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On Infinite-dimensional Algebras Satisfying the Maximal Condition for Subalgebras

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1.

It has been an open question whether there exists an infinite-dimensional Lie algebra satisfying the maximal condition for subalgebras. Recently in the paper [3], we have given an affirmative answer to this question by showing that the Lie algebra W introduced in [1, p. 177] is such a Lie algebra.

On the other hand, in the paper [2] R. K. Amayo has constructed a countable infinity of pair-wise non-isomorphic Lie algebras satisfying the maximal condition for subalgebras. The reasoning is however much complicated.

Thus in this paper we shall present simple and brief proofs of the results in [2] by reasoning along the same lines as in [3].

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2.

The fundamental tool which we employ is the lemma in [3]. We state it without proof in the following

LEMMA. *Let S be a subset of $\mathbf{N}$ satisfying the condition: If $s, t \in S$ and $s \neq t$, $s+t \in S$. Then there exists a finite number of different elements $s_1, s_2, \ldots, s_r$ of S such that*

(i) *s_1 is the smallest element of S,*

(ii) $S = \{s_1\} \cup \{s_2 + ns_1 \mid n = 0, 1, 2, \ldots\} \cup \cdots \cup \{s_r + ns_1 \mid n = 0, 1, 2, \ldots\}$.

Let $\mathfrak{k}$ be a field of characteristic 0. We let $\lambda\colon \mathbf{Z} \times \mathbf{Z} \to \mathfrak{k}$ be a map such that

$$(*) \qquad i \neq j \Rightarrow \lambda(i, j) \neq 0$$

and let $A(\lambda)$ be the infinite-dimensional (not necessarily associative) algebra over $\mathfrak{k}$ with basis $\{w(i) \mid i \in \mathbf{Z}\}$ and bilinear product defined by

$$w(i) \circ w(j) = \lambda(i, j) w(i+j), \qquad i, j \in \mathbf{Z}.$$

For any non-negative integer n, let $A(\lambda, n)$ be a subalgebra of $A(\lambda)$ generated by

$\{w(i)|i\geqq n\}$. For any element x of $A(\lambda)$, we denote by $\min(x)$ and $\max(x)$ respectively the integer k and m such that

$$x=\sum_{i=k}^{m}\alpha_i w(i), \qquad \alpha_k\alpha_m \neq 0.$$

THEOREM 1. *$A(\lambda, 1)$ satisfies the maximal condition for subalgebras.*

PROOF. Let H be any subalgebra of $A(\lambda, 1)$ and S the set of all $\max(x)$ for $x\in H$. If $s, t\in S$ and $s\neq t$, then $s=\max(x)$ and $t=\max(y)$ for $x, y\in H$ and therefore by $(*)$ $s+t=\max(x\circ y)\in S$. Hence by Lemma there exists a finite number of different elements $s_1, s_2,\dots, s_r$ of S satisfying the conditions (i), (ii) in its statement. For $i=1, 2,\dots, r$, take an element z_i of H such that $\max(z_i)=s_i$. We then assert that $H=\langle z_1,\dots, z_r\rangle$.

Suppose on the contrary that $H\neq\langle z_1,\dots, z_r\rangle$. Take an x in $H\backslash\langle z_1,\dots, z_r\rangle$. We shall find elements x_i in $H\backslash\langle z_1,\dots, z_r\rangle$ for $i\geqq 0$ as follows. Put $x_0=x$. Assume that x_i is already found in $H\backslash\langle z_1,\dots, z_r\rangle$. If $\max(x_i)=s_1$, then there exists a $\beta\in\mathfrak{k}$ such that $\max(x_i-\beta z_1)<s_1$. Since $x_i-\beta z_1\in H$, by the minimality of s_1 we have $x_i-\beta z_1=0$, which is a contradiction. Therefore we have $\max(x_i)\neq s_1$ and

$$\max(x_i)=s_{\mu(i)}+n_i s_1, \qquad \mu(i)\neq 1.$$

By $(*)$ there exists a $\gamma_i\in\mathfrak{k}$ such that

$$\max(x_i-\gamma_i(z_{\mu(i)}\circ_{n_i}(z_1))<\max(x_i).$$

Now we put $x_{i+1}=x_i-\gamma_i(z_{\mu(i)}\circ_{n_i}z_1)$. Then x_{i+1} is in $H\backslash\langle z_1,\dots, z_r\rangle$. Thus we have found a sequence of different elements x_i of $H\backslash\langle z_1,\dots, z_r\rangle$ such that

$$\max(x_0)>\max(x_1)>\max(x_2)>\cdots.$$

This is a contradiction.

Therefore we have $H=\langle z_1,\dots, z_r\rangle$ as asserted. Every subalgebra of $A(\lambda, 1)$ is thus finitely generated. Hence $A(\lambda, 1)$ satisfies the maximal condition for subalgebras.

3.

As a consequence of Theorem 1 we have the following theorem which is Theorem B in [2].

THEOREM 2. *$A(\lambda, n)$ satisfies the maximal condition for subalgebras.*

PROOF. If $n=0$, then every subalgebra H of $A(\lambda, 0)$ is either $H\cap A(\lambda, 1)$ or $H\cap A(\lambda, 1)+\mathfrak{k}x$ for some $x\in H$. By Theorem 1 H is therefore finitely generated. If $n \geqq 1$, then $A(\lambda, n)$ is a subalgebra of $A(\lambda, 1)$. Hence all $A(\lambda, n)$ satisfy the maximal condition for subalgebras.

Theorem C in [2] is a part of the following

COROLLARY. *Over any field of characteristic* 0 *there exists a countable infinity of pair-wise non-isomorphic infinite-dimensional algebras, Lie algebras, Jordan algebras, associative algebras satisfying the maximal condition for subalgebras.*

PROOF. If $\lambda(i, j)=i-j$, $A(\lambda, n)$ is a Lie algebra. Since the dimension of $A(\lambda, n)/(A(\lambda, n)\circ A(\lambda, n))$ is precisely $n+1$, $A(\lambda, n) \not\cong A(\lambda, m)$ for $n \neq m$. Hence the assertion holds for algebras and Lie algebras. By putting $\lambda(i, j)=(i-j)^2$ and $\lambda(i, j)=1$, we have the statement for Jordan and associative algebras respectively.

Finally we show the following theorem which is Theorem A in [2].

THEOREM 3. *$A(\lambda)$ satisfies the maximal condition for subalgebras.*

PROOF. Let H be any subalgebra of $A(\lambda)$ and let $M=H \cap A(\lambda, 0)$. By Theorem 2 M is finitely generated. Put $S=\{-\min(x) | \min(x)<0, x \in H\}$. If $s, t \in S$ and $s \neq t$, then $s=-\min(x)$ and $t=-\min(y)$ for $x, y \in H$ and therefore by $(*)$ $s+t=-\min(x\circ y) \in S$. Hence there exists a finite number of different elements $s_1, s_2, \ldots, s_r$ of S satisfying the condition (i), (ii) in Lemma. For $i=1, 2, \ldots, r$, we take an element z_i of H such that $-\min(z_i)=s_i$. We then have $H=\langle z_1, \ldots, z_r, M\rangle$. This can be proved in the same way as in Theorem 1. So its proof will be omitted. Thus we can conclude that $A(\lambda)$ satisfies the maximal condition for subalgebras.

References

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