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Propagation of Chaos for Boltzmann-like Equation of Non-cutoff Type in the Plane

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§ 1. Introduction

Consider a rarefied monoatomic gas composed of a large number of (say, N) molecules moving in space according to the law of classical mechanics and colliding in pairs from time to time. Assume that the motion is specified by giving the intermolecular forces, which are supposed to be given by pair forces only. Let $Nu(t, x)dx$ be the number of molecules with velocities belonging to dx at time t. Then the time evolution of the density $u(t, x)$ in the spatially homogeneous case is given by the following Boltzmann equation:

$$(1.1) \qquad \frac{\partial u(t, x)}{\partial t} = \int_{(0,\pi)\times(0,2\pi)\times R^3} \{u(t, x')u(t, y') - u(t, x)u(t, y)\} \times |x-y| I(|x-y|, \theta) \sin \theta d\theta d\epsilon dy,$$

where x' and y' stand for the velocities after collision of molecules with velocities x and y, I is the differential scattering cross section and θ (ϵ) is the collatitude (the longitude) which measures the scattering angle formed by $y-x$ and $y'-x'$.

When the pair forces are determined by the power-law potential proportional to ρ^{-4} (Maxwellian molecules), $|x-y|I(|x-y|, \theta)$ becomes a function of θ alone; here ρ is the distance of the colliding molecules. For these materials, see Uhlenbeck and Ford [17]. In this case Tanaka [14] constructed the associated Markov process by making use of the stochastic integral equation based upon a Poisson random measure.

The purpose of this paper is to prove the propagation of chaos for two-dimensional analogous model of non-cutoff type by using similar stochastic integral equations.

Propagation of chaos was first discovered by Kac [7] for a model of the Maxwellian gas of cutoff type. The statement of propagation of chaos for (1.1) in the sense of Kac is that if $u_n = u_n(t, x_1, \dots, x_n)$ is the solution of the forward equation of n-molecules

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$$\begin{cases} \dfrac{\partial u}{\partial t} = Gu \\ G\psi = \dfrac{1}{n} \sum\limits_{i<j} \int\limits_{(0,\pi)\times(0,2\pi)} \{\psi(\dots, x_i', \dots, x_j', \dots) - \psi\} \\ \qquad \times |x_i - x_j| I(|x_i - x_j|, \theta) \sin\theta d\theta d\epsilon, \quad \psi \in C_0^\infty(R^{3n})^{*)} \end{cases}$$

with chaotic initial data $f_n = f \otimes \cdots \otimes f$ (n-fold outer product of a probability density f on R^3), then, for each $m \geqq 1$, the marginal density $u_{m|n}$ in $x_1, \dots, x_m$ of $u_n(t, x_1, \dots, x_n)$ splits as $n \to \infty$ into the m-fold product of the solution $u = u(t, x)$ of (1.1) with $u(0, x) = f(x)$. This statement was verified by McKean [9] [10], Johnson [5], Ueno [16], Tanaka [13], Grünbaum [3] and recently by McKean [11], Berresford [1] in various cases. However these results are restricted to the cutoff case; the methods are more or less analytic and do not seem to work for the non-cutoff case.

The emphasis of this paper is on the non-cutoff type. We deal with the following two-dimensional model analogous to Maxwellian molecules, which is still of non-cutoff type. We consider a number of molecules moving on the plane according to the pair forces determined by the potential proportional to ρ^{-2}. The Boltzmann equation is

$$(1.2) \quad \frac{\partial u(t, x)}{\partial t} = \int\limits_{R^1 \times R^2} \{u(t, x')u(t, y') - u(t, x)u(t, y)\} |x - y| dr dy, \quad x \in R^2,$$

where r is the impact parameter which measures the distance from the direction given by $y - x$; the colliding angle $\theta \in (-\pi, \pi)$ is determined by the relation

$$(1.3) \quad \theta = \operatorname{sgn} r \cdot \left\{1 - \left(1 + \frac{2}{|x-y|^2 r^2}\right)^{-1/2}\right\} \pi;$$

the velocities x' and y' after collision of molecules with velocities x and y are expressed by

$$x' = \frac{x+y}{2} + R \cdot \frac{x-y}{2}, \quad y' = \frac{x+y}{2} - R \cdot \frac{x-y}{2}$$

with the rotation R of angle θ:

$$(1.4) \quad R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

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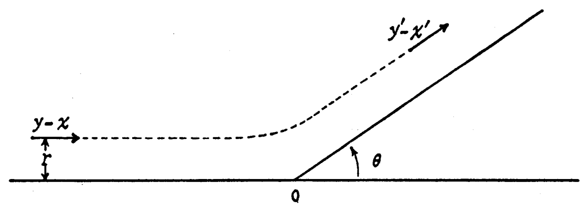
*) $C_0^\infty(R^k)$ denotes the space of real C^∞-functions on R^k with compact supports, throughout.



By (1.3) we have $|x-y|dr=Q(\theta)d\theta$ with some positive (even) function $Q(\theta)$ having the singularity at $\theta=0$:

$$Q(\theta) \sim \text{const.}\, |\theta|^{-3/2}, \qquad \theta \to 0.$$

Therefore the equation (1.2) yields

$$\text{(1.5)} \qquad \frac{\partial u(t, x)}{\partial t} = \int_{(-\pi,\pi)\times R^2} \{u(t, x')u(t, y')-u(t, x)u(t, y)\}Q(d\theta)dy,$$

where $Q(d\theta)$ is a measure on $(-\pi, \pi)$ satisfying

$$\text{(1.6)} \qquad Q(d\theta)=Q(\theta)d\theta, \quad Q(\theta) \geqq 0, \quad \int_{-\pi}^{\pi} |\theta| Q(d\theta) < \infty,$$

$$\int_{-\pi}^{\pi} Q(d\theta) = \infty \qquad \text{(non-cutoff)}.$$

The non-cutoff property makes the situation difficult and even the existence of the solutions of (1.5) seems to be unknown.

In this paper we consider only probability measure solutions of the following equation:

$$\text{(1.7)} \qquad \frac{\partial <u(t, \cdot), \varphi>}{\partial t} = \int_{(-\pi,\pi)\times R^2 \times R^2} \{\varphi(x')-\varphi(x)\}Q(d\theta)u(t, dx)u(t, dy), \qquad \varphi \in C_0^\infty(R^2).$$

As in Tanaka [14] a Markov process $\{X(t), t \geqq 0\}$ associated with (1.7) can be constructed as the solution of the stochastic integral equation

$$\text{(1.8)} \qquad X(t) = X(0) + \int_{S_t} a(X(s-), Y(s, \alpha), \theta)p(dsd\theta d\alpha),$$

where $p(\omega, dsd\theta d\alpha)$ is a Poisson random measure on $R_+^1 \times (-\pi, \pi) \times (0, 1]$ with mean measure $dsQ(d\theta)d\alpha$, $S_t=[0, t]\times(-\pi, \pi)\times(0, 1]$, $a(x, y, \theta)=x'-x$ and $\{Y(t), t \geqq 0\}$ is a process defined on (0, 1] such that $Y(t)$ has the same distribution as that of $X(t-)$ for each fixed t. In the same way (even simpler) as in [14] the existence and the uniqueness (in the law sense) of solutions of (1.8) can be proved under the assumption that the initial distribution f has finite first absolute moment. Denote by P_f the probability measure on the path space of the Markov process $\{X(t), t\geqq 0\}$. Next we consider the Markov process $\bar{X}_n=\{(\bar{X}_1(t),\dots, \bar{X}_n(t)), t\geqq 0\}$ of n-particles with generator given by

$$(1.9)\qquad G_n\psi = \frac{1}{n}\sum_{i<j}\int_{-\pi}^{\pi}\{\psi(\dots, x_i',\dots, x_j',\dots) - \psi(x)\}Q(d\theta),$$

$$x = (x_1,\dots, x_n)\in R^{2n},\quad \psi\in C_0^\infty(R^{2n}),$$

$$x_i' = \frac{x_i+x_j}{2} + R\cdot\frac{x_i-x_j}{2},\quad x_j' = \frac{x_i+x_j}{2} - R\cdot\frac{x_i-x_j}{2}.$$

Let the Markov process $\bar{X}_n$ start with initial distribution $f\otimes\cdots\otimes f$ and, for a positive integer m $(<n)$, denote by $P_{m|n}$ the probability measure on the path space of $\{(\bar{X}_1(t),\dots, \bar{X}_m(t)), t\geqq 0\}$. Then our main result is stated as follows.

If $\int_{R^2}|x|^{2+\delta}f(dx) < \infty$ for some $\delta > 0$, then for each m

$$(1.10)\qquad P_{m|n} \longrightarrow \overbrace{P_f\otimes\cdots\otimes P_f}^{m},\qquad n\to\infty.$$

(1.10) implies the propagation of chaos: The (marginal) distribution $u_{m|n}(t)$ of $(\bar{X}_1(t),\dots, \bar{X}_m(t))$ splits as $n\to\infty$ into the m-fold product of the probability distribution $u(t)$ of $X(t)$.

The main part of the paper is § 6 in which the proof of (1.10) is given. The preceding sections are preparations for this. In § 2 we introduce a metric ρ between probability distributions in R^2 and prove some properties of ρ which will be used in § 6. We list some general (known) properties of Poisson random measures in § 3. § 4 and § 5 are for the existence and the uniqueness of the solutions of the stochastic integral equations associated with (1.7), (1.9), and some moment estimates of the solutions. As an immediate consequence of the results in § 6 the law of large numbers is stated in § 7, and finally in § 8 a remark to the one-dimensional analogous model is given.

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§ 2. A metric between probability distributions in R^2

Let $\mathscr{P}$ be the class of all probability distributions f in R^2 with $\int_{R^2}|x|f(dx) < \infty$, and define

$$(2.1)\qquad \rho(f, g) = \inf_{h\in\mathscr{H}_{f,g}}\int_{R^4}|x-y|\,h(dxdy),\qquad f, g\in\mathscr{P},$$

where $\mathscr{H}_{f,g}$ is the class of probability distributions h in R^4 with $h(A\times R^2)=f(A)$ and $h(R^2\times A)=g(A)$, $A\in\mathscr{B}(R^2)$. An alternative expression of ρ is

$$(2.2)\qquad \rho(f, g) = \inf E\{|X - Y|\}, \qquad f, g\in\mathscr{P},$$

where the infimum is taken over all pairs of R^2-valued random variables X and Y defined on a suitable probability space $(\Omega, \mathscr{F}, P)$ whose probability distributions are f and g, respectively.

It is easily proved that ρ gives a metric on $\mathscr{P}$, and so in this section we discuss some other properties of ρ for the later use.

Let $\mathscr{P}_L=\{f\in\mathscr{P}: f(|x|\leqq L)=1\}$. For any $f\in\mathscr{P}$, we define a probability measure $f_L\in\mathscr{P}_L$ by the relation

$$\int_{R^2}\psi(x)f_L(dx) = \int_{R^2}\psi_L(x)f(dx)$$

for any bounded continuous function ψ in R^2, where

$$\psi_L(x) = \begin{cases} \psi(x), & |x|\leqq L, \\ \psi(Lx/|x|), & |x|>L. \end{cases}$$

Let Φ_L be the set of all Lipschitz continuous functions φ: $R^2\rightarrow R^1$ with Lipschitz constant $\leqq 1$, $\varphi(0)=0$ and $\varphi(x)=\varphi(Lx/|x|)$ for $|x|>L$. Then we can prove the following

PROPOSITION 2.1. *For given $\varepsilon>0$ and $L>0$, there exist a positive constant $K=K(\varepsilon, L)$ and a finite subset Φ_L^ε of Φ_L such that*

$$(2.3)\qquad \rho(f, g)\leqq K\max_{\varphi\in\Phi_L^\varepsilon}\left[\int_{R^2}\varphi(x)f(dx) - \int_{R^2}\varphi(x)g(dx)\right]$$

$$+ \rho(f, f_L) + \rho(g, g_L) + \varepsilon, \quad f, g\in\mathscr{P}.$$

PROOF. First we prove that

$$(2.4)\qquad \rho(f_L, g_L)\leqq K\sup_{\varphi\in\Phi_L}\left[\int_{R^2}\varphi(x)f(dx) - \int_{R^2}\varphi(x)g(dx)\right] + \varepsilon/2.$$

Choose a countable family $\{\psi_n\}_{n\geqq 1}$ which is dense in $C_0^\infty(R^2)$ with respect to the uniform topology, and put

$$d(f, g) = \sum_{n=1}^{\infty}\frac{1}{2^n}\left[\left|\int_{R^2}\psi_n(x)f(dx) - \int_{R^2}\psi_n(x)g(dx)\right| \wedge 1\right]$$

for $f, g\in\mathscr{P}_L$. Then $\mathscr{P}_L$ is a compact metric space with this metric d and we can easily show that d-convergence is equivalent to ρ-convergence in $\mathscr{P}_L$. Since

$\rho(f, g)/d(f, g)$ is a continuous function on the compact set $(\mathscr{P}_L \times \mathscr{P}_L)_\varepsilon = \{(f, g) \in \mathscr{P}_L \times \mathscr{P}_L : \rho(f, g) \geqq \varepsilon/4\}$, there exists a positive constant $K_1 = K_1(\varepsilon, L)$ such that $\rho(f, g) \leqq K_1 d(f, g)$ for $(f, g) \in (\mathscr{P}_L \times \mathscr{P}_L)_\varepsilon$. Hence we have

$$\rho(f, g) \leqq K_1 d(f, g) + \varepsilon/4 \tag{2.5}$$

for any $f, g \in \mathscr{P}_L$. Because of the relation

$$d(f, g) \leqq \sum_{n=1}^{m} \left| \int_{R^2} \psi_n(x) f(dx) - \int_{R^2} \psi_n(x) g(dx) \right| + \varepsilon/4$$

for some $m = m(\varepsilon)$, we have

$$d(f, g) \leqq \int_{R^2} \psi^\varepsilon(x) f(dx) - \int_{R^2} \psi^\varepsilon(x) g(dx) + \varepsilon/4$$

with $\psi^\varepsilon = \sum_{n=1}^{m} \pm \psi_n \in C_0^\infty(R^2)$, where $\pm$ is chosen according to

$$\int_{R^2} \psi_n(x) f(dx) - \int_{R^2} \psi_n(x) g(dx) \geqq 0 \quad \text{or} \quad < 0.$$

If we put

$$\varphi^\varepsilon(x) = (\psi^\varepsilon(x) - \psi^\varepsilon(0))/M, \quad M = \max_{x \in R^2} \left| \frac{d\psi^\varepsilon(x)}{dx} \right|,$$

then $\varphi^\varepsilon(x)$ is a Lipschitz continuous function with Lipschitz constant $\leqq 1$, $\varphi^\varepsilon(0) = 0$ and we have

$$d(f, g) \leqq M\left[\int_{R^2} \varphi^\varepsilon(x) f(dx) - \int_{R^2} \varphi^\varepsilon(x) g(dx)\right] + \varepsilon/4. \tag{2.6}$$

Inserting (2.6) to (2.5) and taking $K = K_1 \cdot M$, we have

$$\rho(f, g) \leqq K\left[\int_{R^2} \varphi^\varepsilon(x) f(dx) - \int_{R^2} \varphi^\varepsilon(x) g(dx)\right] + \varepsilon/2$$

for each $f, g \in \mathscr{P}_L$ and hence

$$\rho(f_L, g_L) \leqq K\left[\int_{R^2} \varphi_L^\varepsilon(x) f(dx) - \int_{R^2} \varphi_L^\varepsilon(x) g(dx)\right] + \varepsilon/2$$

for any $f, g \in \mathscr{P}$ which implies (2.4). Finally if we choose a finite subset $\Phi_L^\varepsilon = \{\varphi_1, \dots, \varphi_n\}$ of Φ_L so that $\min_{1 \leqq k \leqq n} \max_{|x| \leqq L} |\varphi(x) - \varphi_k(x)| < \varepsilon/(4K)$ for any $\varphi \in \Phi_L$, then (2.3) follows immediately from (2.4).

PROPOSITION 2.2. *For given $\varepsilon > 0$ there exists a transition function $P_{f,g}^\varepsilon(x, B)$, $x \in R^2$, $B \in \mathscr{B}(R^2)$ $(f, g \in \mathscr{P})$ which has the following properties.*

(2.7) $P^{\varepsilon}_{f,g}(x, B)$ *is a (Borel) measurable function of* $(x, f, g) \in R^2 \times \mathscr{P} \times \mathscr{P}$ *for each fixed* $B \in \mathscr{B}(R^2)$.

(2.8) $P^{\varepsilon}_{f,g}(x, \cdot)$ *is a probability measure on* R^2 *for each fixed* (x, f, g).

(2.9) $$\int_{R^2} f(dx) P^{\varepsilon}_{f,g}(x, B) = g(B), \qquad B \in \mathscr{B}(R^2).$$

(2.10) $$\int_{R^2}\int_{R^2} |x - y| f(dx) P^{\varepsilon}_{f,g}(x, dy) \leqq \rho(f, g) + \varepsilon.$$

If f and g are fixed, then it is evident that there exists $P^{\varepsilon}_{f,g}(x, \cdot)$ satisfying (2.8)~(2.10). The crucial point of this proposition is how to choose a transition function which is measurable as stated in (2.7). This might be rather technical matter, and its exact construction is somewhat complicated as will be given below. First we prepare an elementary lemma.

LEMMA 2.3. *Let* $(\Omega, \mathscr{F}, P)$ *be a probability space and* $(\Lambda, \mathscr{A})$ *be a measurable space. Let* X_λ *and* Y_λ *be* R^2*-valued random variables on* $(\Omega, \mathscr{F}, P)$ *such that* $X_\lambda(\omega)$ *and* $Y_\lambda(\omega)$ *are* $\mathscr{A} \otimes \mathscr{F}$*-measurable. Then there exists a (regular) conditional probability distribution* $P_\lambda(x, \cdot)$ *of* Y_λ *given* X_λ *such that* $P_\lambda(x, A)$ *is* $\mathscr{A} \otimes \mathscr{B}(R^2)$*-measurable for each fixed* $A \in \mathscr{B}(R^2)$.

PROOF. For each integer $n \geqq 1$ and for any lattice point $(l, m) \in \boldsymbol{Z}^2$, we set $C_n(l, m) = [l2^{-n}, (l+1)2^{-n}) \times [m2^{-n}, (m+1)2^{-n})$ and denote by $C_n(x)$ the cube containing x. We put

$$P^{(n)}_\lambda(x, A) = \frac{P\{X_\lambda \in C_n(x), Y_\lambda \in A\}}{P\{X_\lambda \in C_n(x)\}}, \qquad A \in \mathscr{B}(R^2),$$

$$\Psi^n_\lambda(x) = \int_{R^2} \psi(y) P^{(n)}_\lambda(x, dy), \qquad \psi \in C^\infty_0(R^2).$$

Then $\Psi^n_\lambda(x)$ is $\mathscr{A} \otimes \mathscr{B}(R^2)$-measurable, and so the set

$$E_\psi = \{(\lambda, x) \in \Lambda \times R^2 : \lim_{n\to\infty} \Psi^n_\lambda(x) \text{ exists}\}$$

is also measurable. Using the convergence theorem of martingales, we can prove that the λ-section $E^\lambda_\psi = \{x \in R^2 : (\lambda, x) \in E_\psi\}$ has full f_λ-measure, where f_λ is the probability distribution of X_λ. Taking a countable family $\{\psi_k\}$, $k \geqq 1$, which is dense in $C^\infty_0(R^2)$ with respect to the uniform topology, we put $E = \cap_{k \geqq 1} E_{\psi_k}$ and $E^\lambda = \cap_{k \geqq 1} E^\lambda_{\psi_k}$. Then $E \in \mathscr{A} \otimes \mathscr{B}(R^2)$ and $f_\lambda(E^\lambda) = 1$. Moreover for each $(\lambda, x) \in E$ there exists a unique measure $P_\lambda(x, \cdot)$ which is defined by

$$\int_{R^2} \psi(y) P_\lambda(x, dy) = \lim_{n\to\infty} \int_{R^2} \psi(y) P^{(n)}_\lambda(x, dy), \qquad \psi \in C^\infty_0(R^2).$$

We next define $P_\lambda(x, \cdot)$ for $(\lambda, x) \notin E$ to be an arbitrarily fixed probability measure $\bar{P}(\cdot)$ on R^2, and finally redefine $P_\lambda(x, \cdot) = \bar{P}(\cdot)$ for (λ, x) such that $P_\lambda(x, R^2) \neq 1$. Thus constructed $P_\lambda(x, \cdot)$ is the conditional probability distribution we were looking for.

Now we prove Proposition 2.2. For any $\varepsilon > 0$ which is fixed throughout the proof, we put

$$\begin{cases} N(\varepsilon, f) = \min\{n \in \boldsymbol{N}: \int_{|x|>n} |x| f(dx) < \varepsilon/8\}, \\ \mathscr{P}_n = \{f \in \mathscr{P}: N(\varepsilon, f) = n\}, \qquad n \geqq 1. \end{cases}$$

Then each $\mathscr{P}_n$ is a Borel set and $\mathscr{P}$ is the disjoint sum $\cup_{n \geq 1} \mathscr{P}_n$. Therefore it is sufficient to prove that, for each pair (m, n), there exists $P^\varepsilon_{f,g}(x, B)$ satisfying (2.7)$\sim$(2.10) when (f, g) is restricted to $\mathscr{P}_m \times \mathscr{P}_n$.

Let ξ be a Borel isomorphism: $R^2 \to R^1$; it is fixed throughout the proof. Define a probability distribution f^* in R^1 by $f^*(A) = f(\xi^{-1}(A))$, $A \in \mathscr{B}(R^1)$, let $X^*_f(\alpha)$ be the right continuous inverse of the distribution function of f^*, and put $X_f(\alpha) = \xi^{-1}(X^*_f(\alpha))$. Then $X_f(\alpha)$ is an f-distributed, R^2-valued random variable defined on the probability space $((0, 1], d\alpha)$; it is also a measurable function of $(f, \alpha) \in \mathscr{P} \times (0, 1]$, because

$$\begin{aligned} \{(f, \alpha): X_f(\alpha) \in \xi^{-1}((-\infty, t])\} &= \{(f, \alpha): X^*_f(\alpha) \leqq t\} \\ &= \{(f, \alpha): f^*((-\infty, t]) \geqq \alpha\} \\ &= \{(f, \alpha): f(\xi^{-1}(-\infty, t]) \geqq \alpha\}. \end{aligned}$$

From now on we assume that $f \in \mathscr{P}_m$ and $g \in \mathscr{P}_n$, m and n being fixed. Let $N = \max(m, n)$ and put

$$\tilde{X}_f(\alpha) = \begin{cases} X_f(\alpha), & |X_f(\alpha)| \leqq N, \\ 0, & |X_f(\alpha)| > N. \end{cases}$$

Taking a positive constant δ, we also put

$$X^\delta_f(\alpha) = [\tilde{X}_f(\alpha)]_\delta,$$

where $[x]_\delta = (k\delta, l\delta)$ for $x \in c_\delta(k, l) \equiv [k\delta, (k+1)\delta) \times [l\delta, (l+1)\delta)$. Then the mapping $\mathscr{P}_m \times (0, 1] \ni (f, \alpha) \mapsto X^\delta_f(\alpha) \in \Gamma$ is Borel measurable. Here Γ is the finite set defined by

$$\Gamma = \Gamma_\delta = \{\gamma = (k\delta, l\delta) : c_\delta(k, l) \cap (|x| \leqq N) \neq \phi\}.$$

Next for $\tau > 0$ which will be determined later and for each Borel subset A of $(0, 1]$ we denote by $[A]_\tau$ the Borel set $A \cap (0, t_A)$, where

$$t_A = \inf\{t > 0: |A \cap (0, t]| \geqq |A|_\tau\}.^{*)}$$

Putting $A_f^\gamma = \{\alpha \in (0, 1]: X_f^\delta(\alpha) = \gamma\}$ for $\gamma \in \Gamma$ and $A = \cup_{\gamma \in \Gamma}[A_f^\gamma]_\tau$, we define $X_f^{\delta,\tau}(\alpha)$ by

$$X_f^{\delta,\tau}(\alpha) = X_f^\delta(\alpha)\chi_A(\alpha).$$

Since $|X_f^\delta(\alpha) - X_f^{\delta,\tau}(\alpha)| \leqq (N + \sqrt{2}\,\delta)\chi_{A^c}(\alpha)$, we have

$$\text{(2.11)} \qquad \begin{aligned} \int_0^1 |X_f^\delta(\alpha) - X_f^{\delta,\tau}(\alpha)| d\alpha &\leqq (N + \sqrt{2}\,\delta)(1 - |A|) \\ &\leqq (N + \sqrt{2}\,\delta) \cdot \#(\Gamma) \cdot \tau \\ &\leqq (N + \sqrt{2}\,\delta) \cdot (2N + 2\delta)^2 \cdot \tau/\delta^2. \end{aligned}$$

Moreover, the mapping $\mathscr{P}_m \times (0, 1] \ni (f, \alpha) \mapsto X_f^{\delta,\tau}(\alpha) \in \Gamma$ is Borel measurable; in fact, the set $\{(f, \alpha): \chi_A(\alpha) = 1\}$ is measurable, since it is equal to

$$\begin{aligned} &\bigcup_{\gamma \in \Gamma} \{(f, \alpha): \alpha \in [A_f^\gamma]_\tau\} \\ &= \bigcup_{\gamma \in \Gamma} \{(f, \alpha): \alpha \in A_f^\gamma, |A_f^\gamma \cap (0, \alpha]| < |A_f^\gamma|_\tau\} \\ &= \bigcup_{\gamma \in \Gamma} \left\{(f, \alpha): X_f^\delta(\alpha) = \gamma, \int_0^\alpha \chi_{\{\gamma\}}(X_f^\delta(t)) dt < \left(\int_0^1 \chi_{\{\gamma\}}(X_f^\delta(t))\, dt\right)_\tau\right\}. \end{aligned}$$

We can also define $X_g^{\delta,\tau}$ for $g \in \mathscr{P}_n$ and prove that it has similar properties. Denote by $f^{\delta,\tau}$ ($g^{\delta,\tau}$) the probability distribution of $X_f^{\delta,\tau}$ ($X_g^{\delta,\tau}$) and put $\mathscr{P}_m^\# = \{f^{\delta,\tau}: f \in \mathscr{P}_m\}$, $\mathscr{P}_n^\# = \{g^{\delta,\tau}: g \in \mathscr{P}_n\}$. Then obviously $\mathscr{P}_m^\#$ and $\mathscr{P}_n^\#$ are finite sets. It is also obvious that for each pair $(f^\#, g^\#)$ from $\mathscr{P}_m^\# \times \mathscr{P}_n^\#$ there exists a transition function $Q_{f^\#,g^\#}(u, \cdot)$ satisfying the following conditions.

(i) $\displaystyle\int_{R^2} f^\#(du) Q_{f^\#,g^\#}(u, \cdot) = g^\#(\cdot).$

(ii) $\displaystyle\int_{R^2}\int_{R^2} |u - v| f^\#(du) Q_{f^\#,g^\#}(u, dv) = \rho(f^\#, g^\#).$

Because the mapping $\mathscr{P}_m \times \mathscr{P}_n \ni (f, g) \mapsto (f^{\delta,\tau}, g^{\delta,\tau}) \in \mathscr{P}_m^\# \times \mathscr{P}_n^\#$ is measurable, $\bar{Q}_{f,g}(u, B) \equiv Q_{f^{\delta,\tau},g^{\delta,\tau}}(u, B)$ is jointly measurable in $(f, g, u) \in \mathscr{P}_m \times \mathscr{P}_n \times R^2$ for each $B \in \mathscr{B}(R^2)$. Next we apply Lemma 2.3 to have a conditional probability distribution $R_f(x, \cdot)$ of $X_f^{\delta,\tau}$ given X_f, which has the following properties.

(iii) $R_f(x, B)$ is Borel measurable in $(f, x) \in \mathscr{P}_m \times R^2$ for each $B \in \mathscr{B}(R^2)$.

(iv) $\displaystyle\int_{R^2} f(dx) R_f(x, \cdot) = f^{\delta,\tau}(\cdot).$

$*)$ $|A|$ denotes the Lebesgue measure of A, and $r_\tau = \max\{n\tau: n\tau \leqq r\}$ for a real number r.

Using Lemma 2.3 for $X_g^{\delta,\tau}$ and X_g, we can also choose a conditional probability distribution $\bar{R}_g(v, \cdot)$ of X_g given $X_g^{\delta,\tau}$ such that

(v) $\bar{R}_g(v, B)$ is Borel measurable in $(g, v) \in \mathscr{P}_n \times R^2$ for each $B \in \mathscr{B}(R^2)$.

(vi) $\int_{R^2} g^{\delta,\tau}(dv)\bar{R}_g(v, \cdot) = g(\cdot)$.

Choosing first $\delta = \varepsilon/16\sqrt{2}$ and then $\tau > 0$ so that

$$(N + \sqrt{2}\,\delta)\cdot(2N + 2\delta)^2\cdot\tau/\delta^2 \leqq \varepsilon/16,$$

we put

$$P^{\varepsilon}_{f,g}(x, B) = \int_{R^2}\int_{R^2} R_f(x, du)\bar{Q}_{f,g}(u, dv)\bar{R}_g(v, B).$$

From the construction it is clear that $P^{\varepsilon}_{f,g}(x, \cdot)$ satisfies (2.7) and (2.8) for $(f, g, x) \in \mathscr{P}_m \times \mathscr{P}_n \times R^2$. The relations (2.9) and (2.10) are also valid because

$$\begin{aligned}\int_{R^2} f(dx)P^{\varepsilon}_{f,g}(x, B) &= \int_{R^2}\int_{R^2}\int_{R^2} f(dx)R_f(x, du)\bar{Q}_{f,g}(u, dv)\bar{R}_g(v, B)\\ &= g(B)\end{aligned}$$

and

$$\begin{aligned}&\int_{R^2}\int_{R^2} |x - y| f(dx)P^{\varepsilon}_{f,g}(x, dy)\\ &\leqq \int_{R^2}\int_{R^2}\int_{R^2}\int_{R^2} (|x - u| + |u - v| + |v - y|)\\ &\qquad\qquad \cdot f(dx)R_f(x, du)\bar{Q}_{f,g}(u, dv)\bar{R}_g(v, dy)\\ &= \int_{R^2}\int_{R^2} |x - u| f(dx)R_f(x, du) + \int_{R^2}\int_{R^2} |u - v| f^{\delta,\tau}(du)\bar{Q}_{f,g}(u, dv)\\ &\qquad + \int_{R^2}\int_{R^2} |v - y| g^{\delta,\tau}(dv)\bar{R}_g(v, dy)\\ &= \int_0^1 |X_f(\alpha) - X_f^{\delta,\tau}(\alpha)|d\alpha + \rho(f^{\delta,\tau}, g^{\delta,\tau}) + \int_0^1 |X_g^{\delta,\tau}(\alpha) - X_g(\alpha)|d\alpha\\ &\leqq \rho(f, g) + 2\int_0^1 |X_f(\alpha) - X_f^{\delta,\tau}(\alpha)|d\alpha + 2\int_0^1 |X_g(\alpha) - X_g^{\delta,\tau}(\alpha)|d\alpha\\ &\leqq \rho(f, g) + 2\Big\{\int_0^1 |X_f(\alpha) - \tilde{X}_f(\alpha)|\,d\alpha + \int_0^1 |\tilde{X}_f(\alpha) - X_f^{\delta}(\alpha)|d\alpha\\ &\qquad + \int_0^1 |X_f^{\delta}(\alpha) - X_f^{\delta,\tau}(\alpha)|d\alpha\Big\} + \text{similar terms}\end{aligned}$$

$$\leqq \rho(f, g) + 4\{\varepsilon/8 + \sqrt{2}\cdot\varepsilon/16\sqrt{2} + \varepsilon/16\}$$

$$= \rho(f, g) + \varepsilon.$$

Here we have used (i), (ii), (iv), (vi) and (2.11). The proof of Proposition 2.2 is completed.

§3. Poisson random measures

In this section we state preliminalies from Poisson random measures for later sections.

Given a σ-finite measure space $(\boldsymbol{S}, \lambda)$ in which each single point set is measurable, we denote by $\mathscr{M}_\lambda(\boldsymbol{S})$ the space of all measures μ on $\boldsymbol{S}$ which are expressed as (at most) countable sums of δ-measures and satisfying $\mu(A)<\infty$ for any λ-finite set A. Here a δ-measure means a probability measure on $\boldsymbol{S}$ with unit mass at some point of $\boldsymbol{S}$. Let $(\Omega, \mathscr{F}, P)$ be a probability space.

Definition. A mapping $p\colon \Omega\to\mathscr{M}_\lambda(\boldsymbol{S})$ is called a Poisson random measure with mean measure λ, if the following two conditions are satisfied.

(i) $p(A)$ is $\mathscr{F}$-measurable for each λ-finite set A.

(ii) For any disjoint λ-finite sets $A_1,\dots, A_k$,

$$P\{p(A_j) = n_j, j = 1,\dots, k\} = \prod_{j=1}^{k} \lambda(A_j)^{n_j} \exp\{-\lambda(A_j)\}/n_j!,\qquad n_j = 0, 1, 2,\dots.$$

Here $p(A)=p(\omega, A)$ is a short for $(p(\omega))(A)$.

The existence of a Poisson random measure with given mean measure is well-known.

In this paper we are concerned only with the case $\boldsymbol{S}=R_+^1\times S$ and $d\lambda=dt\mu_t(d\sigma)$, where $\{\mu_t, t\geqq 0\}$ are σ-finite measures on some measurable space $(S, \mathscr{S})$. Suppose that an increasing family $\{\mathscr{F}_t, t\geqq 0\}$ of sub-σ-fields is given in $(\Omega, \mathscr{F}, P)$. A real valued function $a(t, \sigma, \omega)$ defined on $R_+^1\times S\times\Omega$ is called $\mathscr{F}_t$-predictable, if $a(t, \sigma, \omega)$ is $\mathscr{K}$-measurable; here $\mathscr{K}$ is the (smallest) sub-σ-field on $R_+^1\times S\times\Omega$ generated by all functions $a(t, \sigma, \omega)$ satisfying the following conditions.

(i) $a(t, \cdot, \cdot)$ is $\mathscr{S}\otimes\mathscr{F}_t$-measurable for each fixed $t\in R_+^1$.

(ii) $a(\cdot, \sigma, \omega)$ is left-continuous for each fixed $(\sigma, \omega)\in S\times\Omega$.

A Poisson random measure p on $R_+^1\times S$ defined on $(\Omega, \mathscr{F}, P)$ is said to be $\mathscr{F}_t$-adapted, if (i) $p(A)$ is $\mathscr{F}_t$-measurable for each $A\in\mathscr{B}([0, t]\times S)$, $t\in R_+^1$, and (ii) for each $t\geqq 0$ the σ-field induced by $\{p(A), A\in\mathscr{B}((t, \infty)\times S)\}$ is independent of $\mathscr{F}_t$. If p is such a Poisson random measure with mean measure $d\lambda=dt\mu_t(d\sigma)$ and if an $\mathscr{F}_t$-predictable $a(t, \sigma, \omega)$ satisfies

(3.1) $$E\Big\{\int_{[0,t]\times S} |a(s, \sigma, \omega)|\mu_s(d\sigma)ds\Big\} < \infty, \qquad t \in R_+^1,$$

then

(3.2) $$\int_{[0,t]\times S} a(s, \sigma, \omega)p(dsd\sigma) - \int_{[0,t]\times S} a(s, \sigma, \omega)\mu_s(d\sigma)ds$$

is an $\mathscr{F}_t$-martingale, and in particular

$$E\Big\{\int_{[0,t]\times S} a(s, \sigma, \omega)p(dsd\sigma)\Big\} = E\Big\{\int_{[0,t]\times S} a(s, \sigma, \omega)\mu_s(d\sigma)ds\Big\}.$$

The following martingale characterization of $\mathscr{F}_t$-adapted Poisson random measures is well-known.

THEOREM 3.1 (for example, see [18]). *Let p be a mapping*: $\Omega \to \mathscr{M}_\lambda(R_+^1 \times S)$ *and assume that* $p(\omega, \{t\}\times S)=0$ *or* 1 *for any* $t \in R_+^1$ *with probability one. Let* $\{\mathscr{F}_t, t \geqq 0\}$ *be an increasing family of sub-σ-fields of* $\mathscr{F}$. *Then*, (3.2) *is an* $\mathscr{F}_t$*-martingale for any* $\mathscr{F}_t$*-predictable* $a(t, \sigma, \omega)$ *satisfying* (3.1) *if and only if p is an* $\mathscr{F}_t$*-adapted Poisson random measure with mean measure* $d\lambda = dt\mu_t(d\sigma)$.

§4. Stochastic integral equation and the Markov process associated with (1.7)

Given a measure $Q(d\theta)$ on $(-\pi, \pi)$ satisfying (1.6), we consider the equation

(4.1) $$\frac{\partial \langle u(t, \cdot), \varphi \rangle}{\partial t} = \int_{(-\pi,\pi)\times R^2\times R^2} \{\varphi(x') - \varphi(x)\}Q(d\theta)u(t, dx)u(t, dy),$$

$$x' = \frac{x+y}{2} + R\cdot\frac{x-y}{2}, \quad R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \varphi \in C_0^\infty(R^2),$$

and construct a Markov process associated with (4.1)=(1.7) making use of the stochastic integral equation as in H. Tanaka [14].

We put $a(x, y, \theta) = x' - x$, $S = (-\pi, \pi)\times(0, 1]$ and $S_t = [0, t]\times S$. Taking a suitable probability space $(\Omega, \mathscr{F}, P)$ with an increasing family of sub-σ-fields $\{\mathscr{F}_t\}$, we suppose that there is given an $\mathscr{F}_t$-adapted Poisson random measure $p(\omega, dsd\theta d\alpha)$ on $R_+^1 \times S$ with mean measure $dsQ(d\theta)d\alpha$. Given an $\mathscr{F}_0$-measurable random variable $X(0)$ with values in R^2, we consider the stochastic integral equation

(4.2) $$X(t) = X(0) + \int_{S_t} a(X(s-), Y(s, \alpha), \theta)p(dsd\theta d\alpha), \qquad t \geqq 0.$$

If there exist an $\mathscr{F}_t$-adapted process $\{X(t, \omega), t \geqq 0\}$ on R^2 and a process $\{Y(t, \alpha), t \geqq 0\}$ defined on the probability space $((0, 1], d\alpha)$ such that (i) $Y(t)$ has the same probability distribution as that of $X(t-) \equiv \lim_{\varepsilon \downarrow 0} X(t-\varepsilon)$ for each t and (ii) (4.2) holds with probability one, then $\{X(t, \omega)\}$ is called an $\mathscr{F}_t$-adapted solution of (4.2). The stochastic integral equation of this type was treated by [14] in the case of the 3-dimensional Maxwellian molecules. Owing to the estimates

$$(4.3)\qquad \begin{cases} |a(x, y, \theta)| \leqq \left|\sin \dfrac{\theta}{2}\right| (|x| + |y|) \\ |a(x, y, \theta) - a(x_1, y_1, \theta)| \leqq \left|\sin \dfrac{\theta}{2}\right| (|x - x_1| + |y - y_1|), \end{cases}$$

the method of [14] can be applied, even in a much simpler way, to prove the following

THEOREM 4.1. *Let f be a probability distribution belonging to $\mathscr{P}$, and assume that $X(0)$ is an $\mathscr{F}_0$-measurable, f-distributed, R^2-valued random variable. Then there exists an $\mathscr{F}_t$-adapted solution $X(t)$ of* (4.2) *satisfying $\int_0^t E\{|X(s)|\} ds < \infty$ for $t < \infty$. The uniqueness in the law sense holds: If $\{X(t)\}$ and $\{\tilde{X}(t)\}$ are $\mathscr{F}_t$-adapted solutions of* (4.2) *with the same initial distribution $f \in \mathscr{P}$ and satisfying the above integrability condition, then $\{X(t)\}$ and $\{\tilde{X}(t)\}$ are equivalent, that is, they have the same finite dimensional distributions. The probability distribution $u(t)$ of $X(t)$ solves* (4.1) *with $u(0) = f$.*

We make a remark about the stochastic integral equation (4.2). Given $f \in \mathscr{P}$, we denote by $u(t)$ the solution of (4.1) with $u(0) = f$ constructed in Theorem 4.1. Let $\{Y(t, \omega, \alpha), t \geqq 0\}$ be any $\mathscr{F}_t$-predictable process such that, for each fixed t and ω, the probability distribution of $Y(t, \omega, \cdot)$ as a random variable on the probability space $((0, 1], d\alpha)$ is $u(t)$ $(= u(t-))$. Owing to (4.3), the stochastic integral equation

$$(4.4)\qquad \hat{X}(t) = \hat{X}(0) + \int_{S_t} a(\hat{X}(s-), Y(s, \omega, \alpha), \theta) p(ds d\theta d\alpha)$$

can easily be solved by iteration for any $\mathscr{F}_0$-measurable $\hat{X}(0)$ with $E\{|\hat{X}(0)|\} < \infty$.

THEOREM 4.2. *Let $\hat{X}(0)$ and $\{Y(t, \omega, \alpha), t \geqq 0\}$ be as above, and $\{\hat{X}(t), t \geqq 0\}$ the solution of* (4.4). *Then any finite dimensional probability distribution of $\{\hat{X}(t), t \geqq 0\}$ is determined only by f and the probability distribution of $\hat{X}(0)$, that is, it is independent of the choice of $\hat{X}(0)$ and $\{Y(t, \omega, \alpha), t \geqq 0\}$ as far as they satisfy the stated conditions. In particular, if $\hat{X}(0)$ is f-distributed, then $\{\hat{X}(t)\}$ is equivalent to $\{X(t)\}$ of Theorem* 4.1.

PROOF. Take a partition Δ of a finite interval $[0, T]$: $0 = t_0 < t_1 < \cdots < t_n$

$=T$, and put $\Delta(0)=0$, $\Delta(t)=t_{k-1}$ for $t_{k-1}<t\leqq t_k$ $(1\leqq k\leqq n)$. We consider the stochastic equation

$$\hat{X}_\Delta(t) = \hat{X}(0) + \int_{S_t} a(\hat{X}_\Delta(\Delta(s)),\ Y(s,\omega,\alpha),\ \theta)p(dsd\theta d\alpha), \qquad 0\leqq t\leqq T.$$

As in [6] for the diffusion case, we can prove that $E\{|\hat{X}(t)-\hat{X}_\Delta(t)|\}\to 0$ as $|\Delta| \equiv \max_{1\leqq k\leqq n}(t_k-t_{k-1})\to 0$ for $0\leqq t\leqq T$. Since for $s<t$

$$E\{\exp i(\xi,\ \hat{X}_\Delta(t))|\mathscr{F}_s\}$$
$$= \exp\Big[i(\xi,\ \hat{X}_\Delta(s)) + \int_{(s,t]\times(-\pi,\pi)\times R^2}\{\exp i(\xi,\ a(\hat{X}_\Delta(\Delta(\tau)),\ y,\ \theta)) - 1\}d\tau Q(d\theta)u(\tau, dy)\Big],$$

any finite dimensional probability distribution of $\{\hat{X}_\Delta(t),\ 0\leqq t\leqq T\}$ does not depend upon the choice of $\{Y(t,\omega,\alpha)\}$ and hence the same holds for $\{\hat{X}(t),\ 0\leqq t \leqq T\}$. The proof is finished.

Let $\{X(t),\ t\geqq 0\}$ be an $\mathscr{F}_t$-adapted solution of (4.2) with initial distribution $f\in\mathscr{P}$. Then it is a Markov process associated with (1.7) in the sense of McKean [8]. In fact, if $u(t,\cdot)=P\{X(t)\in\cdot\}$ and if $P_f(t,x,\cdot)$ denotes the probability distribution, at time t, of the solution $\hat{X}(t)$ of (4.4) with initial value $\hat{X}(0)=x$, then

$$(4.5)\qquad P_f(t,x,\cdot) = \int_{R^2} P_f(s,x,dy)P_{u(s)}(t-s,y,\cdot), \quad 0\leqq s\leqq t,$$

$$(4.6)\qquad P\{X(t)\in A|\mathscr{F}_s\} = P_{u(s)}(t-s,\ X(s),\ A), \quad \text{a.s.},\quad 0\leqq s\leqq t,\ A\in\mathscr{B}(R^2),$$

and $u(t)$ is a solution of (4.1) with initial distribution f.

The following estimate (4.7) will be used in §6.

LEMMA 4.3. *Assume that* $\int_{R^2}|x|^{2+\delta}f(dx)<\infty$ *for some* $\delta\geqq 0$ *and let* $\{X(t),\ t\geqq 0\}$ *be an* $\mathscr{F}_t$*-adapted solution of* (4.2) *with an f-distributed initial value* $X(0)$. *Then*

$$(4.7)\qquad E\{|X(t)|^{2+\delta}\} \leqq E\{|X(0)|^{2+\delta}\}\exp(2^{4+3\delta}Mt), \qquad t\geqq 0,$$

where $M=\int_{-\pi}^{\pi}\left|\sin\frac{\theta}{2}\right|Q(d\theta)$. *When* $\delta=0$, *we have* $E\{|X(t)|^2\}=E\{|X(0)|^2\}$.

PROOF. Define $\{X_k(t),\ t\geqq 0\}$, $k=0, 1, 2,\ldots$, by

$$\begin{cases} X_0(t) = X(0) \\ X_k(t) = X(0) + \int_{S_t} a(X_{k-1}(s-),\ Y_{k-1}(s,\alpha),\ \theta)p(dsd\theta d\alpha), \qquad k\geqq 1, \end{cases}$$

where $\{Y_{k-1}(s, \alpha), s \geqq 0, \alpha \in (0, 1]\}$ is chosen so that (i) it is Borel measurable in the pair (s, α) and (ii) $Y_{k-1}(s, \cdot)$ has the same probability distribution as that of $X_{k-1}(s-)$ for each s. Then, by the same method as in the proof of Theorem A of [14], we can prove that a suitable choice of the sequence $\{Y_k(s, \alpha)\}_{k=0}^{\infty}$ implies the almost sure convergence of $\{X_k(t)\}_{k=0}^{\infty}$ to a solution $X(t)$ of (4.2). Therefore (4.7) follows immediately from

$$E\{|X_k(t)|^{2+\delta}\} \leqq E\{|X(0)|^{2+\delta}\} \exp(2^{4+3\delta}Mt), \qquad t \geqq 0,\ k \geqq 0 \tag{4.8}$$

which can be proved in the following way. Using the transformation formula of the stochastic integrals

$$\varphi(X_k(t)) = \varphi(X(0)) + \int_{S_t} \{\varphi(X_{k-1}(s-) + a(X_{k-1}(s-), Y_{k-1}(s, \alpha), \theta)) - \varphi(X_{k-1}(s-))\} p(dsd\theta d\alpha), \qquad k \geqq 0,$$

with $\varphi(x) = |x|^{2+\delta}$, we have

$$\begin{aligned} &E\{|X_k(t)|^{2+\delta}\} \\ &\leqq E\{|X(0)|^{2+\delta}\} + E\Big\{\int_{S_t} \Big| |X_{k-1}(s-) + a(X_{k-1}(s-), Y_{k-1}(s, \alpha), \theta)|^{2+\delta} - |X_{k-1}(s-)|^{2+\delta}\Big| dsQ(d\theta)d\alpha\Big\} \\ &\leqq E\{|X(0)|^{2+\delta}\} + 2^{4+3\delta}M\int_0^t E\{|X_{k-1}(s)|^{2+\delta}\}ds \end{aligned}$$

which yields (4.8). In the above we have used the estimate

$$\begin{aligned} &\Big| |x + a(x, y, \theta)|^{2+\delta} - |x|^{2+\delta}\Big| \\ &\leqq |a(x, y, \theta)|(|x + a(x, y, \theta)| + |x|)^{1+\delta} \\ &\leqq \Big|\sin\frac{\theta}{2}\Big|(|x| + |y|)(3|x| + |y|)^{1+\delta} \\ &\leqq 2^{2+2\delta}\Big|\sin\frac{\theta}{2}\Big|(|x| + |y|)^{2+\delta} \\ &\leqq 2^{3+3\delta}\Big|\sin\frac{\theta}{2}\Big|(|x|^{2+\delta} + |y|^{2+\delta}). \end{aligned} \tag{4.9}$$

Finally, when $\delta = 0$, we have

$$E\{|X(t)|^2\} - E\{|X(0)|^2\}$$

$$= E\left\{\int_{S_t}[|X(s-) + a(X(s-), Y(s, \alpha), \theta)|^2 - |X(s-)|^2]dsQ(d\theta)d\alpha\right\}$$

$$= \int_{(-\pi,\pi)\times R^2\times R^2} \{|x'|^2 - |x|^2\}Q(d\theta)u(s, dx)u(s, dy)$$

$$= \frac{1}{2}\int_{(-\pi,\pi)\times R^2\times R^2} \{|x'|^2+|y'|^2-|x|^2-|y|^2\}Q(d\theta)u(s, dx)u(s, dy)$$

$$= 0.$$

The proof is finished.

§5. n-particle motion

In this section we consider the time evolution of the velocities of n-particles. It is described as a Markov process on R^{2n} determined by the forward equation:

$$(5.1) \qquad \frac{d}{dt}<u(t, \cdot), \psi(\cdot)> = <u(t, \cdot), G_n\psi(\cdot)>$$

$$= \frac{1}{n}\sum_{i<j}\int_{(-\pi,\pi)\times R^{2n}} \{\psi(x_1,\dots, x'_i,\dots, x'_j,\dots, x_n) - \psi(x)\}Q(d\theta)u(t, dx_1\dots dx_n),$$

$$x = (x_1,\dots, x_n)\in R^{2n}, \qquad \psi\in C_0^\infty(R^{2n}),$$

where $x'_i = \frac{x_i+x_j}{2} + R\cdot\frac{x_i-x_j}{2}$, $x'_j=\frac{x_i+x_j}{2} - R\cdot\frac{x_i-x_j}{2}$ with the rotation $R=R(\theta)$ given by (1.4), and $Q(d\theta)$ is the measure on $(-\pi, \pi)$ satisfying (1.6).

On a suitable probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ we can construct a Markov process associated with (5.1) by solving the following stochastic integral equation:

$$(5.2) \quad \bar{X}_i(t) = \bar{X}_i(0) + \sum_{j\neq i}\int_{U_t} a(\bar{X}_i(s-), \bar{X}_j(s-), \theta)\bar{p}_{ij}(dsd\theta), \qquad 1\leqq i\leqq n,$$

where $U_t=[0, t]\times(-\pi, \pi)$, $a(x_i, x_j, \theta)=x'_i-x_i$ and $\{\bar{p}_{ij}(dtd\theta), 1\leqq i, j\leqq n\}$ is a system of $\mathscr{F}_t$-adapted Poisson random measures on $R^1_+\times(-\pi, \pi)$ with common mean measure $dtQ(d\theta)/n$ satisfying the following properties.

$$(5.3) \qquad \{\bar{p}_{ij}(dtd\theta), i\leqq j\} \text{ is an independent family.}$$

$$(5.4) \qquad \bar{p}_{ij}(\Gamma) = \bar{p}_{ji}(\Gamma), i\neq j, \quad \text{for} \quad \Gamma\in\mathscr{B}(R^1_+\times(-\pi, \pi)).$$

THEOREM 5.1. *Let $\{\bar{p}_{ij}(dtd\theta), 1\leqq i, j\leqq n\}$ be as above, and assume that $\bar{X}_i(0)$, $1\leqq i\leqq n$, are $\mathscr{F}_0$-measurable independent R^2-valued random variables with common probability distribution f in $\mathscr{P}$. Then we have the following* *assertions.*

(i) *There exists a unique* $\mathscr{F}_t$*-adapted solution* $\bar{\mathbf{X}}_n(t)=(\bar{X}_1(t),\dots,\bar{X}_n(t))$ *of* (5.2), *and the probability distribution* $u(t,\cdot)$ *of* $\bar{\mathbf{X}}_n(t)$ *satisfies* (5.1) *for* $\psi\in C_0^\infty(R^{2n})$.

(ii) *If* $\int_{R^2}|x|^{2+\delta}f(dx)<\infty$ *for some* $\delta\geqq 0$, *then*

$$(5.5)\quad E\{|\bar{X}_i(t)|^{2+\delta}\}\leqq E\{|\bar{X}_i(0)|^{2+\delta}\}\exp(2^{4+3\delta}Mt),\qquad t>0,\quad 1\leqq i\leqq n,$$

where $M=\int_{-\pi}^{\pi}\left|\sin\frac{\theta}{2}\right|Q(d\theta)$. *In the case* $\delta=0$, *we have*

$$\sum_{i=1}^n|\bar{X}_i(t)|^2=\sum_{i=1}^n|\bar{X}_i(0)|^2.$$

PROOF. (i) The existence and the uniqueness of the solution follow from the general theory of stochastic differential equations due to K. Itô [4]. In fact, if we put for $1\leqq i\leqq n$

$$\begin{cases}\bar{X}_i^0(t)=\bar{X}_i(0)\\ \bar{X}_i^{k+1}(t)=\bar{X}_i(0)+\sum_{j\neq i}\int_{U_t}a(\bar{X}_i^k(s-),\bar{X}_j^k(s-),\theta)\bar{p}_{ij}(dsd\theta),\qquad k\geqq 0,\end{cases}$$

then we can prove that $\bar{\mathbf{X}}_n^k(t)=(\bar{X}_1^k(t),\dots,\bar{X}_n^k(t))$ converges uniformly on each finite t-interval as $k\to\infty$ with probability one, and that the limit $\bar{\mathbf{X}}_n(t)=(\bar{X}_1(t),\dots,\bar{X}_n(t))$ is the unique $\mathscr{F}_t$-adapted solution of (5.2). The last part of (i) follows from

$$\psi(\bar{\mathbf{X}}_n(t))=\psi(\bar{\mathbf{X}}_n(0))+\sum_{i<j}\int_{U_t}\{\psi(\bar{X}_1(s-),\dots,\bar{X}_i'(s-),\dots,\bar{X}_j'(s-),\dots,\bar{X}_n(s-))\\ -\psi(\bar{\mathbf{X}}_n(s-))\}\bar{p}_{ij}(dsd\theta),$$

where $\psi\in C_0^\infty(R^{2n})$; here we have put $\bar{X}_i'(s-)=\bar{X}_i(s-)+a(\bar{X}_i(s-),\bar{X}_j(s-),\theta)$ and $\bar{X}_j'(s-)=\bar{X}_j(s-)-a(\bar{X}_i(s-),\bar{X}_j(s-),\theta)$.

(ii) Using the formula

$$\varphi(\bar{X}_i^{k+1}(t))=\varphi(\bar{X}_i(0))+\sum_{j\neq i}\int_{U_t}\{\varphi(\bar{X}_i^k(s-)+a(\bar{X}_i^k(s-),\bar{X}_j^k(s-),\theta))\\ -\varphi(\bar{X}_i^k(s-))\}\bar{p}_{ij}(dsd\theta)$$

with $\varphi(x)=|x|^{2+\delta}$ and then noting (4.9), we have

$$E\{|\bar{X}_i^{k+1}(t)|^{2+\delta}\}\\ \leqq E\{|\bar{X}_i(0)|^{2+\delta}\}+2^{3+3\delta}M\cdot\frac{1}{n}\sum_{j\neq i}\int_0^t E\{|\bar{X}_i^k(s)|^{2+\delta}+|\bar{X}_j^k(s)|^{2+\delta}\}ds$$

$$\leqq E\{|\bar{X}_i(0)|^{2+\delta}\} + 2^{4+3\delta}M\int_0^t E\{|\bar{X}_i^k(s)|^{2+\delta}\}ds, \qquad k \geqq 0.$$

Therefore we obtain

$$E\{|\bar{X}_i^{k+1}(t)|^{2+\delta}\} \leqq E\{|\bar{X}_i(0)|^{2+\delta}\}\exp(2^{4+3\delta}Mt),$$

and hence (5.5). When $\delta=0$, we use

$$\varphi(\bar{X}_i(t)) = \varphi(\bar{X}_i(0)) + \sum_{j\neq i}\int_{U_t}\{\varphi(\bar{X}_i(s-) + a(\bar{X}_i(s-), \bar{X}_j(s-), \theta))$$
$$- \varphi(\bar{X}_i(s-))\}\bar{p}_{ij}(dsd\theta), \qquad 1 \leqq i \leqq n,$$

with $\varphi(x)=|x|^2$, and obtain

$$\sum_{i=1}^n |\bar{X}_i(t)|^2 - \sum_{i=1}^n |\bar{X}_i(0)|^2$$
$$= \sum_{i<j}\int_{U_t}\{|\bar{X}_i(s-) + a(\bar{X}_i(s-), \bar{X}_j(s-), \theta)|^2 - |\bar{X}_i(s-)|^2\}\bar{p}_{ij}(dsd\theta)$$
$$+ \sum_{i>j}\int_{U^t}\{|\bar{X}_i(s-) + a(\bar{X}_i(s-), \bar{X}_j(s-), \theta)|^2 - |\bar{X}_i(s-)|^2\}\bar{p}_{ij}(dsd\theta)$$
$$= \sum_{i<j}\int_{U_t}\{|\bar{X}_i(s-) + a(\bar{X}_i(s-), \bar{X}_j(s-), \theta)|^2 - |\bar{X}_i(s-)|^2$$
$$+ |\bar{X}_j(s-) + a(\bar{X}_j(s-), \bar{X}_i(s-), \theta)|^2 - |\bar{X}_j(s-)|^2\}\bar{p}_{ij}(dsd\theta)$$
$$= \sum_{i<j}\int_{U_t}\{|\bar{X}'_i(s-)|^2 + |\bar{X}'_j(s-)|^2 - |\bar{X}_i(s-)|^2 - |\bar{X}_j(s-)|^2\}\bar{p}_{ij}(dsd\theta)$$
$$= 0.$$

This completes the proof.

§6. Propagation of chaos

Let f be a probability measure in R^2 belonging to $\mathscr{P}$, that is, $\int_{R^2}|x|f(dx) < \infty$. For a given positive integer n, let $\bar{\mathbf{X}}_n(t)=(\bar{X}_1(t),\dots, \bar{X}_n(t))$ be a Markov process with generator

$$G_n\psi(x) = \frac{1}{n}\sum_{i<j}\int_{-\pi}^{\pi}\{\psi(x_1,\dots, x'_i,\dots, x'_j,\dots, x_n) - \psi(x)\}Q(d\theta),$$
$$x = (x_1,\dots, x_n)\in R^{2n}, \qquad \psi\in C_0^\infty(R^{2n})$$

and with initial distribution $f\otimes\cdots\otimes f$ (n-fold product); it was constructed in §5 as a solution of (5.2). For fixed positive integer $m\,(\leqq n)$, we denote by $\overline{\boldsymbol{X}}_{m|n}(t)$ the first m coordinates of $\overline{\boldsymbol{X}}_n(t)$. Then almost all sample paths of $\overline{\boldsymbol{X}}_{m|n}$ belong to the space W_m of all R^{2m}-valued right continuous functions on R_+^1 having left limits. On this space we consider the Skorohod topology induced by the metric

$$s_m(w, \tilde{w}) = \sum_{k=1}^{\infty} \frac{1}{2^k} \{\inf [\sup_{0\leqq t\leqq k} |w(t) - \tilde{w}(\lambda(t))| + \sup_{0\leqq t\leqq k} |t-\lambda(t)|] \wedge 1\}\,,$$

$$w,\ \tilde{w} \in W_m,$$

where the infimum is taken over all homeomorphisms λ: $[0, k]\to[0, k]$ with $\lambda(0)=0$ and $\lambda(k)=k$. Then the topological Borel field $\mathscr{B}_m$ of W_m coincides with the σ-field generated by cylinder sets. We denote by $P_{m|n}$ the probability measure on $(W_m, \mathscr{B}_m)$ induced by the process $\overline{\boldsymbol{X}}_{m|n}$. We also consider the Markov process $\boldsymbol{X}=\{X(t), t\geqq 0\}$ associated with (4.1) having initial distribution f, and denote by P_f the probability measure on $(W_1, \mathscr{B}_1)$ induced by $\boldsymbol{X}$. We have the following propagation of chaos.

THEOREM 6.1. *Assume that* $\int_{R^2} |x|^{2+\delta} f(dx)<\infty$ *for some* $\delta>0$. *Then for each fixed* $m\geqq 1$

$$P_{m|n} \longrightarrow \overbrace{P_f\otimes\cdots\otimes P_f}^{m} \quad \text{as} \quad n\to\infty.$$

For the proof, it is enough to show the following theorem, which is slightly stronger.

THEOREM 6.2. *Let f be the same as in Theorem* 6.1 *and m be a fixed positive integer. Then for any $\varepsilon>0$ and $T>0$ there exists a positive integer n_0 $(\geqq m)$ such that the following statement holds: If $n\geqq n_0$, then on a suitable probability space we can construct*

(i) *n independent solutions $X_1(t),\dots, X_n(t)$ of* (4.2) *with initial distribution f, and*

(ii) *a solution $(\overline{X}_1(t),\dots, \overline{X}_n(t))$ of* (5.2) *with initial distribution* $\overbrace{f\otimes\cdots\otimes f}^{n}$ *such that*

$$E\{\sup_{0\leqq t\leqq T} |X_i(t) - \overline{X}_i(t)|\} < \varepsilon, \qquad 1\leqq i\leqq m.$$

The proof of Theorem 6.2 is divided into some steps. Let $n\,(\geqq m)$ be fixed. On a suitable probability space $(\Omega, \mathscr{F}, P)$ we choose an independent family $\{p_{ij}(dtd\theta d\alpha),\ 1\leqq i, j\leqq n\}$ of Poisson random measures on $R_+^1\times(-\pi, \pi)\times(0, 1/n]$ with common mean measure $dtQ(d\theta)d\alpha$, and independent, identically f-distributed random variables $\{\overline{X}_i(0),\ 1\leqq i\leqq n\}$. We also assume that $\{\overline{X}_i(0),\ 1\leqq i\leqq n\}$ and $\{p_{ij}(dtd\theta d\alpha),\ 1\leqq i, j\leqq n\}$ are independent. If we put

$$\mathscr{F}_t = \sigma(\{\bar{X}_i(0),\ p_{ij}(A),\ 1 \leqq i, j \leqq n;\ A \in \mathscr{B}([0, t] \times (-\pi, \pi) \times (0, 1/n])\}),$$

then $p_{ij}(dtd\theta d\alpha)$, $1 \leqq i, j \leqq n$, are $\mathscr{F}_t$-adapted. We define

$$\bar{p}_{ij}(\Gamma) = \begin{cases} p_{ij}(\Gamma), & i \leqq j, \\ p_{ji}(\Gamma), & i > j, \end{cases}$$

$$\Gamma \in \mathscr{B}(R_+^1 \times (-\pi, \pi) \times (0, 1/n]), \qquad 1 \leqq i, j \leqq n,$$

and also

$$p_i(\Gamma) = \sum_{j=1}^{n} p_{ij}(\Gamma_j), \quad \bar{p}_i(\Gamma) = \sum_{j=1}^{n} \bar{p}_{ij}(\Gamma_j), \qquad 1 \leqq i \leqq n$$

for $\Gamma \in \mathscr{B}(R_+^1 \times (-\pi, \pi) \times (0, 1])$, where $\Gamma_j = \{(t, \theta, \alpha) \in R_+^1 \times (-\pi, \pi) \times (0, 1/n]:$ $(t, \theta, \alpha + (j-1)/n) \in \Gamma\}$. Note that $p_i(dtd\theta d\alpha)$, $1 \leqq i \leqq n$, are independent, $\mathscr{F}_t$-adapted Poisson random measures on $R_+^1 \times (-\pi, \pi) \times (0, 1]$ with common mean measure $dtQ(d\theta)d\alpha$, and that

$$\bar{p}_{ij}(dtd\theta) \equiv \bar{p}_{ij}(dtd\theta \cdot (0, 1/n]), \qquad 1 \leqq i, j \leqq n$$

are $\mathscr{F}_t$-adapted Poisson random measures on $R_+^1 \times (-\pi, \pi)$ with mean measure $dtQ(d\theta)/n$. Therefore, from the results in § 5, there exists a unique $\mathscr{F}_t$-adapted solution $\bar{\boldsymbol{X}}_n(t) = (\bar{X}_1(t), \dots, \bar{X}_n(t))$ of

$$(6.1) \qquad \bar{X}_i(t) = \bar{X}_i(0) + \sum_{j \neq i} \int_{U_t} a(\bar{X}_i(s-), \bar{X}_j(s-), \theta) \bar{p}_{ij}(dsd\theta), \qquad 1 \leqq i \leqq n.$$

For the proof of Theorem 6.2 it is convenient to express (6.1) in the following form:

$$(6.2) \quad \bar{X}_i(t) = \bar{X}_i(0) + \int_{S_t} a(\bar{X}_i(s-), \bar{X}(s-, \omega, \alpha), \theta) \bar{p}_i(dsd\theta d\alpha), \qquad 1 \leqq i \leqq n,$$

where $\bar{X}(t, \omega, \alpha) = \bar{X}_j(t)$ for $\alpha \in I_j \equiv ((j-1)/n, j/n]$. Let $u(t)$ be the probability distribution at time t of the Markov process with initial distribution f which is associated with (4.1). Then we can prove the following

LEMMA 6.3. *Let $\bar{X}_i(t)$, $1 \leqq i \leqq n$, $\bar{X}(t, \omega, \alpha)$ and $u(t)$ be the same as above. Then for any $\varepsilon > 0$ there exists a process $Y^\varepsilon(t, \omega, \alpha)$ having the following properties.*

(6.3) *$Y^\varepsilon(t, \omega, \alpha)$ is $\mathscr{F}_t$-predictable.*

(6.4) *The probability distribution of $Y^\varepsilon(t, \omega, \cdot)$ is $u(t-)$ for fixed ω.*

(6.5) $E\left\{\int_0^1 |\bar{X}(t-, \omega, \alpha) - Y^\varepsilon(t, \omega, \alpha)| d\alpha\right\} \leqq E\{\rho(\bar{f}(t-, \omega), u(t-))\} + \varepsilon,$

where $\bar{f}(t-, \omega)$ is the probability distribution of $\bar{X}(t-, \omega, \cdot)$ on $\alpha \in (0, 1]$ for *each fixed* ω.

PROOF. Take a Borel isomorphism $\xi: R^2 \to R^1$. Since $\bar{f}(t-, \omega)$ and $u(t-)$ belong to $\mathscr{P}$, we can apply Proposition 2.2 to have a transition function $P^{\varepsilon}_{\bar{f}(t-,\omega),u(t-)}(x, B)$, $x \in R^2$, $B \in \mathscr{B}(R^2)$ satisfying (2.7)$\sim$(2.10). $\bar{X}(t-, \omega, \alpha)$, $\alpha \in I_j$, is the constant $\bar{X}_j(t-)$ for fixed (t, ω). Therefore we may consider

$$P^{\varepsilon}_j(t, \omega, \cdot) \equiv P^{\varepsilon}_{\bar{f}(t-,\omega),u(t-)}(\bar{X}_j(t-), \cdot), \qquad 1 \leqq j \leqq n,$$

as probability measures in R^2. Denote by $Y^*_j(t, \omega, \beta)$, $\beta \in (0, 1]$, the right continuous inverse function of the distribution function $P^{\varepsilon}_j(t, \omega, \xi^{-1}((-\infty, x]))$ and put

$$Y^{\varepsilon}(t, \omega, \alpha) = \xi^{-1}(Y^*_j(t, \omega, n\alpha - j + 1)), \qquad \alpha \in I_j.$$

Then $Y^{\varepsilon}(t, \omega, \alpha)$ satisfies (6.3)$\sim$(6.5). In fact, (6.3) is clear from the construction, and as for the rest we use (2.9), (2.10) and the relation

$$n|\{\alpha \in I_j;\ Y^{\varepsilon}(t, \omega, \alpha) \in \Gamma\}|^{*)} = P^{\varepsilon}_j(t, \omega, \Gamma), \qquad \Gamma \in \mathscr{B}(R^2),$$

to obtain

$$\begin{aligned} |\{\alpha \in (0, 1];\ Y^{\varepsilon}(t, \omega, \alpha) \in \Gamma\}| &= \frac{1}{n}\sum_{j=1}^{n} P^{\varepsilon}_j(t, \omega, \Gamma) \\ &= \int_{R^2} P^{\varepsilon}_{\bar{f}(t-,\omega),u(t-)}(x, \Gamma)\bar{f}(t-, \omega, dx) \\ &= u(t-, \Gamma), \end{aligned}$$

and

$$\begin{aligned} \int_0^1 |\bar{X}(t-, \omega, \alpha) - Y^{\varepsilon}(t, \omega, \alpha)|\, d\alpha &= \sum_{j=1}^{n}\frac{1}{n}\int_{R^2}|\bar{X}_j(t-) - y|\, P^{\varepsilon}_j(t, \omega, dy) \\ &= \int_{R^2}\int_{R^2}|x - y| P^{\varepsilon}_{\bar{f}(t-,\omega),u(t-)}(x, dy)\bar{f}(t-, \omega, dx) \\ &\leqq \rho(\bar{f}(t-, \omega), u(t-)) + \varepsilon. \end{aligned}$$

Therefore the lemma is proved.

Using this process $Y^{\varepsilon}(t, \omega, \alpha)$ we consider the following integral equations:

$$(6.6)\quad X_i(t) = \bar{X}_i(0) + \int_{S_t} a(X_i(s-), Y^{\varepsilon}(s, \omega, \alpha), \theta)p_i(ds d\theta d\alpha), \qquad 1 \leqq i \leqq n.$$

*) $|A|$ denotes the Lebesgue measure of $A \subset (0, 1]$.

By Theorem 4.2, for each i, there exists a unique $\mathscr{F}_t$-adapted solution $\{X_i(t), t\geqq 0\}$ of (6.6) which is equivalent in the law sense to the Markov process associated with (4.1) having initial distribution f.

LEMMA 6.4. *$\{X_i(t), t\geqq 0\}$, $1\leqq i\leqq n$, are independent.*

PROOF. For each i we define

$$p_i^{\#}(A)=\int_{R_+^1\times(-\pi,\pi)\times(0,1]}\chi_A(s,\theta,Y^{\varepsilon}(s,\omega,\alpha))p_i(dsd\theta d\alpha),\qquad A\in\mathscr{B}(R_+^1\times(-\pi,\pi)\times R^2).$$

Then $\{p_i^{\#}(dtd\theta dy), 1\leqq i\leqq n\}$ are $\mathscr{F}_t$-adapted random measures on $R_+^1\times(-\pi,\pi)\times R^2$ with the common mean measure $dtQ(d\theta)u(t, dy)$:

$$E\{p_i^{\#}(A)\} = E\Big\{\int_{R_+^1\times(-\pi,\pi)\times(0,1]}\chi_A(s,\theta,Y^{\varepsilon}(s,\omega,\alpha))d\alpha Q(d\theta)ds\Big\}$$

$$= \int_{R_+^1\times(-\pi,\pi)\times R^2}\chi_A(s,\theta,y)u(s,dy)Q(d\theta)ds.$$

Since the equation (6.6) can be written in the form

$$X_i(t) = \overline{X}_i(0) + \int_{[0,t]\times(-\pi,\pi)\times R^2} a(X_i(s-), y, \theta)p_i^{\#}(dsd\theta dy),\quad 1\leqq i\leqq n,$$

for the proof of Lemma 6.4, it is enough to show that $\{p_i^{\#}(dtd\theta dy), 1\leqq i\leqq n\}$ are independent Poisson random measures. Put $S=\sum_{i=1}^n S_i$ (direct sum) for $S_i=(-\pi,\pi)\times R^2$, and define $p^{\#}(A)=\sum_{i=1}^n p_i^{\#}(A_i)$, $A_i=A\cap(R_+^1\times S_i)$, for each Borel set $A\subset R_+^1\times S$. Since

$$a(s,\theta,\alpha,\omega)\equiv\chi_{B_i}(\theta, Y^{\varepsilon}(s,\omega,\alpha))$$

is $\mathscr{F}_t$-predictable for $B_i\in\mathscr{B}((-\pi,\pi)\times R^2)$ and $\{p_i(dtd\theta d\alpha), 1\leqq i\leqq n\}$ are independent $\mathscr{F}_t$-adapted Poisson random measures with mean measure $dtQ(d\theta)d\alpha$,

$$\sum_{i=1}^n\Big\{\int_{[0,t]\times S_i}\chi_{B_i}(\theta, Y^{\varepsilon}(s,\omega,\alpha))p_i(dsd\theta d\alpha)$$

$$-\int_{[0,t]\times S_i}\chi_{B_i}(\theta, Y^{\varepsilon}(s,\omega,\alpha))d\alpha Q(d\theta)ds\Big\}$$

is $\mathscr{F}_t$-martingale. Hence, for $B=\sum_{i=1}^n B_i$,

$$p^{\#}([0,t]\times B)-\int_0^t u(s,B)Q(d\theta)ds$$

$$= \sum_{i=1}^{n}\left\{\int_{[0,t]\times S_i} \chi_{B_i}(\theta, y)p_i^{\#}(dsd\theta dy) - \int_{[0,t]\times S_i} \chi_{B_i}(\theta, y)u(s, dy)Q(d\theta)ds\right\}$$

$$= \sum_{i=1}^{n}\left\{\int_{[0,t]\times S_i} \chi_{B_i}(\theta, Y^{\varepsilon}(s, \omega, \alpha))p_i(dsd\theta d\alpha)\right.$$

$$\left. - \int_{[0,t]\times S_i} \chi_{B_i}(\theta, Y^{\varepsilon}(s, \omega, \alpha))d\alpha Q(d\theta)ds\right\}$$

is also $\mathscr{F}_t$-martingale. Therefore Theorem 3.1 implies that $\{p_i^{\#}(dtd\theta dy), 1 \leqq i \leqq n\}$ are independent Poisson random measures. This implies the independence of $\{X_i(t), t \geqq 0\}$, $1 \leqq i \leqq n$.

To estimate the difference between $X_i(t)$ of (6.6) and $\bar{X}_i(t)$ of (6.2), it is convenient to consider the following auxiliary stochastic integral equations:

$$(6.7) \quad \tilde{X}_i(t) = \bar{X}_i(0) + \int_{S_t} a(\tilde{X}_i(s-), Y^{\varepsilon}(s, \omega, \alpha), \theta)\bar{p}_i(dsd\theta d\alpha), \qquad 1 \leqq i \leqq n.$$

By Theorem 4.2, (6.7) has a unique solution which is equivalent in the law sense to $\{X_i(t)\}$.

LEMMA 6.5. *For each $T>0$ and an integer $m>0$, there exists a positive constant c, independent of n and ε, such that*

$$E\{\sup_{0\leqq t\leqq T}|X_i(t) - \tilde{X}_i(t)|\} \leqq c\cdot n^{-1/2}, \qquad 1 \leqq i \leqq m.$$

PROOF. By the smoothness of $a(x, y, \theta)$ expressed in (4.3) and the definition of Poisson random measures $\{p_i(dtd\theta d\alpha), 1 \leqq i \leqq n\}$ and $\{\bar{p}_i(dtd\theta d\alpha), 1 \leqq i \leqq n\}$, we have

$$(6.8) \quad |X_i(t) - \tilde{X}_i(t)|$$

$$\leqq \int_{[0,t]\times(-\pi,\pi)\times((i-1)/n,1]} |a(X_i(s-), Y^{\varepsilon}(s, \omega, \alpha), \theta) - a(\tilde{X}_i(s-), Y^{\varepsilon}(s, \omega, \alpha), \theta)|p_i(dsd\theta d\alpha)$$

$$+ \sum_{j<i}\int_{[0,t]\times(-\pi,\pi)\times(0,1/n]} |a(X_i(s-), Y^{\varepsilon}(s, \omega, \alpha + (j-1)/n), \theta)|p_{ij}(dsd\theta d\alpha)$$

$$+ \sum_{j<i}\int_{[0,t]\times(-\pi,\pi)\times(0,1/n]} |a(\tilde{X}_i(s-), Y^{\varepsilon}(s, \omega, \alpha + (j-1)/n), \theta)|p_{ji}(dsd\theta d\alpha)$$

$$\leqq \int_{S_t} |X_i(s-) - \tilde{X}_i(s-)|\cdot\left|\sin\frac{\theta}{2}\right| p_i(dsd\theta d\alpha)$$

$$+\sum_{j<i}\int_{[0,t]\times(-\pi,\pi)\times(0,1/n]}[|X_i(s-)|+|Y^\varepsilon(s,\omega,\alpha+(j-1)/n)]\cdot\left|\sin\frac{\theta}{2}\right|p_{ij}(dsd\theta d\alpha)$$

$$+\sum_{j<i}\int_{[0,t]\times(-\pi,\pi)\times(0,1/n]}[|\tilde{X}_i(s-)|+|Y^\varepsilon(s,\omega,\alpha+(j-1)/n)|]\cdot\left|\sin\frac{\theta}{2}\right|p_{ji}(dsd\theta d\alpha)$$

for each $1\leqq i\leqq m$. Taking the expectation of both sides of (6.8) and putting $M=\int_{-\pi}^{\pi}\left|\sin\frac{\theta}{2}\right|Q(d\theta)$, we have

$$E\{|X_i(t)-\tilde{X}_i(t)|\}$$

$$\leqq M\int_0^t E\{|X_i(s)-\tilde{X}_i(s)|\}ds+\frac{mM}{n}\int_0^t E\{|X_i(s)|+|\tilde{X}_i(s)|\}ds$$

$$+2M\sum_{j<i}\int_0^t E\left\{\int_0^1\chi_{I_j}(\alpha)|Y^\varepsilon(s,\omega,\alpha)|d\alpha\right\}ds.$$

Therefore Schwarz's inequality implies that

$$E\{|X_i(t)-\tilde{X}_i(t)|\}\leqq M\int_0^t E\{|X_i(s)-\tilde{X}_i(s)|\}ds$$

$$+2mM\mu^{1/2}t\cdot n^{-1}+2mM\mu^{1/2}t\cdot n^{-1/2};$$

here we put $\mu=E\{|\bar{X}_i(0)|^2\}=\int_{R^2}|x|^2f(dx)$. Hence we have

$$E\{|X_i(t)-\tilde{X}_i(t)|\}\leqq c_1\exp(c_2t)\cdot n^{-1/2}\tag{6.9}$$

for $1\leqq i\leqq m$ by Gronwall's inequality, where c_1 and c_2 are positive constants depending only on m, M and f. Taking the supremum of (6.8) for $0\leqq t\leqq T$ and using (6.9), we have

$$E\{\sup_{0\leqq t\leqq T}|X_i(t)-\tilde{X}_i(t)|\}\leqq c_3\exp(c_4T)\cdot n^{-1/2}\tag{6.10}$$

for each $1\leqq i\leqq m$. (c_3 and c_4 are positive constants depending on m, M and f.) Thus the lemma is proved.

Next let us estimate the quantity $E\{\sup_{0\leqq t\leqq T}|\bar{X}_i(t)-\tilde{X}_i(t)|\}$ for a given $T>0$. For this purpose it is essential to prove the following lemma.

LEMMA 6.6. *For fixed $1\leqq j<k\leqq n$ and $\varphi\in\Phi_L^\varepsilon$**), *there exists a positive constant c' depending only on M and f such that*

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*) See §2 for terminology.

(6.11) $$|E\{\varphi(\tilde{X}_j(t))\varphi(\tilde{X}_k(t))\} - M_\varphi^2(t)| \leqq c' \exp(c't)\cdot n^{-\delta/2(2+\delta)},$$

where $M_\varphi(t) = E\{\varphi(X(t))\} \left(=\int_{R^2}\varphi(y)u(t, dy)\right)$.

PROOF. Take a new Poisson random measure

$$\bar{p}_k^j(\Gamma) = \sum_{i\neq j}\bar{p}_{ki}(\Gamma_i) + p_{kj}(\Gamma_j), \quad \Gamma\in\mathscr{B}(R_+^1\times(-\pi, \pi)\times(0, 1]),$$

on $R_+^1\times(-\pi, \pi)\times(0, 1]$ with mean measure $dtQ(d\theta)d\alpha$, where $\Gamma_i = \{(t, \theta, \alpha)\in R_+^1\times(-\pi, \pi)\times(0, 1]; (t, \theta, \alpha+(i-1)/n)\in\Gamma\}$. We consider the stochastic integral equation:

(6.12) $$\tilde{X}_k^j(t) = \bar{X}_k(0) + \int_{S_t} a(\tilde{X}_k^j(s-), Y^\varepsilon(s, \omega, \alpha), \theta)\bar{p}_k^j(dsd\theta d\alpha).$$

The existence and the uniqueness of $\mathscr{F}_t$-adapted solution are easily proved. We will notice that the probability distribution of $\tilde{X}_k^j(t)$ is the same as that of $\tilde{X}_k(t)$ (and of $X(t)$), and that $\tilde{X}_k^j(t)$ is independent of $\tilde{X}_j(t)$ because the Poisson random measures $\bar{p}_k^j$ and $\bar{p}_j$ are independent. (See the argument used in the proof of Lemma 6.4.) Therefore $M_\varphi^2(t) = E\{\varphi(\tilde{X}_j(t))\varphi(\tilde{X}_k^j(t))\}$, and hence

(6.13) $$\begin{aligned}&|E\{\varphi(\tilde{X}_j(t))\varphi(\tilde{X}_k(t))\} - M_\varphi^2(t)|\\ &= |E\{\varphi(\tilde{X}_j(t))\cdot(\varphi(\tilde{X}_k(t)) - \varphi(\tilde{X}_k^j(t)))\}|\\ &\leqq [E\{|\tilde{X}_j(t)|^2\}\cdot E\{|\tilde{X}_k(t) - \tilde{X}_k^j(t)|^2\}]^{1/2}\end{aligned}$$

for $\varphi\in\Phi_L^\varepsilon$. Next applying the transformation formula of stochastic integrals for

$$\begin{aligned}\tilde{X}_k(t) - \tilde{X}_k^j(t) &= \int_{[0,t]\times(-\pi,\pi)\times((0,1]-I_j)} a(\tilde{X}_k(s-) - \tilde{X}_k^j(s-), 0, \theta)\,\bar{p}_k(dsd\theta d\alpha)\\ &\quad + \int_{[0,t]\times(-\pi,\pi)\times(0,1/n]} a(\tilde{X}_k(s-), Y^\varepsilon(s, \omega, \alpha+(j-1)/n), \theta)\bar{p}_{kj}(dsd\theta d\alpha)\\ &\quad - \int_{[0,t]\times(-\pi,\pi)\times(0,1/n]} a(\tilde{X}_k^j(s-), Y^\varepsilon(s, \omega, \alpha+(j-1)/n), \theta)p_{kj}(dsd\theta d\alpha),\end{aligned}$$

we obtain

$$\begin{aligned}&|\tilde{X}_k(t) - \tilde{X}_k^j(t)|^2\\ &= \int_{[0,t]\times(-\pi,\pi)\times((0,1]-I_j)} \{|\tilde{X}_k(s-) - \tilde{X}_k^j(s-) + a(\tilde{X}_k(s-) - \tilde{X}_k^j(s-), 0, \theta)|^2\end{aligned}$$

$$- |\tilde{X}_k(s-) - \tilde{X}_k^j(s-)|^2\}\bar{p}_k(dsd\theta d\alpha)$$

$$+ \int_{[0,t]\times(-\pi,\pi)\times(0,1/n]} \{|\tilde{X}_k(s-) - \tilde{X}_k^j(s-) + a(\tilde{X}_k(s-), Y^\varepsilon(s, \omega, \alpha + (j-1)/n), \theta)|^2$$

$$- |\tilde{X}_k(s-) - \tilde{X}_k^j(s-)|^2\}\bar{p}_{kj}(dsd\theta d\alpha)$$

$$+ \int_{[0,t]\times(-\pi,\pi)\times(0,1/n]} \{|\tilde{X}_k(s-) - \tilde{X}_k^j(s-) - a(\tilde{X}_k^j(s-), Y^\varepsilon(s, \omega, \alpha + (j-1)/n), \theta)|^2$$

$$- |\tilde{X}_k(s-) - \tilde{X}_k^j(s-)|^2\}p_{kj}(dsd\theta d\alpha)$$

$$\leqq \int_{S_t} |\tilde{X}_k(s-) - \tilde{X}_k^j(s-)|^2 \cdot \left|\sin\frac{\theta}{2}\right| \bar{p}_k(dsd\theta d\alpha)$$

$$+ \int_{[0,t]\times(-\pi,\pi)\times(0,1/n]} [4\{|\tilde{X}_k(s-)|^2 + |Y^\varepsilon(s, \omega, \alpha + (j-1)/n)|^2\}$$

$$+ |\tilde{X}_k(s-) - \tilde{X}_k^j(s-)|^2] \cdot \left|\sin\frac{\theta}{2}\right| \bar{p}_{kj}(dsd\theta d\alpha)$$

$$+ \int_{[0,t]\times(-\pi,\pi)\times(0,1/n]} [4\{|\tilde{X}_k^j(s-)|^2 + |Y^\varepsilon(s, \omega, \alpha + (j+1)/n)|^2\}$$

$$+ |\tilde{X}_k(s-) - \tilde{X}_k^j(s-)|^2] \cdot \left|\sin\frac{\theta}{2}\right| p_{kj}(dsd\theta d\alpha);$$

in the above we have used

$$\left| |x + a(x, 0, \theta)|^2 - |x|^2 \right| = \left|\sin\frac{\theta}{2}\right|^2 \cdot |x|^2,$$

$$\begin{aligned} \left| |z + a(x, y, \theta)|^2 - |z|^2 \right| &\leqq |a(x, y, \theta)| \{ |a(x, y, \theta)| + 2|z| \} \\ &\leqq \left|\sin\frac{\theta}{2}\right| (|x| + |y|)(|x| + |y| + 2|z|) \\ &\leqq \{4(|x|^2 + |y|^2) + |z|^2\} \left|\sin\frac{\theta}{2}\right|. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} &E\{|\tilde{X}_k(t) - \tilde{X}_k^j(t)|^2\} \\ &\leqq 3M\int_0^t E\{|\tilde{X}_k(s) - \tilde{X}_k^j(s)|^2\}ds + 4Mn^{-1}\int_0^t E\{|\tilde{X}_k(s)|^2 + |\tilde{X}_k^j(s)|^2\}ds \\ &\quad + 8M\int_0^t E\left\{\int_0^1 \chi_{I_j}(\alpha) \cdot |Y^\varepsilon(s, \omega, \alpha)|^2 d\alpha\right\} ds. \end{aligned}$$

Using Hölder's inequality and the estimates (4.7), we have

$$E\{|\tilde{X}_k(t) - \tilde{X}_k^j(t)|^2\} \leqq 3M\int_0^t E\{|\tilde{X}_k(s) - \tilde{X}_k^j(s)|^2\}ds + 8M\mu t\cdot n^{-1}$$
$$+ 8Mv^{2/(2+\delta)}t\cdot\exp(2^{4+3\delta}Mt)\cdot n^{-\delta/(2+\delta)},$$

where we put $\mu=\int_{R^2}|x|^2 f(dx)$ and $v=\int_{R^2}|x|^{2+\delta}f(dx)$. Hence there exist positive constants c_5 and c_6 depending only on M and f such that

$$E\{|\tilde{X}_k(t) - \tilde{X}_k^j(t)|^2\} \leqq c_5\exp(c_6 t)\cdot n^{-\delta/(2+\delta)}.$$

Combining this with (6.13), we obtain

$$|E\{\varphi(\tilde{X}_j(t))\varphi(\tilde{X}_k(t))\} - M_\varphi^2(t)| \leqq c'\exp(c't)\cdot n^{-\delta/2(2+\delta)}$$

with $c'=\max\{(c_5\mu)^{1/2}, c_6/2\}$, which implies the lemma.

LEMMA 6.7. *For each $T>0$ and $\varepsilon>0$,*

$$E\{\sup_{0\leqq t\leqq T}|\bar{X}_i(t) - \tilde{X}_i(t)|\} \leqq K_\varepsilon\cdot n^{-\delta/4(2+\delta)} + c''\cdot\varepsilon,$$

where K_ε is a positive constant depending on M, f, T and ε, and c'' is a positive constant depending on M, f and T.

PROOF. Using the smoothness of $a(x, y, \theta)$, we have

$$(6.14)\quad |\bar{X}_i(t) - \tilde{X}_i(t)|$$
$$\leqq \int_{S_t}|a(\bar{X}_i(s-), \bar{X}(s-, \omega, \alpha), \theta) - a(\tilde{X}_i(s-), Y^\varepsilon(s, \omega, \alpha), \theta)|\bar{p}_i(dsd\theta d\alpha)$$
$$\leqq \int_{S_t}[|\bar{X}_i(s-)-\tilde{X}_i(s-)| + |\bar{X}(s-, \omega, \alpha)-Y^\varepsilon(s, \omega, \alpha)|]\cdot\left|\sin\frac{\theta}{2}\right|\bar{p}_i(dsd\theta d\alpha).$$

By the relation (6.5) we have

$$E\{|\bar{X}_i(t) - \tilde{X}_i(t)|\}$$
$$\leqq M\int_0^t E\{|\bar{X}_i(s-) - \tilde{X}_i(s-)| + \int_0^1|\bar{X}(s-, \omega, \alpha) - Y^\varepsilon(s, \omega, \alpha)|d\alpha\}ds$$
$$\leqq M\int_0^t E\{|\bar{X}_i(s-) - \tilde{X}_i(s-)| + \rho(\bar{f}(s-, \omega), u(s-))\}ds + Mt\varepsilon,$$

where $\bar{f}(s-, \omega)$ and $u(s-)$ are probability distributions of $\bar{X}(s-, \omega, \cdot)$ on (0, 1] and $X(s-)$ on Ω respectively. Let $\tilde{f}(s-, \omega)$ be the probability distribution of $\tilde{X}(s-, \omega, \alpha)$, where $\tilde{X}(s-, \omega, \alpha)=\tilde{X}_j(s-)$ for $\alpha\in I_j$, $1\leqq j\leqq n$. Then

$$E\{\rho(\bar{f}(s-,\omega),u(s-))\}$$
$$\leqq E\{\rho(\bar{f}(s-,\omega),\tilde{f}(s-,\omega))\}+E\{\rho(\tilde{f}(s-,\omega),u(s-))\}$$
$$\leqq \frac{1}{n}\sum_{j=1}^{n}E\{|\bar{X}_j(s-)-\tilde{X}_j(s-)|\}+E\{\rho(\tilde{f}(s-,\omega),u(s-))\},$$

and so

$$(6.15)\qquad E\{|\bar{X}_i(t)-\tilde{X}_i(t)|\}$$
$$\leqq M\int_0^t E\{|\bar{X}_i(s)-\tilde{X}_i(s)|+\frac{1}{n}\sum_{j=1}^{n}|\bar{X}_j(s)-\tilde{X}_j(s)|\}ds$$
$$+M\int_0^t E\{\rho(\tilde{f}(s-,\omega),u(s-))\}ds+Mt\varepsilon.$$

Put $d(\bar{X}(t),\tilde{X}(t))=\sum_{i=1}^{n}E\{|\bar{X}_i(t)-\tilde{X}_i(t)|\}$ and sum up (6.15) in i. Then Gronwall's inequality says that

$$(6.16)\qquad d(\bar{X}(t),\tilde{X}(t))\leqq M\left\{\sum_{i=1}^{n}\int_0^t E\{\rho(\tilde{f}(s-,\omega),u(s-))\}ds+nt\varepsilon\right\}\exp(2Mt).$$

Inserting (6.16) to (6.15), we have

$$(6.17)\qquad E\{|\bar{X}_i(t)-\tilde{X}_i(t)|\}$$
$$\leqq M\int_0^t E\{|\bar{X}_i(s)-\tilde{X}_i(s)|\}ds+M\int_0^t E\{\rho(\tilde{f}(s-,\omega),u(s-))\}ds$$
$$+M^2\int_0^t\left\{\int_0^s E\{\rho(\tilde{f}(v-,\omega),u(v-))\}dv+s\varepsilon\right\}\cdot\exp(2Ms)ds$$
$$+Mt\varepsilon.$$

Therefore for the estimate of the quantity $E\{|\bar{X}_i(t)-\tilde{X}_i(t)|\}$ it is enough to calculate $E\{\rho(\tilde{f}(s-,\omega),u(s-))\}$, $0\leqq s\leqq T$. Proposition 2.1 implies that there exists a constant $K=K(\varepsilon,L)$ such that

$$E\{\rho(\tilde{f}(s-,\omega),u(s-))\}$$
$$\leqq K\cdot E\left\{\max_{\varphi\in\Phi_L^{\varepsilon}}\left[\left[\int_{R^2}\varphi(x)\tilde{f}(s-,\omega,dx)-\int_{R^2}\varphi(x)u(s-,dx)\right]\right]\right\}$$
$$+E\{\rho(\tilde{f}(s-,\omega),\tilde{f}(s-,\omega)_L)\}+\rho(u(s-),u(s-)_L)+\varepsilon.$$

Since

$$E\{\rho(\tilde{f}(s-,\omega),\tilde{f}(s-,\omega)_L)\} \leqq E\left\{\int_{|x|>L}|x|\tilde{f}(s-,\omega,dx)\right\}$$

$$\leqq E\left\{\frac{1}{L}\int_{R^2}|x|^2\tilde{f}(s-,\omega,dx)\right\}$$

$$= E\left\{\frac{1}{L}\cdot\frac{1}{n}\sum_{j=1}^{n}|\tilde{X}_j(s-)|^2\right\} = \frac{\mu}{L},$$

and similarly $\rho(u(s-), u(s-)_L) \leqq \mu/L$, we can choose large $L=L(\varepsilon)$ so that

$$E\{\rho(\tilde{f}(s-,\omega), u(s-))\}$$

$$\leqq K\cdot E\left\{\max_{\varphi\in\Phi_L^\varepsilon}\left[\int_{R^2}\varphi(x)\tilde{f}(s-,\omega,dx)-\int_{R^2}\varphi(x)u(s-,dx)\right]\right\}+3\varepsilon.$$

Thus we can write

$$E\{\rho(\tilde{f}(s-,\omega), u(s-))\} \leqq K'\cdot E\left\{\left|\frac{1}{n}\sum_{j=1}^{n}\varphi(\tilde{X}_j(s-))-M_\varphi(s-)\right|\right\}+3\varepsilon \tag{6.18}$$

for $\varphi\in\Phi_L^\varepsilon$, where K' is a positive constant depending only on ε and the number of elements of Φ_L^ε, and $M_\varphi(s-)=E\{\varphi(X(s-))\}$. By Lemma 6.6, (6.18) and the relation

$$E\left\{\left|\frac{1}{n}\sum_{j=1}^{n}\varphi(\tilde{X}_j(s-))-M_\varphi(s-)\right|^2\right\}$$

$$\leqq \frac{1}{n^2}\sum_{j=1}^{n}E\{|\tilde{X}_j(s-)|^2\}+\frac{1}{n^2}\sum_{j\neq k}\left|E\{\varphi(\tilde{X}_j(s-))\varphi(\tilde{X}_k(s-))\}-M_\varphi^2(s-)\right|,$$

we have

$$\int_0^t E\{\rho(\tilde{f}(s-,\omega), u(s-))\}ds \tag{6.19}$$

$$\leqq \int_0^t K'(\mu\cdot n^{-1}+c'\exp(c's)\cdot n^{-\delta/2(2+\delta)})^{1/2}ds+3\varepsilon t$$

$$\leqq K'c_7\exp(c_8t)\cdot n^{-\delta/4(2+\delta)}+3\varepsilon t$$

for some positive constants c_7 and c_8 depending only on M and f. Inserting (6.19) to (6.17) we have

$$E\{|\bar{X}_i(t)-\tilde{X}_i(t)|\} \leqq M\int_0^t E\{|\bar{X}_i(s)-\tilde{X}_i(s)|\}ds$$

$$+K'\cdot c_9\exp(c_{10}t)\cdot n^{-\delta/4(2+\delta)}+c_{11}\exp(c_{12}t)\cdot\varepsilon,$$

and hence we obtain

(6.20) $E\{|\bar{X}_i(t)-\tilde{X}_i(t)|\} \leqq K'\cdot c_{13}\exp(c_{14}t)\cdot n^{-\delta/4(2+\delta)}+c_{15}\exp(c_{16}t)\cdot\varepsilon;$

here, and from now on, c_9, c_{10},... are used for positive constants which are independent of ε and n. Taking the supremum of (6.14) for $0\leqq t\leqq T$ and using (6.19) and (6.20), we obtain

$$(6.21)\qquad E\{\sup_{0\leqq t\leqq T}|\bar{X}_i(t)-\tilde{X}_i(t)|\}$$
$$\leqq K'\cdot c_{17}\exp(c_{18}T)\cdot n^{-\delta/4(2+\delta)}+c_{19}\exp(c_{20}T)\cdot\varepsilon,$$

which proves the lemma.

The proof of Theorem 6.2 is now completed, because by Lemma 6.5 and 6.7 there exist a constant $\bar{K}_\varepsilon>0$ depending on M, f, T, m and ε and a constant $\bar{c}$ depending only on M, f and T such that

$$E\{\sup_{0\leqq t\leqq T}|X_i(t)-\bar{X}_i(t)|\} \leqq \bar{K}_\varepsilon\cdot n^{-\delta/4(2+\delta)}+\bar{c}\cdot\varepsilon.$$

§7. Law of large numbers

In this section we deal with the empirical distribution, of n molecules, defined by

$$\bar{f}_n(t,\omega,\cdot)=\frac{1}{n}\sum_{j=1}^{n}\delta_{\bar{X}_j(t,\omega)}(\cdot),\quad t\geqq 0.$$

Here $(\bar{X}_1(t),\dots,\bar{X}_n(t))$ denotes the Markov process determined by the forward equation (5.1). Let it be defined as the solution of (6.1), assuming that the initial distribution is $f\otimes\cdots\otimes f$ with $\int_{R^2}|x|^{2+\delta}f(dx)<\infty$ for some $\delta>0$. $\bar{f}_n(t,\omega,\cdot)$ is nothing but $\bar{f}(t,\omega,\cdot)$ of Lemma 6.3. As in §6, we consider the auxiliary process $(\tilde{X}_1(t),\dots,\tilde{X}_n(t))$ defined by (6.7). Then, we have proved the following estimates in (6.11) and (6.20): For any $\varepsilon>0$ and n, there exist positive constants c_1, c_2, c_3 depending on M and f and a positive constant K_ε depending on ε such that

$$(7.1)\quad |E\{\varphi(\tilde{X}_j(t))\varphi(\tilde{X}_k(t))\}-M_\varphi^2(t)|\leqq L_\varphi^2\cdot c_1\exp(c_1t)\cdot n^{-\delta/2(2+\delta)},$$
$$\varphi\in C_0^\infty(R^2),\qquad 1\leqq j\neq k\leqq n,$$

$$(7.2)\quad E\{|\bar{X}_j(t)-\tilde{X}_j(t)|\}\leqq K_\varepsilon\cdot c_2\exp(c_2t)\cdot n^{-\delta/4(2+\delta)}+c_3\exp(c_3t)\cdot\varepsilon,$$
$$1\leqq j\leqq n,$$

where $M_\varphi(t)=E\{\varphi(\tilde{X}_j(t))\}\left(=\int_{R^2}\varphi(x)u(t,dx)\right)$ and L_φ is the Lipschitz constant of φ.

Using the above inequalities, we can prove the following law of large numbers.

THEOREM 7.1. *Assume that* $\int_{R^2}|x|^{2+\delta}f(dx)<\infty$ *for some* $\delta>0$. *Then,*

$$\bar{f}_n(t, \omega, \cdot) \longrightarrow u(t, \cdot) \quad (\textit{in probability}), \qquad n \to \infty,$$

where $u(t)$ *is the solution of* (1.7) $(=(4.1))$ *with* $u(0)=f$.

PROOF. For $\varphi \in C_0^\infty(R^2)$ we estimate

$$
\begin{aligned}
(7.3)\qquad \bar{I}_n(\varphi) &\equiv E\left\{\left|\int_{R^2}\varphi(x)\bar{f}_n(t, \omega, dx) - \int_{R^2}\varphi(x)u(t, dx)\right|\right\} \\
&= E\left\{\left|\frac{1}{n}\sum_{j=1}^n \varphi(\bar{X}_j(t)) - M_\varphi(t)\right|\right\} \\
&\leqq \frac{1}{n}\sum_{j=1}^n E\{|\varphi(\bar{X}_j(t)) - \varphi(\tilde{X}_j(t))|\} + E\left\{\left|\frac{1}{n}\sum_{j=1}^n \varphi(\tilde{X}_j(t)) - M_\varphi(t)\right|\right\} \\
&\equiv I_n(\varphi) + \tilde{I}_n(\varphi).
\end{aligned}
$$

By (7.2) the first term $I_n(\varphi)$ has the estimate

$$(7.4)\qquad I_n(\varphi) \leqq L_\varphi\{K_\varepsilon \cdot c_2 \exp(c_2 t)\cdot n^{-\delta/4(2+\delta)} + c_3\exp(c_3 t)\cdot \varepsilon\}.$$

On the other hand, using the relation

$$
\begin{aligned}
&E\left\{\left|\frac{1}{n}\sum_{j=1}^n \varphi(\tilde{X}_j(t)) - M_\varphi(t)\right|^2\right\} \\
&\leqq \frac{1}{n^2}\sum_{j=1}^n E\{|\varphi(\tilde{X}_j(t))|^2\} + \frac{1}{n^2}\sum_{j\neq k}|E\{\varphi(\tilde{X}_j(t))\varphi(\tilde{X}_k(t))\} - M_\varphi^2(t)|
\end{aligned}
$$

and (7.1), we have

$$(7.5)\qquad \tilde{I}_n(\varphi)^2 \leqq L_\varphi^2\{\mu\cdot n^{-1} + c_1\exp(c_1 t)\cdot n^{-\delta/2(2+\delta)}\},$$

where $\mu=\int_{R^2}|x|^2 f(dx)$. Inserting (7.4) and (7.5) to (7.3), we obtain

$$
\begin{aligned}
(7.6)\qquad &\bar{I}_n(\varphi) \leqq L_\varphi\cdot c(n, \varepsilon) \\
&c(n, \varepsilon) = K_\varepsilon\cdot c_2\exp(c_2 t)\cdot n^{-\delta/4(2+\delta)} + c_3\exp(c_3 t)\cdot\varepsilon \\
&\qquad\qquad + \{\mu\cdot n^{-1} + c_1\exp(c_1 t)\cdot n^{-\delta/2(2+\delta)}\}^{1/2}.
\end{aligned}
$$

Using the metric d in $\mathscr{P}$ defined in § 2, for any $\varepsilon'>0$ we have

$$P\{d(\bar{f}_n(t,\omega), u(t)) > \varepsilon'\} \leqq \frac{1}{\varepsilon'} E\{d(\bar{f}_n(t,\omega), u(t))\}$$

$$\leqq \frac{1}{\varepsilon'} \sum_{k=1}^{\infty} \frac{1}{2^k} \left[E\left\{ \left| \int_{R^2} \psi_k(x) \bar{f}_n(t,\omega,dx) - \int_{R^2} \psi_k(x) u(t,dx) \right| \right\} \wedge 1 \right]$$

$$\leqq \frac{1}{\varepsilon'} \sum_{k=1}^{\infty} \frac{1}{2^k} [(L_{\psi_k} \cdot c(n,\varepsilon)) \wedge 1],$$

where $\{\psi_k\}_{k \geqq 1}$ is a countable family which is dense in $C_0^\infty(R^2)$ with respect to the uniform topology. On the other hand, from the expression of $c(n, \varepsilon)$ we see that there exist $\varepsilon_n > 0$, $n = 1, 2, \ldots$, such that $c(n, \varepsilon_n) \to 0$ as $n \to \infty$. Thus, by the dominated convergence theorem we have

$$\lim_{n\to\infty} P\{d(\bar{f}_n(t,\omega), u(t)) > \varepsilon'\} \leqq \frac{1}{\varepsilon'} \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{2^k} [(L_{\psi_k} \cdot c(n,\varepsilon_n)) \wedge 1] = 0,$$

completing the proof.

REMARK. The above convergence is also true under ρ-metric. In fact, using Proposition 2.1, for any $\varepsilon' > 0$, $\varepsilon > 0$ and $L > 0$ we have

$$P\{\rho(\bar{f}_n(t,\omega), u(t)) > \varepsilon'\} \leqq \frac{1}{\varepsilon'} E\{\rho(\bar{f}_n(t,\omega), u(t))\}$$

$$\leqq \frac{1}{\varepsilon'} E\left\{ K \cdot \max_{\varphi \in \Phi_L^\varepsilon} \left| \int_{R^2} \varphi(x) \bar{f}_n(t,\omega,dx) - \int_{R^2} \varphi(x) u(t,dx) \right| \right.$$

$$\left. + \rho(\bar{f}_n(t,\omega), (\bar{f}_n(t,\omega))_L) + \rho(u(t), (u(t))_L) + \varepsilon \right\}$$

$$\leqq \frac{1}{\varepsilon'} \left[K \cdot \sum_{\varphi \in \Phi_L^\varepsilon} \bar{I}_n(\varphi) + E\left\{ \frac{1}{L} \cdot \frac{1}{n} \sum_{j=1}^{n} |\bar{X}_j(t)|^2 \right\} + \frac{1}{L} \int_{R^2} |x|^2 u(t,dx) + \varepsilon \right]$$

$$= \frac{1}{\varepsilon'} \left[K \cdot \sum_{\varphi \in \Phi_L^\varepsilon} c(n,\varepsilon) + \frac{2}{L} \cdot \mu + \varepsilon \right]$$

which implies the ρ-convergence of $\bar{f}_n(t, \omega)$.

§8. Remarks to the one-dimensional analogous problem

Let us consider the following one-dimensional analogy of Boltzmann-type equation:

$$(8.1) \quad \frac{\partial u(t,x)}{\partial t} = \int_{(0,2\pi)\times R^1} \{u(t,x')u(t,y') - u(t,x)u(t,y)\} Q(d\theta) dy, \quad x \in R^1,$$

$$x' = x\cos\theta - y\sin\theta, \quad y' = x\sin\theta + y\cos\theta,$$

(8.2) $$Q(d\theta)=Q(\theta)d\theta,\quad Q(\theta)=Q(2\pi-\theta)>0,\quad \int_0^{2\pi}\theta Q(d\theta)<\infty.$$

If $Q(d\theta)=d\theta/2\pi$, then (8.1) is the well-known Kac's one-dimensional model of Maxwellian gas. This is the case in which Kac first considered the propagation of chaos.

The purpose of this section is to remark that the case $\int_0^{2\pi}Q(d\theta)=\infty$ can also be treated by the method of stochastic integral equations as we have done for the two-dimensional case. The discussions here are much simpler owing to the following inequality (8.3).

Let $\beta\geqq 1$, and f, g be one-dimensional probability distributions with the finite β-th absolute moments. Denote by F^{-1} and G^{-1} the right continuous inverse of distribution functions of f and g, respectively, and by $\mathscr{H}_{f,g}$ the class of all two-dimensional probability distributions whose marginal ones are f and g. Then we have

(8.3) $$\int_0^1|F^{-1}(x)-G^{-1}(x)|^\beta dx\leqq\int_{R^2}|x-y|^\beta h(dxdy),\qquad h\in\mathscr{H}_{f,g}.$$

The above inequality (8.3) for $\beta=2$ was used by H. Tanaka [15] in the study of the trend to the equilibrium for Kac's model; for several dimensional case some basic properties of the quantity $\mathfrak{e}(f,g)$ itself, defined by the infimum of the right hand side of the inequality (8.3), was investigated in H. Murata and H. Tanaka [12]. The inequality (8.3) is probably known, but for completeness we will remark that the inequality (8.3) is an immediate consequence of the following identity (G. Dall'Aglio [2]).

(8.4) $$\int_{R^2}|x-y|^\beta h(dxdy)$$
$$=\begin{cases}\displaystyle\int_{R^1}\{F(x)+G(x)-2H(x,x)\}dx, & \beta=1,\\[2ex] \displaystyle\beta(\beta-1)\int_{x>y}(x-y)^{\beta-2}\{G(y)-H(x,y)\}dxdy & \\ \displaystyle\qquad+\beta(\beta-1)\int_{x<y}(y-x)^{\beta-2}\{F(x)-H(x,y)\}dxdy, & \beta>1.\end{cases}$$

Here F, G and H are probability distribution functions of f, g and h, respectively.

Proof of (8.4): Let φ be a function on R^2 with the following properties.

(i) φ is a non-negative, continuous function on R^2, vanishing on the diagonal.

(ii) $\varphi\in C^2$ off diagonal and $\varphi_{12}\leqq 0$.

(iii) For each $x\in R^1$ the following (finite) limits exist:

$$\varphi_1(x, x-) = \lim_{\varepsilon \downarrow 0} \varphi_1(x, x-\varepsilon), \quad \varphi_2(x-, x) = \lim_{\varepsilon \downarrow 0} \varphi_2(x-\varepsilon, x)^{*)}.$$

Then we have

$$\varphi(x, y) = \int_{y<s<t<x} -\varphi_{12}(t, s)\,dtds + \int_{y<t<x} \varphi_1(t, t-)dt, \qquad y<x,$$

and a similar relation for $x<y$. Hence by Fubini's theorem we can write

$$(8.5) \quad \int_{R^2} \varphi(x, y)h(dxdy)$$

$$= \int_{t>s} -\varphi_{12}(t, s)dtds \int_{\substack{y\leqq s\\ x>t}} h(dxdy) + \int_{t<s} -\varphi_{12}(t, s)dtds \int_{\substack{x\leqq t\\ y>s}} h(dxdy)$$

$$+\int_{R^1} \varphi_1(t, t-)dt \int_{\substack{y\leqq t\\ x>t}} h(dxdy) + \int_{R^1} \varphi_2(t-, t)dt \int_{\substack{x\leqq t\\ y>t}} h(dxdy)$$

$$= \int_{t<s} -\varphi_{12}(t, s)\{G(s) - H(t, s)\}dtds$$

$$+ \int_{t<s} -\varphi_{12}(t, s)\{F(t) - H(t, s)\}dtds$$

$$+ \int_{R^1} [\varphi_1(t, t-)\{G(t) - H(t, t)\} + \varphi_2(t-, t)\{F(t) - H(t, t)\}]dt.$$

Putting $\varphi(x, y) = |x-y|^\beta$ in (8.5), we obtain (8.4).

Proof of (8.3): If $H_1(x, y)$ denotes the joint distribution function of the random variables F^{-1} and G^{-1} on the probability space $((0, 1], dx)$, then we have

$$\int_{R^2} |x-y|^\beta dH_1(x, y) = \int_0^1 |F^{-1}(x) - G^{-1}(x)|^\beta dx,$$

$$H_1(x, y) = \min(F(x), G(y)) \geqq H(x, y)$$

for any H corresponding to $h \in \mathscr{H}_{f,g}$, and then from the identity (8.4) we obtain (8.3).

As in (1.7) we consider

*) φ_1, φ_2 and φ_{12} denote the partial derivatives $\frac{\partial}{\partial x}\varphi(x, y)$, $\frac{\partial}{\partial y}\varphi(x, y)$ and $\frac{\partial^2}{\partial x \partial y}\varphi(x, y)$, respectively.

$$(8.6)\qquad \frac{\partial<u(t,\cdot),\varphi>}{\partial t}=\int_{(0,2\pi)\times R^1\times R^1}\{\varphi(x')-\varphi(x)\}Q(d\theta)u(t,dx)u(t,dy),\qquad \varphi\in C_0^\infty(R^1).$$

The Markov process $X(t)$ associated with (8.6) is given by the following stochastic integral equation:

$$(8.7)\qquad X(t)=X(0)+\int_{S_t}a(X(s-),Y(s-,\alpha),\theta)p(dsd\theta d\alpha),$$

where $\{Y(t), t\geqq 0\}$ is a process defined on the probability space $((0,1], d\alpha)$ and such that the probability distribution of $Y(t)$ is equal to that of $X(t)$ for each fixed $t\geqq 0$, $S_t=[0,t]\times(0,2\pi)\times(0,1]$, $a(x,y,\theta)=x'-x$ and p is an $\mathscr{F}_t$-adapted Poisson random measure on $R_+^1\times(0,2\pi)\times(0,1]$ with mean measure $dsQ(d\theta)d\alpha$.

A merit in one-dimensional case is that we can choose $Y(t-,\alpha)$ to be the right continuous inverse of the distribution function of $X(t-)$.

THEOREM 8.1. *Let f be the probability distribution with $\int_{R^1}|x|f(dx)<\infty$, and assume that $X(0)$ is f-distributed and $\mathscr{F}_0$-measurable. Then there exists a unique $\mathscr{F}_t$-adapted solution $X(t)$ of* (8.7) *such that $\int_0^t E\{|X(s)|\}ds<\infty$ for each $t<\infty$. Therefore the probability distribution $u(t)$ of $X(t)$ is a solution of* (8.6) *with $u(0)=f$.*

PROOF. We put

$$\begin{cases} X^0(t)=X(0)\\ X^{k+1}(t)=X(0)+\int_{S_t}a(X^k(s-),Y^k(s-,\alpha),\theta)p(dsd\theta d\alpha), & k\geqq 0,\end{cases}$$

where $Y^k(s-,\alpha)$ is the right continuous inverse of the distribution function of $X^k(s-)$, $k\geqq 0$. We notice that

$$(8.8)\qquad \int_0^1|Y^k(s-,\alpha)-Y^{k-1}(s-,\alpha)|d\alpha\leqq E\{|X^k(s-)-X^{k-1}(s-)|\},\qquad k\geqq 1,$$

by the inequality (8.3). Using first the estimates

$$(8.9)\qquad \begin{cases} |a(x,y,\theta)|\leqq 2\sin\frac{\theta}{2}(|x|+|y|),\\ |a(x,y,\theta)-a(x_1,y_1,\theta)|\leqq 2\sin\frac{\theta}{2}(|x-x_1|+|y-y_1|),\end{cases}$$

and then by (8.8), we have

(8.10) $E\{|X^{k+1}(t)-X^{k}(t)|\}$

$$\leqq M\int_0^t\left[E\{|X^k(s-)-X^{k-1}(s-)|\}+\int_0^1|Y^k(s-,\alpha)-Y^{k-1}(s-,\alpha)|d\alpha\right]ds$$

$$\leqq 2M\int_0^t E\{|X^k(s)-X^{k-1}(s)|\}ds$$

for $k\geqq 1$, and

$$E\{|X^1(t)-X^0(t)|\}\leqq 2MtE\{|X(0)|\},$$

where $M=2\int_0^{2\pi}\sin\frac{\theta}{2}Q(d\theta)$. Therefore

$$E\{|X^{k+1}(t)-X^k(t)|\}\leqq E\{|X(0)|\}\cdot(2Mt)^{k+1}/(k+1)!,$$

and hence by (8.10) we have

$$E\{\sup_{0\leqq t\leqq T}|X^{k+1}(t)-X^k(t)|\}\leqq E\{|X(0)|\}\cdot(2MT)^{k+1}/(k+1)!$$

for each fixed $T>0$. Therefore $\{X^k(t)\}_{k=0}^{\infty}$ converges uniformly on each finite t-interval with probability one, and the limit $X(t)$ is an $\mathscr{F}_t$-adapted solution of (8.7). The uniqueness can be proved similarly. The last assertion can also be proved by making use of the transformation formula for stochastic integrals.

The propagation of chaos for this model can also be proved as in §6, again with some simplification.

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