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A Note on Witt Algebras

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Introduction

Let $\mathfrak{k}$ be a field and let G be a subgroup of the additive group $\mathfrak{k}^+$. Let $W(G)$ be a Lie algebra with basis $\{w(g)|g \in G\}$ and multiplication

$$[w(g), w(h)] = (g - h)w(g + h), \qquad g, h \in G.$$

This Lie algebra $W(G)$ is called the Witt algebra for G. These Lie algebras were studied by R. K. Amayo and I. Stewart in their book [1, pp. 206–212]. Theorem 3.4 in [1, p. 208] states as follows.

THEOREM A. *If $\mathfrak{k}$ has characteristic* 0 *and $0 \neq G \leqq \mathfrak{k}^+$, then the finite-dimensional subalgebras of $W(G)$ are precisely* (a) *the* 1*-dimensional subalgebras $<x>$ for $x \in W(G)$ and* (b) *the subalgebras of* 3*-dimensional algebras spanned by $\{w(-g), w(0), w(g)\}$ for $0 \neq g \in G$.*

In [1, p. 210] they made a

CONJECTURE. *Residually nilpotent Lie algebras satisfying the maximal condition for subideals may have no infinite-dimensional abelian subalgebras.*

In this paper, we shall point out that Theorem A is not correct, and establish the correct form of Theorem A. We shall also give an example to illustrate that the above conjecture is not true.

1. Notations and preliminary results

Let $\mathfrak{k}$ be of characteristic 0 and G be a fixed subgroup of $\mathfrak{k}^+$ with $|G| > 1$. Any torsion-free abelian group can be linearly ordered (Neumann [2]), so we may equip G with a linear ordering $<$. For any element

$$0 \neq x = \Sigma x_g w(g)$$

of $W(G)$, where $x_g \in \mathfrak{k}$ and the summation is over a finite number of $g \in G$, we define

$$\max(x) = \max\{g|x_g \neq 0\}, \quad \min(x) = \min\{g|x_g \neq 0\},$$

and for any subset $S \neq 0$ of $W(G)$ we define

$$\max(S) = \{\max(s) | s \in S\}, \quad \min(S) = \{\min(s) | s \in S\}.$$

A subset U of G is said to be closed if $u, v \in U$ and $u \neq v$ implies $u+v \in U$. The following two results are shown in [1, p. 208]:

(R. 1) *If V is a Lie subalgebra of $W(G)$ then* $\max(V)$ *and* $\min(V)$ *are closed subsets of G.*

(R. 2) *Any non-empty finite closed subset of G takes one of the following forms,*

(1) $\{g\}$ $(g \in G)$,
(2) $\{0, g\}$ $(g \in G)$,
(3) $\{-g, 0, g\}$ $(g > 0, g \in G)$.

2. On Theorem A

Let $W(\boldsymbol{Z})$ be the Witt algebra for the additive group $\boldsymbol{Z}$ of all integers. Let $x = w(-1) + w(0)$ and $y = 2w(0) + 3w(1) + w(2)$ and let $H = \langle x, y \rangle$. Then $[x, y] = -2(x+y)$, whence H is a 2-dimensional subalgebra of $W(\boldsymbol{Z})$. We obtain the following

(R. 3) *H is not contained in any subalgebra $K(p)$ spanned by $\{w(-p), w(0), w(p)\}$ with a positive integer p.*

PROOF. Suppose that $H \subseteq K(p)$ for some positive integer p. If $p=1$ or 2, then $w(2) \in K(p)$. If $p>2$, then

$$[w(p), {}_{p-2}w(-1)] = (p+1)p \cdots 4w(2) \in K(p),$$

and so $w(2) \in K(p)$. In any case $w(2) \in K(p)$, whence $W(\boldsymbol{Z}) = \langle w(-1), w(0), w(2) \rangle \subseteq K(p)$, which is a contradiction.

This result tells us that Theorem A is not correct. We show the following theorem corrects Theorem A.

THEOREM 1. *If $\mathfrak{k}$ has characteristic 0 and $0 \neq G \leqq \mathfrak{k}^+$, then the finite-dimensional subalgebras of $W(G)$ are precisely*

(a) *the subalgebras of dimension $\leqq 2$,*

(b) *the 3-dimensional subalgebras spanned by $\{w(-g), w(0), w(g)\}$ for some positive element $g \in G$.*

PROOF. Let H be a finite-dimensional subalgebra of $W(G)$. Then H lies inside a subspace of $W(G)$ spanned by a finite set of elements $w(g)$, so $\max(H)$ and $\min(H)$ are finite sets. By (R. 1) they are closed and so must be of type (1), (2) or (3) in (R. 2). If $\max(H)$ is of type (1), then $\dim H = 1$. Suppose $\max(H)$

$=\{0, g\}$ for a positive element $g \in G$. Let x and y be elements of H such that $\max(x)=g$ and $\max(y)=0$. Then there exists an $\alpha \in \mathfrak{k}$ such that $\max([x, y]-\alpha x)=0$. Since $\max(H)=\{0, g\}$, $[x, y]-\alpha x=\beta y$ for some $\beta \in \mathfrak{k}$. This implies $\dim H=2$. Similarly we may deal with negative components. Suppose that $\max(H)=\{-g, 0, g\}$ for some positive element $g \in G$. Let x, y and z be elements of H such that $\max(x)=-g$, $\max(y)=0$ and $\max(z)=g$. Clearly we can write $\min(x)=-h$, $\min(y)=0$, $\min(z)=h$ for some positive element $h \in G$. Evidently $y=\alpha w(0)$ for $0 \neq \alpha \in \mathfrak{k}$. Let

$$x = x_{-h}w(-h) + \cdots + x_{-g}w(-g), \quad z = z_h w(h) + \cdots + z_g w(g),$$

where $x_{-h}x_{-g}z_h z_g \neq 0$. Then

$$[z, y] = \alpha z_h h w(h) + \cdots + \alpha z_g g w(g) \in H,$$

so that from $\max(H)=\{-g, 0, g\}$

$$\max([z, y] - \alpha g z) = 0 \quad \text{and} \quad \min([z, y] - \alpha g z) \geqq h.$$

This implies $[z, y]-\alpha g z=0$, whence $\alpha h z_h=\alpha g z_h$ and so $h=g$. Therefore we get $x=x_{-g}w(g)$ and $z=z_g w(g)$. This concludes $H=<w(-g), w(0), w(g)>$.

Theorem A is not correct as we have shown. However the Corollary [1, p. 209] of Theorem A is itself correct. We now show the Corollary by making use of our Theorem 1.

COROLLARY. (a) *Every abelian subalgebra of $W(G)$ is of dimension* $\leqq 1$.
(b) *Every nilpotent subalgebra of $W(G)$ is of dimension* $\leqq 1$.
(c) *Every soluble subalgebra of $W(G)$ is of dimension* $\leqq 2$.

PROOF. It is sufficient to show (b) and (c).

(b): Let H be a nilpotent subalgebra of $W(G)$. Let x be a non-zero element of H and let y be any element of H. Since H is nilpotent, $<x, y>$ is nilpotent and finite-dimensional. By Theorem 1 $<x, y>$ must be of type (a) or (b) in Theorem 1. The subalgebra of type (b) is not nilpotent. Then $\dim <x, y> \leqq 2$. Suppose that $\dim <x, y> = 2$. Evidently $\max(<x, y>)=\{0, g\}$ for some $g \in G$. Then $\max(<x, y>^n) \supseteq \{g\}$ for any positive integer n. This implies $<x, y>^n \neq 0$ for any positive integer n, which is a contradiction.

(c): Let H be a soluble subalgebra of $W(G)$. Suppose that H is not finite-dimensional. $\max(H)$ is an infinitely closed subset of G. Then we may suppose that $\max(H)$ includes a subset $\{h_1, h_2, h_3, \ldots\}$ such that $h_1 < h_2 < h_3 < \cdots$. Therefore

$$\max(H^{(n)}) \supseteq \{h_1 + h_2 + \cdots + h_n, h_2 + h_3 + \cdots + h_{n+1}, \ldots\}$$

for any positive integer n. This implies $H^{(n)} \neq 0$ for any positive integer n, which is a contradiction. Thus H is finite-dimensional. As in the proof of (b) we have $\dim H \leqq 2$.

3. On the Conjecture

A Lie algebra L is residually nilpotent if it possesses a family $\{N_\lambda\}$ of ideals such that $\cap N_\lambda = 0$ and L/N_λ is nilpotent. We denote by $\mathrm{R}\mathfrak{N}$ the class of all residually nilpotent Lie algebras. We denote by Max, Min-si, and Max-$\lhd$ respectively the classes of Lie algebras satisfying the maximal condition for subalgebras, for subideals and for ideals.

Let $W(\boldsymbol{N})$ be the subalgebra of $W(\boldsymbol{Z})$ generated by $\{w(1), w(2), \dots\}$. The conjecture in Introduction was done by comparing the following three properties:

(1) $W(\boldsymbol{N})$ satisfies the maximal condition for subideals ([1, p. 178]).

(2) $W(\boldsymbol{N})$ is residually nilpotent.

(3) The abelian subalgebras of $W(\boldsymbol{N})$ is of dimension $\leqq 1$.

The next result gives the negative answer to the above conjecture.

THEOREM 2. *Over any field* $\mathfrak{k}$ *of characteristic* 0 *there exists a Lie algebra in the class* Max-si $\cap$ $\mathrm{R}\mathfrak{N}$ *which has infinite-dimensional abelian subalgebras.*

PROOF. Let (t) be the ideal of $\mathfrak{k}[t]$ generated by t and B be the split simple algebra $\langle u, v, w \rangle$:

$$[u, v] = w, \quad [v, w] = -2v, \quad [u, w] = 2u.$$

Let $L = (t) \otimes B$ be a Lie algebra with multiplication

$$[f(t) \otimes a, g(t) \otimes b] = f(t)g(t) \otimes [a, b],$$

where $f(t), g(t) \in (t)$ and $a, b \in B$. Then in [3, p. 13] I. Stewart has shown that $L \in$ Max-si $\cap$ $\mathrm{R}\mathfrak{N}$. It is clear that $(t) \otimes \langle u \rangle$ is an infinite-dimensional abelian subalgebra of L.

For the Lie algebra L in Theorem 2 it is obvious that $L \notin$ Max. This shows the first part of the following

THEOREM 3. Max $\cap$ $\mathrm{R}\mathfrak{N} \subsetneqq$ Max-si $\cap$ $\mathrm{R}\mathfrak{N} \subsetneqq$ Max-$\lhd$ $\cap$ $\mathrm{R}\mathfrak{N}$.

PROOF. Let $\mathfrak{k}$ be a field of characteristic 0 and let M be a Lie algebra over $\mathfrak{k}$ with basis $\{x_1, x_2, \dots, z\}$ and multiplication

$$[x_i, x_j] = 0 \quad \text{and} \quad [x_i, z] = x_{i+1} \quad \text{for } i, j = 1, 2, \dots.$$

Let N be a non-zero ideal of M. Then $N \not\subseteq \langle z \rangle$. Let x be a non-zero element of N such that

$$x = \sum_{i=k}^{m} \alpha_i x_i + \beta z, \quad \alpha_k \alpha_m \neq 0.$$

Since N is an ideal of M, we have

$$[x, {}_n z] = \sum_{i=k}^{m} \alpha_i x_{i+n} \in N \qquad \text{for } n = 1, 2, \dots,$$

and so

$$<[x, {}_n z], x | n = 1, 2, \dots> \subseteq N.$$

Therefore we have

$$M = kx_1 + kx_2 + \cdots + kx_{m-1} + kz + <[x, {}_n z], x | n = 1, 2, \dots>,$$

so that

$$M = kx_1 + kx_2 + \cdots + kx_{m-1} + kz + N.$$

Thus N is an ideal of finite codimension. This implies $M \in \text{Max-}\lhd$.

Let $N_m = <x_i | i = m, m+1, \dots>$ for $m = 1, 2, \dots$. Clearly $\cap N_m = 0$ and M/N_m is nilpotent. Thus $M \in \mathrm{R}\mathfrak{N}$. It is however clear that $M \notin \text{Max-si}$.

References

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