1} + j_{s-1}$, $i_{s-1} = i_s p + k_s$. *Then* $\theta(b;\, s)/p$ *in* (i) *satisfies the following properties*:

$$\theta(b;\, s)/p \equiv \begin{cases} 0 \bmod p^{p-k_s-1}, \\ (-1)^{k_s-1} p^{p-k_s-1} \bmod p^{p-k_s}, & \textit{if } j_{s-1} \leqq (p^{s-1}-1)/(p-1). \end{cases}$$

(iv) *If* $k_s = p-1$ *in* (iii), *then* $b = p^s + (p^{s-1} - j_{s-1})(p-1) - 1$ *and*

$$\theta'(n_i;\, s-1) \equiv (-1)^{\tilde{a}_s - i_s - 1}(\theta(b;\, s)/p - \theta(b;\, s-1)/p^2) \bmod p^{\lambda},$$
$$\theta(b;\, s-1)/p^2 = 0 \quad \textit{if} \quad s = 1, \qquad \equiv 0 \bmod p^{2p-2-k_{s-1}} \quad \textit{if} \quad s \geqq 2.$$

Proof. (i) By Lemma 7.1 (ii), (6.10) and the assumption, we see that

$$\theta(n_i;\, s+1) \equiv (-1)^l p^{l-1}(p\theta(b;\, s+1) - \theta(b;\, s)) \bmod p^{l+\lambda}, \quad l = \tilde{a}_s - 1 - i_s.$$

This shows (i) by (6.9) and the second equality in Lemma 4.8 (i), since $b < p^{s+1}$.

(ii) follows from (i) and the first equality in Lemma 4.8 (i).

(iii) is a consequence of Lemma 4.8 (ii), since $b = (p-k_s)p^{s-1}(p-1) + p^{s-1} + (p^{s-1} - j_{s-1})(p-1) - 1$.

(iv) By Lemma 7.1 (ii), (6.10) and the assumption, we see that

$$\theta(n_i;\, s) \equiv (-1)^{lp} p^{lp-1}(p\theta(b;\, s) - \theta(b;\, s-1)) \bmod p^{lp+\lambda+1},$$

where $lp+1 = \tilde{a}_{s-1} - 1 - i_{s-1}$ since $k_s = p-1$ and $\tilde{a}_{s-1} = p\tilde{a}_s + 1$. Hence we have (iv) by (6.9) and Lemma 4.8. q. e. d.

Lemma 7.4. *Assume* $\rho = 2$ *in* (6.4). *Then*

(i) $$\tilde{a}_s = a_s \quad \textit{if} \quad s > 2, \qquad \tilde{a}_2 = a_2 + 1, \qquad \tilde{a}_0 - a_0 \leqq p+1,$$
$$\tilde{a}_1 = a_1 + 1 \quad \textit{if} \quad \tilde{a}_0 - a_0 \leqq p, \quad = a_1 + 2 \quad \textit{if} \quad \tilde{a}_0 - a_0 = p+1.$$

(ii) *Under the relations* (6.8.1), *the relations in* (6.8.2) *for* $0 \leqq i \leqq q = q_1 p + q_0$ $(0 \leqq q_0 < p)$ *are reduced to the following relations*:

$$p^{\tilde{a}_2-1}\alpha_2 - p^{\tilde{a}_1-1}\alpha_1 + p^{\tilde{a}_0-1}\alpha_0 = 0, \qquad \textit{if} \quad q = 0,$$
$$p^{\tilde{a}_2-1}\alpha_2 - p^{\tilde{a}_1-1}\alpha_1 = p^{\tilde{a}_0-q-1}\alpha_0 = 0, \qquad \textit{if} \quad 0 < q < p-1,$$
$$p^{\tilde{a}_2-1}\alpha_2 - p^{\tilde{a}_1-1}\alpha_1 = p^{\tilde{a}_1-q_1-1}\alpha_1 - p^{\tilde{a}_0-q-1}\alpha_0 = p^{\tilde{a}_0-p+1}\alpha_0 = 0,$$
$$\textit{if} \quad q = p-1,\, p.$$

PROOF. (i) The results are seen easily by Lemma 6.6.

(ii) If $q=0$, the relation is obtained by (6.9) and Lemma 6.11. We can prove the desired results successively for $q\geqq 1$, showing that the relation for $i=q$ in (6.8.2) is reduced to the following under the relations (6.8.1) and the desired one for $q-1$, by using (6.9), Lemmas 6.11, 7.2 and 7.3 (ii), (iii):

$$p^{\tilde{a}_2-1}\alpha_2-p^{\tilde{a}_1-1}\alpha_1-p^{\tilde{a}_0-2}(1-p)\alpha_0=0, \qquad \text{if } q=1,$$

$$p^{\tilde{a}_2-1}\alpha_2-p^{\tilde{a}_1-1}\alpha_1+(-1)^q p^{\tilde{a}_0-q-1}\alpha_0=0, \qquad \text{if } 1<q<p-1,$$

$$p^{\tilde{a}_2-1}\alpha_2-2p\alpha_1+p^{\tilde{a}_0-p}\alpha_0=0, \qquad \text{if } q=p-1,$$

$$p^{\tilde{a}_2-1}\alpha_2+(1-p)p^{\tilde{a}_1-2}\alpha_1-p^{\tilde{a}_0-p-1}\alpha_0=0, \qquad \text{if } q=0. \qquad \text{q.e.d.}$$

The following is the result for the case $r=3$.

PROPOSITION 7.5. *Let a_s be the integers in* (1.5) *and consider $a_s=0$ if $s>r(n)$ and $a_{r(n)}=0$. Then $\tilde{J}(L_0^n(p^3))$ is the direct sum*

(1) $Z(p^{a_0})\oplus Z(p^{a_1})\oplus Z(p^{a_2})$,

if $a_s\not\equiv 0 \bmod p$ *for* $0\leqq s<\min\{2, r(n)\}$;

(2) $Z(p^{a_0+1})\oplus Z(p^{a_1+1})\oplus Z(p^{a_2})$, *if* $a_0=(a_2+1)p^2+p$,

(3) $Z(p^{a_0})\oplus Z(p^{a_1+1})\oplus Z(p^{a_2})$, *if* $(a_2+1)p^2+p>a_0>(a_2+1)p^2+1$,

(4) $Z(p^{a_0+1})\oplus Z(p^{a_1})\oplus Z(p^{a_2})$, *if* $a_0=(a_2+1)p^2+1$,

(5) $Z(p^{a_0+2})\oplus Z(p^{a_1})\oplus Z(p^{a_2})$, *if* $a_0=(a_2+1)p^2$;

(6) $Z(p^{a_0+1})\oplus Z(p^{a_1})\oplus Z(p^{a_2})$, *if* $a_0=(a_1+1)p<(a_2+1)p^2$,

and the first summand is generated by α_0, the second one by $\alpha_1-p^{a_0-a_1}\alpha_0$, which can be replaced by α_1 for (1)–(3), *and the third one by α_2 for* (1) *and* (6), *$\alpha_2-p^{a_1-a_2}\alpha_1+p^{a_0-a_2}\alpha_0$ for* (2), *$\alpha_2-p^{a_1-a_2}\alpha_1$ for* (3) *and* (4) *or $\alpha_2-p^{a_1-a_2+1}\alpha_1$ for* (5), *respectively.*

PROOF. The case (1) is Theorem 1.9. If $\rho=2$ in (6.4), we obtain (2)–(5) by Proposition 6.7 and the above lemma. If $\rho=1$ in (6.4), we obtain (6) by Proposition 6.7 and Lemma 6.12. q.e.d.

LEMMA 7.6. *Assume $\rho=3$ in* (6.4). *Then*

(i) *$\tilde{a}_s=a_s$ if $s>3$, $\tilde{a}_3=a_3+1$, $\tilde{a}_0-a_0\leqq p^2+p+1$, and*

$$\tilde{a}_s-a_s\leqq j \quad \textit{if and only if} \quad \tilde{a}_0-a_0\leqq jp^s \qquad \textit{for} \quad s=1, 2.$$

(ii) *Under the relations* (6.8.1), *the relations* (6.8.2) *for $0\leqq i\leqq q=q_2p^2$*

$+q_1p+q_0$ $(0\leqq q_j<p)$ *are reduced to the following relations, where* $\alpha=p^{\tilde{a}_3-1}\alpha_3-p^{\tilde{a}_2-1}\alpha_2$:

$$\alpha+p^{\tilde{a}_1-1}\alpha_1-p^{\tilde{a}_0-1}\alpha_0=0, \qquad \textit{if} \quad q=0,$$

$$\alpha+p^{\tilde{a}_1-1}\alpha_1=p^{\tilde{a}_0-q-1}\alpha_0=0, \qquad \textit{if} \quad 0<q<p-1,$$

$$\alpha+p^{\tilde{a}_1-1}\alpha_1=p^{\tilde{a}_1-q-1}\alpha_1-p^{\tilde{a}_0-q-1}\alpha_0=0, \qquad \textit{if} \quad q=p-1 \textit{ or } p,$$

$$\alpha=p^{\tilde{a}_1-q_1-1}\alpha_1=p^{\tilde{a}_0-q-1}\alpha_0=0,$$

$$\textit{if} \quad q_2=0, \quad 0<q_1<p-1, \quad 0<q_0<p,$$

$$\alpha=p^{\tilde{a}_1-q_1-1}\alpha_1-p^{\tilde{a}_0-q-1}\alpha_0=p^{\tilde{a}_0-q}\alpha_0=0,$$

$$\textit{if} \quad q_2=0, \quad 1<q_1<p-1, \quad q_0=0,$$

$$\alpha=p^{\tilde{a}_2-q_2-1}\alpha_2-p^{\tilde{a}_1-q_2p-q_1-1}\alpha_1+p^{\tilde{a}_0-q-1}\alpha_0=p^{\tilde{a}_0-q}\alpha_0=0,$$

$$\textit{if} \quad q=(p-1)p \textit{ or } p^2,$$

$$\alpha=p^{\tilde{a}_2-q_2-1}\alpha_2-p^{\tilde{a}_1-q_2p-q_1-1}\alpha_1=p^{\tilde{a}_0-q-1}\alpha_0=0,$$

$$\textit{if} \quad (p-1)p<q<p^2+p-2, \quad q\neq p^2,$$

$$\alpha=p^{\tilde{a}_2-2}\alpha_2-p^{\tilde{a}_1-p-1}\alpha_1=p^{\tilde{a}_1-(q-p^2+2)}\alpha_1-p^{\tilde{a}_0-q-1}\alpha_0=p^{\tilde{a}_0-p^2-p+2}\alpha_0$$

$$=0, \qquad \textit{if} \quad p^2+p-2\leqq q\leqq p^2+p.$$

PROOF. (i) is seen by Lemma 6.6. By (6.9), Lemmas 6.11, 7.2 and 7.3, we can show that the relation for $i=q$ in (6.8.2) is reduced to the following under the relations (6.8.1) and the desired one for $q-1$ (if $q>0$), and hence we see (ii) by induction:

$$\alpha+p^{\tilde{a}_1-1}\alpha_1-p^{\tilde{a}_0-1}\alpha_0=0, \qquad \text{if} \quad q=0,$$

$$\alpha+p^{\tilde{a}_1-1}\alpha_1+p^{\tilde{a}_0-2}(1-p)\alpha_0=0, \qquad \text{if} \quad q=1,$$

$$\alpha+p^{\tilde{a}_1-1}\alpha_1-(-1)^q p^{\tilde{a}_0-q-1}\alpha_0=0, \qquad \text{if} \quad 1<q<p-1,$$

$$\alpha+2p^{\tilde{a}_1-1}\alpha_1-p^{\tilde{a}_0-p}\alpha_0=0, \qquad \text{if} \quad q=p-1,$$

$$\alpha-(1-p)p^{\tilde{a}_1-2}\alpha_1+p^{\tilde{a}_0-p-1}\alpha_0=0, \qquad \text{if} \quad q=p,$$

$$\alpha-(1-p)p^{\tilde{a}_1-2}\alpha_1-(1-p)p^{\tilde{a}_0-p-2}\alpha_0=0, \qquad \text{if} \quad q=p+1,$$

$$-(-1)^q p^{\tilde{a}_0-q-1}\alpha_0=0, \qquad \text{if} \quad p+1<q<(p-1)p,\ q_0\neq 0,\ 1,$$

$$(-1)^{q_1}p^{\tilde{a}_1-q_1-1}\alpha_1-(-1)^q p^{\tilde{a}_0-q-1}\alpha_0=0,$$

$$\text{if} \quad q = q_1 p, \quad 1 < q_1 < p-1,$$

$$(-1)^{q_1} p^{\tilde{a}_1-q_1-1}\alpha_1-(-1)^q(1-p)p^{\tilde{a}_0-q-1}\alpha_0 = 0,$$

$$\text{if} \quad q = q_1 p+1, \quad 1 < q_1 < p-1,$$

$$-p^{\tilde{a}_2-1}\alpha_2+p^{\tilde{a}_1-p}\alpha_1-p^{\tilde{a}_0-q-1}\alpha_0 = 0, \qquad \text{if} \quad q = (p-1)p,$$

$$-p^{\tilde{a}_2-1}\alpha_2+p^{\tilde{a}_1-p}\alpha_1+(1-p)p^{\tilde{a}_0-q-1}\alpha_0 = 0, \quad \text{if} \quad q = (p-1)p+1,$$

$$(\theta(p^2+p-2;\,2)/p)(p^{\tilde{a}_2-1}\alpha_2-p^{\tilde{a}_1-p}\alpha_1)-(-1)^q p^{\tilde{a}_0-q-1}\alpha_0 = 0,$$

$$\text{if} \quad q = (p-1)p+q_0, \quad 1 < q_0 < p,$$

$$p^{\tilde{a}_2-2}\alpha_2-p^{\tilde{a}_1-p-1}\alpha_1+p^{\tilde{a}_0-q-1}\alpha_0 = 0, \qquad \text{if} \quad q = p^2,$$

$$p^{\tilde{a}_2-2}\alpha_2-p^{\tilde{a}_1-p-1}(1-p^{p-2})\alpha_1-(1-p)p^{\tilde{a}_0-q-1}\alpha_0 = 0, \qquad \text{if} \quad q = p^2+1,$$

$$p^{\tilde{a}_2-2}\alpha_2-p^{\tilde{a}_1-p-1}\alpha_1-(-1)^q p^{\tilde{a}_0-q-1}\alpha_0 = 0,$$

$$\text{if} \quad q = p^2+q_0, \quad 1 < q_0 < p-2 \qquad (p>3),$$

$$p^{\tilde{a}_2-2}\alpha_2-(1-p)p^{\tilde{a}_1-p-1}\alpha_1-p^{\tilde{a}_0-q-1}\alpha_0 = 0, \qquad \text{if} \quad q = p^2+p-2 \qquad (p>3),$$

$$p^{\tilde{a}_2-2}\alpha_2-p^{\tilde{a}_1-p-1}\alpha_1-(\theta(2p-2;\,1)/p)(p^{\tilde{a}_1-p-1}\alpha_1-p^{\tilde{a}_0-q-1}\alpha_0) = 0,$$

$$\text{if} \quad q = p^2+p-1,$$

$$p^{\tilde{a}_2-2}\alpha_2+(1-p)p^{\tilde{a}_1-p-2}\alpha_1-p^{\tilde{a}_0-q-1}\alpha_0 = 0, \qquad \text{if} \quad q = p^2+p. \qquad \text{q.e.d.}$$

The following result for $r=4$ is proved by Theorem 1.9, Proposition 6.7 and Lemmas 7.6, 7.4 and 6.12.

PROPOSITION 7.7. *Let a_s be the integers of* (1.5) *and consider $a_s=0$ if $s>r(n)$ and $a_{r(n)}=0$. Also, consider the integer*

$$\tilde{a}_0 = a_\rho p^\rho+(p^{\rho+1}-1)/(p-1) \leqq a_0+(p^\rho-1)/(p-1)$$

of (1.14) *if there exists an integer ρ satisfying* (1.13). *Then*

$$\tilde{J}(L_0^n(p^4)) = Z(p^{a_0+\varepsilon_0}) \oplus Z(p^{a_1+\varepsilon_1}) \oplus Z(p^{a_2+\varepsilon_2}) \oplus Z(p^{a_3}),$$

where

$$\varepsilon_0 = \begin{cases} 3 & \textit{if} \quad \tilde{a}_0-a_0 = p^2+p+1, \\ 2 & \textit{if} \quad \tilde{a}_0-a_0 = p+1,\ p^2+p, \\ 1 & \textit{if} \quad \tilde{a}_0-a_0 = 1,\ p,\ qp+1\ (1 < q \leqq p),\ p^2+p-1, \\ 0 & \textit{otherwise}, \end{cases}$$

$$\varepsilon_1 = \begin{cases} 2 & \text{if} \quad p^2 < \tilde{a}_0 - a_0 \leqq p^2 + p - 2, \\ 1 & \text{if} \quad 0 < \tilde{a}_0 - a_0 < p \ (\rho \geqq 2),\ (p-1)p < \tilde{a}_0 - a_0 \leqq p^2, \\ & \qquad\qquad \tilde{a}_0 - a_0 = p^2 + p - 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$\varepsilon_2 = \begin{cases} 1 & \text{if} \quad 0 < \tilde{a}_0 - a_0 \leqq (p-1)p \ (\rho = 3), \\ 0 & \text{otherwise}. \end{cases}$$

Also, the first summand is generated by α_0; *the second one by*

$$\alpha_1 - p^{a_0 - a_1 - 1}\alpha_0 \quad \text{if} \quad \tilde{a}_0 - a_0 = p^2 + p - 1,$$

$$\alpha_1 - p^{a_0 - a_1}\alpha_0 \quad \text{if} \quad \tilde{a}_0 - a_0 = p,\ qp+1\ (1 \leqq q < p-1),\ p^2+p,\ p^2+p+1, \ \text{or}\ 1\ (\rho = 1),$$

and α_1 *otherwise*; *the third one by*

$$\alpha_2 - p^{a_1 - a_2}\alpha_1 + p^{a_0 - a_2}\alpha_0 \quad \text{if} \quad \tilde{a}_0 - a_0 = (p-1)p+1,\ p^2+1,\ \text{or}\ 1\ (\rho = 2),$$

$$\alpha_2 - p^{a_1 - a_2 + 1}\alpha_1 \quad \text{if} \quad \tilde{a}_0 - a_0 = p^2 + p + 1,\ \text{or}\ \tilde{a}_0 - a_0 = p + 1\ (\rho = 2),$$

$$\alpha_2 - p^{a_1 - a_2}\alpha_1 \quad \text{if} \quad (p-1)p+1 < \tilde{a}_0 - a_0 \leqq p^2,\ p^2+1 < \tilde{a}_0 - a_0 \leqq p^2 + p, \ \text{or}\ 1 < \tilde{a}_0 - a_0 \leqq p\ (\rho = 2),$$

and α_2 *otherwise*; *and the fourth one by*

$$\begin{array}{ll} \alpha_3 - p^{a_2 - a_3}\alpha_2 + p^{a_1 - a_3}\alpha_1 - p^{a_0 - a_3}\alpha_0 & \text{if} \quad \tilde{a}_0 - a_0 = 1, \\ \alpha_3 - p^{a_2 - a_3}\alpha_2 + p^{a_1 - a_3}\alpha_1 & \text{if} \quad 1 < \tilde{a}_0 - a_0 \leqq p, \\ \alpha_3 - p^{a_2 - a_3}\alpha_2 + p^{a_1 - a_3 + 1}\alpha_1 & \text{if} \quad \tilde{a}_0 - a_0 = p + 1, \\ \alpha_3 - p^{a_2 - a_3}\alpha_2 & \text{if} \quad p + 2 \leqq \tilde{a}_0 - a_0 \leqq p^2, \\ \alpha_3 - p^{a_2 - a_3 + 1}\alpha_2 & \text{if} \quad p^2 < \tilde{a}_0 - a_0 \leqq p^2 + p + 1, \end{array}$$

for the case $\rho = 3$, *and* α_3 *for the case* $\rho = 2$, *respectively.*

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