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Potential Theoretic Properties of the Subdifferential of a Convex Function

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1. Introduction

Throughout this paper, let X be a locally compact Hausdorff space and ξ be a positive (Radon) measure on X. We denote by $L^2=L^2(X;\xi)$ the Hilbert space of all real-valued square ξ-integrable functions on X with the inner product $(u, v)=\int_X u\cdot v d\xi$ and the norm $\|u\|=\sqrt{(u, u)}$. For measurable functions u and v on X we define $u\vee v=\max\{u, v\}$, $u\wedge v=\min\{u, v\}$, $u^+=u\vee 0$ and $u^-=-(u\wedge 0)$ and simply write "$u\leqq v$" for "$u\leqq v$ ξ-a.e. on X".

Let ϕ be a lower semicontinuous convex function on L^2 with values in $(-\infty, +\infty]$ and proper on L^2, i.e., $\phi\not\equiv+\infty$ on L^2. Then the subdifferential $\partial\phi$ of ϕ is the (multivalued) operator in L^2 defined by the following: $\partial\phi(u)=\emptyset$ if $u\notin D(\phi)$ and

$$\partial\phi(u)=\{w\in L^2;\ (w, v-u)\leqq\phi(v)-\phi(u)\qquad\text{for all}\quad v\in L^2\}$$

if $u\in D(\phi)$, where $D(\phi)=\{v\in L^2;\ \phi(v)<\infty\}$. We put

$$D(\partial\phi)=\{u\in L^2;\ \partial\phi(u)\neq\emptyset\}$$

and

$$G(\partial\phi)=\{[u, v]\in L^2\times L^2;\ u\in D(\partial\phi),\ v\in\partial\phi(u)\},$$

which are called the domain and the graph of $\partial\phi$, respectively.

In the Dirichlet space theory, contractions on the real line play an important role in connection with potential theoretic properties. Among them the following are the most fundamental:

(a) $\quad T^+t=\max\{t, 0\}$ $\qquad$ (*positive contraction*)

(b) $\quad T_1^+t=\min\{T^+t, 1\}$ $\qquad$ (*unit contraction*).

In case ϕ is Gâteaux-differentiable on a certain functional space, Kenmochi-Mizuta [6, 7] discussed relations between the above contractions and potential theoretic properties, e.g., the maximum principle, the principle of lower envelope, the complete maximum principle and the strong principle of lower envelope.

These potential theoretic properties in nonlinear case were introduced by Calvert [3] for a class of monotone operators in Sobolev spaces. We also refer to Calvert [4].

For a non-negative measurable function g on X, we consider the operator $T_g^+: L^2 \to L^2$ defined by $T_g^+ u = u^+ \wedge g$ for $u \in L^2$. Especially T_∞^+ ($= T_g^+$ with $g \equiv \infty$) is the positive contraction and T_1^+ ($= T_g^+$ with $g \equiv 1$) is the unit contraction. Our aim in this paper is to investigate relations between the following properties:

(1) *T_g^+ operates on L^2 with respect to ϕ*, i.e.,

$$\phi(u + T_g^+(v-u)) + \phi(v - T_g^+(v-u)) \leqq \phi(u) + \phi(v)$$

for any u and $v \in L^2$.

(2) *The strong principle of lower envelope with respect to ϕ and g holds*, i.e., if $[u, u_1]$, $[v, v_1] \in G(\partial\phi)$, then $u \wedge (v+g) \in D(\phi)$ and

$$\phi(u \wedge (v+g) + w) - \phi(u \wedge (v+g)) \geqq (u_1 \wedge v_1, w)$$

for any non-negative function $w \in L^2$.

(3) *The complete maximum principle with respect to ϕ and g holds*, i.e., if $[u, u_1]$, $[v, v_1] \in G(\partial\phi)$ and if there is a measurable function f on X such that $u_1 \wedge v_1 \geqq f$ and $\int_X (u_1 - f)(u - v - g)^+ d\xi = 0$, then $u \leqq v + g$.

(4) For any $\lambda > 0$ and u, $v \in L^2$ with $u \leqq v + g$,

$$J_\lambda u \leqq J_\lambda v + g,$$

where $J_\lambda = (I + \lambda \cdot \partial\phi)^{-1}$ is defined everywhere on L^2 and single-valued (see Brézis [1; Chap. II]).

2. Main theorem

In what follows, let ϕ be a proper lower semicontinuous convex function on L^2 with values in $(-\infty, +\infty]$ and g be a non-negative measurable function on X.

We say that *the positive contraction operates on L^2 with respect to ϕ* if the following is satisfied:

$$(C_0) \qquad \phi(u \vee v) + \phi(u \wedge v) \leqq \phi(u) + \phi(v) \qquad \text{for any } u \text{ and } v \in L^2.$$

Further we shall need the contraction $T_g: L^2 \to L^2$ defined by $T_g u = u \wedge g$ for $u \in L^2$.

We now state our main theorem.

THEOREM. *Assume that ϕ is strictly convex on $D(\phi)$ and the positive con-*

traction operates on L^2 *with respect to* ϕ. *Then the following statements are equivalent to each other*:

(1) *The contraction* T_g^+ *operates on* L^2 *with respect to* ϕ.

(1)′ *The contraction* T_g *operates on* L^2 *with respect to* ϕ, *i.e.*,

$$(C_g) \qquad \phi(u \wedge (v+g))+\phi(v \vee (u-g)) \leqq \phi(u)+\phi(v) \qquad \text{for any} \quad u, v \in L^2.$$

(2) *The strong principle of lower envelope with respect to* ϕ *and* g *holds.*

(3) *The complete maximum principle with respect to* ϕ *and* g *holds.*

(4) $J_\lambda u \leqq J_\lambda v+g$ *for any* $\lambda>0$ *and* $u, v \in L^2$ *with* $u \leqq v+g$.

REMARK 1. The property (C_g) was originally introduced and studied by Brézis [2], in which he investigated the maximum principle and the comparison theorem for the unilateral problems.

REMARK 2. Our theorem is a generalization of a result of Kenmochi-Mizuta [7].

REMARK 3. It is known in [8] that the assertions (1), (1)′ and (4) are equivalent to each other if the positive contraction operates on L^2 with respect to ϕ.

3. Proof of the theorem

We shall prove the theorem through several propositions.

PROPOSITION 1. *Assume that the positive contraction operates on* L^2 *with respect to* ϕ. *If the contraction* T_g *operates on* L^2 *with respect to* ϕ, *then the strong principle of lower envelope with respect to* ϕ *and* g *holds.*

PROOF. Let $[u, u_1], [v, v_1] \in G(\partial\phi)$. Given $\varepsilon>0$, define the function L_ε on L^2 by

$$L_\varepsilon(x) = \varepsilon\|x\|^2/2-((u_1+\varepsilon u) \wedge (v_1+\varepsilon v), x), \qquad x \in L^2,$$

and consider the closed convex set

$$K = \{x \in L^2; x \geqq u \wedge (v+g)\} \qquad (\neq \emptyset).$$

Since ϕ is lower semicontinuous and convex, there are constants C_1 and C_2 such that $\phi(x) \geqq C_1\|x\|+C_2$ for any $x \in L^2$. Consequently,

$$\{\phi(x)+L_\varepsilon(x)\}/\|x\| \longrightarrow \infty \qquad \text{if} \quad \|x\| \longrightarrow \infty.$$

Furthermore, $\phi+L_\varepsilon$ is proper, lower semicontinuous and convex on K. Therefore

$$\inf\{\phi(x)+L_\varepsilon(x);\ x\in K\}$$

can be attained at $x_0\in K$ (cf. [5; Proposition 1.2]). Noting that $tx+(1-t)x_0\in K$ whenever $0\leqq t\leqq 1$ and $x\in K$, we have

$$\phi(tx+(1-t)x_0)-\phi(x_0)\geqq -\{L_\varepsilon(tx+(1-t)x_0)-L_\varepsilon(x_0)\}.$$

By convexity of ϕ,

$$\phi(x)-\phi(x_0)\geqq(-\varepsilon x_0+(u_1+\varepsilon u)\wedge(v_1+\varepsilon v),\ x-x_0)\quad\text{for any}\quad x\in K. \tag{3.1}$$

If we show that $x_0=u\wedge(v+g)$, then we have the proposition by letting $\varepsilon\downarrow 0$ in (3.1) with $x=x_0+w$, where $w\in L^2$ is non-negative.

We first show that $x_0\leqq v+g$. From (C_g) and (3.1) it follows that

$$\begin{aligned}(v_1,\ x_0\wedge(v+g)-x_0)&=(v_1,\ v-v\vee(x_0-g))\\&\geqq\phi(v)-\phi(v\vee(x_0-g))\geqq\phi(x_0\wedge(v+g))-\phi(x_0)\\&\geqq(-\varepsilon x_0+(u_1+\varepsilon u)\wedge(v_1+\varepsilon v),\ x_0\wedge(v+g)-x_0),\end{aligned}$$

which gives

$$(\varepsilon x_0+v_1-(u_1+\varepsilon u)\wedge(v_1+\varepsilon v),\ x_0\wedge(v+g)-x_0)\geqq 0. \tag{3.2}$$

Adding (3.2) and the following inequality

$$((u_1+\varepsilon u)\wedge(v_1+\varepsilon v)-v_1-\varepsilon v,\ x_0\wedge(v+g)-x_0)\geqq 0,$$

we obtain

$$\varepsilon(x_0-v,\ x_0\wedge(v+g)-x_0)\geqq 0,$$

which implies that $x_0\leqq v+g$. In the same manner we see from (3.1) and (C_0) that $x_0\leqq u$. Consequently, $x_0=u\wedge(v+g)$. q.e.d.

PROPOSITION 2. *Assume that ϕ is strictly convex on $D(\phi)$. Then the strong principle of lower envelope with respect to ϕ and g implies the complete maximum principle with respect to ϕ and g.*

PROOF. Let $[u, u_1], [v, v_1]\in G(\partial\phi)$ and f be a measurable function on X such that $u_1\wedge v_1\geqq f$ and $\int_X(u_1-f)(u-v-g)^+d\xi=0$. Since $0\leqq u_1-u_1\wedge v_1\leqq u_1-f$,

$$(u_1-u_1\wedge v_1,\ (u-v-g)^+)=0.$$

From this equality and our assumption, it follows that

$$
\begin{aligned}
(u_1, (u-v-g)^+) &= (u_1 \wedge v_1, (u-v-g)^+) \\
&\leqq \phi(u \wedge (v+g)+(u-v-g)^+)-\phi(u \wedge (v+g)) \\
&= \phi(u)-\phi(u \wedge (v+g)).
\end{aligned}
$$

Noting that u and $u \wedge (v+g) \in D(\phi)$, we have $u=u \wedge (v+g)$ by the strict convexity of ϕ on $D(\phi)$. In fact, if $u \neq u \wedge (v+g)$, then

$$
\begin{aligned}
(u_1, u \wedge (v+g)-u) &= 2\Big(u_1, \frac{u \wedge (v+g)+u}{2}-u\Big) \\
&\leqq 2\Big\{\phi\Big(\frac{u \wedge (v+g)+u}{2}\Big)-\phi(u)\Big\} < \phi(u \wedge (v+g))-\phi(u),
\end{aligned}
$$

and a contradiction follows. q.e.d.

PROPOSITION 3. *If the complete maximum principle with respect to ϕ and g holds, then $J_\lambda u \leqq J_\lambda v+g$ for any $\lambda>0$ and u, $v \in L^2$ with $u \leqq v+g$.*

PROOF. Given $\lambda>0$ and u, $v \in L^2$ such that $u \leqq v+g$, we put $x=J_\lambda u$ and $y=J_\lambda v$. Then there are $x_1 \in \partial\phi(x)$ and $y_1 \in \partial\phi(y)$ such that $u=x+\lambda x_1$ and $v=y+\lambda y_1$. We see that

$$
x_1 \wedge y_1 \geqq \frac{u-x \vee (y+g)}{\lambda}
$$

and that

$$
\int_X \Big(x_1-\frac{u-x \vee (y+g)}{\lambda}\Big)(x-y-g)^+ d\xi = 0.
$$

Hence we obtain by our assumption that $x \leqq y+g$, i.e., $J_\lambda u \leqq J_\lambda v+g$. q.e.d.

PROOF OF THE THEOREM: (1)′ implies (2) by Proposition 1.
(2) implies (3) by Proposition 2.
(3) implies (4) by Proposition 3.
From Remark 3, the assertions (1), (1)′ and (4) are equivalent to each other. Therefore our theorem is completely proved.

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