HIROSHIMA MATH. J.
7 (1977), 833–853

Holomorphic Curves in Algebraic Varieties

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(Received May 19, 1977)

Introduction

Let V be a smooth complex projective algebraic variety of dimension n and Σ a hypersurface in V. We denote by $\Omega^1_V(\log \Sigma)$ the sheaf of germs of logarithmic 1-forms along Σ over V (see section 1 (c)). Let $f: \boldsymbol{C} \to V$ be a holomorphic mapping from the one dimensional complex plane $\boldsymbol{C}$ into V which will be called a holomorphic curve in V, $T_f(r)$ denote the characteristic function of f relative to a Kähler form on V and $\bar{N}_f(r, \Sigma)$ the counting function for Σ without counting multiplicities (see section 1 (a)).

The main purpose of this paper is to establish the following result which may be called a theorem of the second main theorem type (for the precise statement, see Main Theorem in section 3):

Assume that there exists a system $\{\omega_i\}_{i=1,\dots,n+1}$ *of* $n+1$ *logarithmic 1-forms* $\omega_i \in H^0(V, \Omega^1_V(\log \Sigma))$ *such that the n-forms* $\omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge \omega_{i+1} \wedge \cdots \wedge \omega_{n+1}$, $i=1, 2,\dots, n+1$, *are linearly independent over* $\boldsymbol{C}$. *Then there is a positive constant* κ *such that*

$$\kappa T_f(r) < \bar{N}_f(r, \Sigma) + S_f(r), \tag{I}$$

where $S_f(r)$ *is a small term such as* $\underline{\lim}_{r\to\infty} S_f(r)/T_f(r) = 0$ (*see* (2.9)).

Let us recall the well-known case where $f: \boldsymbol{C} \to \boldsymbol{P}^n$ is a holomorphic curve in the n-dimensional complex projective space $\boldsymbol{P}^n$ and Σ a union of hyperplanes Σ_i, $i=1,\dots, q$, in $\boldsymbol{P}^n$. Let $T_f(r)$ be the characteristic function of f relative to the standard Kähler form on $\boldsymbol{P}^n$ and $N_f(r, \Sigma_i)$ the counting function for Σ_i with counting multiplicities. Then we have the following famous theorem:

The second main theorem. *Suppose that* $q > n+1$ *and* Σ_i*'s are in general position, and that the image* $f(\boldsymbol{C})$ *is contained in no hyperplane. Then*

$$(q-n-1)T_f(r) < \sum_{i=1}^{q} N_f(r, \Sigma_i) + S_f(r), \tag{II}$$

where $S_f(r)$ *is a small term as in* (I).

This theorem was first proved by Cartan [3] in 1933 and later by Ahlfors

([1]) in 1941 who completed the work of H. and J. Weyl [21]. As mentioned in Ahlfors [1], it is desirable to deal with holomorphic curves in a general algebraic variety, but their methods are intrinsically based on the facts that $\boldsymbol{P}^n$ and Σ_i are linear. Thereby it is very difficult to generalize that theorem for holomorphic curves in a general algebraic variety in such a complete form.

The second main theorem implies not only that $f(\boldsymbol{C})$ intersects $\Sigma=\bigcup_{i=1}^{q}\Sigma_i$, but much more refined informations on how often the holomorphic curve f intersects Σ. The theory dealing with the magnitude of $f^{-1}(\Sigma)$ is called "equi-distribution theory" (cf. Wu [22, Introduction]). To understand the second main theorem (II) from this viewpoint, one may think it to give a lower bound of $N_f(r, \Sigma)$ with $\Sigma=\bigcup_{i=1}^{q}\Sigma_i$, contrary to

$$N_f(r, \Sigma_i) < T_f(r) + O(1)$$

as $r\to\infty$ for each Σ_i, which is a direct consequence of the first main theorem (cf. [8, section 3]). Therefore one is naturally led to the following problem:

Let $f: \boldsymbol{C}\to V$ be a holomorphic curve in a complex algebraic variety V. Find conditions for a hypersurface Σ in V under which there is a positive constant κ such that

$$\kappa T_f(r) < N_f(r, \Sigma) + S_f(r), \tag{III}$$

where $S_f(r)$ is a small term as in (I) (*see* (2.9)).

This problem may deeply relate to that of Griffiths [8, section 4 (v), Problem].

It was our first aim to obtain (III) for a smooth complex projective algebraic variety V. Since $\bar{N}_f(r, \Sigma)\leqq N_f(r, \Sigma)$ for $r\geqq 1$, (I) implies (III).

The first step of the proof of Main Theorem (I) is a generalization of Nevanlinna's lemma on the logarithmic derivative (see (1.1) and Lemma 2.3). For this aim we shall introduce a sheaf $\mathfrak{A}_V(\log\Sigma)$ $(\subset\Omega_V^1(\log\Sigma))$ which is a sheaf of $\boldsymbol{Z}$-module, such that $H^0(V, \mathfrak{A}_V(\log\Sigma))$ generates $H^0(V, \Omega_V^1(\log\Sigma))$ over the complex number field $\boldsymbol{C}$ (see section 1 (b) and (c)). That generalized version of Nevanlinna's lemma on the logarithmic derivative (Lemma 2.3) and Ochiai's theorem [17, Theorem A] (cf. section 3) will play essential roles in the proof of Main Theorem (I) in section 3.

In section 4 (a) and (b) we shall verify the conditions of Main Theorem in the classical case, i.e., $V=\boldsymbol{P}^n$ and Σ is a union of hyperplanes. An example given in section 4 (b) shows that Main Theorem is slightly different from the second main theorem (II) of Cartan and Ahlfors.

In section 4 (c) we shall give another example which satisfies the conditions of Main Theorem in the case where V is an Abelian variety (see Proposition 4.2). As applications we shall obtain partial answers to the problems of Griffiths

[7, Problem F] and Kobayashi [12, Problem D. 9] (see Theorem 4.1 in section 4(d) and Theorem 4.2 in section 4(e)).

1. Preliminaries

(a) For a meromorphic function α in $\boldsymbol{C}$ we set

$$m(r, \alpha) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\alpha(re^{i\theta})|\, d\theta,$$

$$N(r, \alpha) = \int_0^r \frac{n(t, \alpha) - n(0, \alpha)}{t} dt + n(0, \alpha) \log r,$$

where $\log^+|\alpha| = \max\{\log|\alpha|, 0\}$, $n(t, \alpha)$ denotes the number of poles of α in $\Delta(t) = \{z \in \boldsymbol{C};\ |z| < t\}$ with counting multiplicities and $n(0, \alpha)$ the order of the pole of α at the origin 0 when 0 is a pole of α. Nevanlinna's characteristic function $T(r, \alpha)$ is defined by

$$T(r, \alpha) = m(r, \alpha) + N(r, \alpha).$$

For the elementary properties of $T(r, \alpha)$, see Nevanlinna [13, Chap. I]. The order of α is defined by

$$\varlimsup_{r\to\infty} \frac{\log T(r, \alpha)}{\log r}.$$

Letting α' denote the first derivative of α, we have Nevanlinna's lemma on the logarithmic derivative (cf. [13, Chap. IV]):

$$m\left(r, \frac{\alpha'}{\alpha}\right) = O(\log^+ T(r, \alpha)) + O(\log^+ r) + O(1) \tag{1.1}$$

for all r if α is of finite order, and otherwise except for r belonging to a union of intervals whose total linear measure is finite.

In estimate (1.1) we abbreviated the phrase "*as* $r\to\infty$". This abbreviation will be done throughout the present paper.

Let M be a compact Kähler manifold with Kähler form Ω. Let $f: \boldsymbol{C}\to M$ be a holomorphic curve. Then the characteristic function of f relative to Ω is defined by

$$T_f(r) = \int_0^r \frac{dt}{t}\int_{\Delta(t)} f^*\Omega.$$

Let $T'_f(r)$ be the characteristic function relative to another Kähler form Ω' on M. Then there are positive constants A and A' such that

$$A'T'_f(r) + O(1) < T_f(r) < AT'_f(r) + O(1).$$

Therefore the order of f

$$\overline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log r}$$

is defined independently of the choice of Ω.

In general we denote by $\operatorname{Supp} D$ the support of a divisor D on M and call the irreducible components of $\operatorname{Supp} D$ the components of D. For an effective divisor D on M such that $f(\boldsymbol{C}) \not\subset \operatorname{Supp} D$ we denote by $n_f(t, D)$ the sum of orders of the divisor $f^*D \cap \Delta(t)$ and by $n_f(0, D)$ the order of f^*D at the origin. Without counting multiplicities we define $\bar{n}_f(t, D)$ and $\bar{n}_f(0, D)$ in the same manner as above. Set

$$N_f(r, D) = \int_0^r \frac{n_f(t, D) - n_f(0, D)}{t} dt + n_f(0, D) \log r,$$

$$\bar{N}_f(r, D) = \int_0^r \frac{\bar{n}_f(t, D) - \bar{n}_f(0, D)}{t} dt + \bar{n}_f(0, D) \log r.$$

Then obviously

$$\bar{N}_f(r, D) \leqq N_f(r, D) \qquad \text{for} \quad r \geqq 1.$$

For a hypersurface Σ in M we define $N_f(r, \Sigma)$ and $\bar{N}_f(r, \Sigma)$, regarding Σ itself as an effective divisor on M.

Let $[D] \to M$ be the line bundle over M determined by D. Let $\psi \in c_1(D)$ (=the first Chern class $c_1([D])$ of $[D]$) be the curvature form of a metric $\|\cdot\|$ in $[D]$ and take a section $\sigma \in \Gamma(M, [D])$ such that the divisor (σ) equals D and $\|\sigma\| \leqq 1$. Then we have

$$T_f(r, c_1(D)) = N_f(r, D) + m_f(r, D) + C, \tag{1.2}$$

where

$$T_f(r, c_1(D)) = \int_0^r \frac{dt}{t} \int_{\Delta(t)} f^*\psi,$$

$$m_f(r, D) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{\|\sigma(f(re^{i\theta}))\|} d\theta,$$

and C is a constant (cf. [9]).

Since Ω is positive definite and M is compact, there is a positive constant K such that $K\Omega - \psi$ is semi-positive definite, so that

$$T_f(r, c_1(D)) \leqq K T_f(r). \tag{1.3}$$

For simplicity we assume in the rest of this paper that $f(0)$ is not contained in the supports of divisors or hypersurfaces concerned, knowing that the exceptional case can be treated.

Let M be a projective algebraic manifold V, that is, a smooth projective algebraic variety which is defined over $\boldsymbol{C}$, and $\mathscr{R}(V)$ denote the field of rational functions on V. Let $\{\phi_1,\dots,\phi_l\}$ be a system of generators of $\mathscr{R}(V)$ over $\boldsymbol{C}$ such that $f^*\phi_j$ are defined, and set

$$\tilde{T}_f(r) = \max_{1\leq j\leq l}\{T(r, f^*\phi_j)\}.$$

Then there are positive constants B and B' such that

$$B'\tilde{T}_f(r) + O(1) < T_f(r) < B\tilde{T}_f(r) + O(1). \tag{1.4}$$

(b) Let M be a compact Kähler manifold of dimension n and Σ a hypersurface in M. Let $\mathfrak{M}_M^*(\Sigma)$ denote the sheaf of germs of non-zero meromorphic functions whose zeros and poles are contained in Σ and $\mathfrak{A}_M(\log\Sigma)$ the sheaf of germs of meromorphic closed 1-forms $d\log\zeta$ with $\zeta\in\mathfrak{M}_M^*(\Sigma)$. Then we have the exact sequences:

$$0 \longrightarrow \boldsymbol{C}^* \longrightarrow \mathfrak{M}_M^*(\Sigma) \xrightarrow{d\log} \mathfrak{A}_M(\log\Sigma) \longrightarrow 0,$$

$$H^0(M, \mathfrak{M}_M^*(\Sigma)) \xrightarrow{d\log} H^0(M, \mathfrak{A}_M(\log\Sigma)) \xrightarrow{\delta} H^1(M, \boldsymbol{C}^*) \longrightarrow \cdots, \tag{1.5}$$

where $\boldsymbol{C}^*$ denotes the multiplicative group of non zero complex numbers. One notes that

$$H^1(M, \boldsymbol{C}^*) = \mathrm{Hom}(\pi_1(M), \boldsymbol{C}^*) = \mathrm{Hom}(H_1(M), \boldsymbol{C}^*),$$

where $\pi_1(M)$ is the fundamental group of M and $H_1(M)$ the first homology group of M with integral coefficients.

Let Σ_i, $i=1, 2,\dots$, be the irreducible components of Σ and $\mathring{\Sigma}_i$ denote the set of regular points of Σ_i. In a small neighborhood U of each point of $\mathring{\Sigma}_i$ we can take a local coordinate system $(x_1,\dots, x_n)$ so that $\{x_1=0\}=\Sigma_i\cap U$. Then every global section $\omega\in H^0(M, \mathfrak{A}_M(\log\Sigma))$ is written in U as

$$\omega = \nu_i\frac{dx_1}{x_1} + \eta,$$

where ν_i is an integer and η does not contain the term dx_1/x_1. It is easy to see that the integer ν_i is independent of such a local coordinate system. Since $\mathring{\Sigma}_i$ is connected, ν_i is constant on $\mathring{\Sigma}_i$. We define the residue of ω on Σ_i by

$$\mathrm{res}(\omega, \Sigma_i) = \nu_i.$$

Thus we get a divisor $D=\sum\mathrm{res}(\omega, \Sigma_i)\Sigma_i$. Let $\pi\colon \tilde{M}\to M$ be the universal covering of M with transformation group $\pi_1(M)$ and take a point $\tilde{x}_0\in\tilde{M}-\pi^{-1}(\mathrm{Supp}\,D)$.

Then the function

$$\vartheta(\tilde{x}) = \exp\left(\int_{\tilde{x}_0}^{\tilde{x}} \pi^*\omega\right)$$

is a meromorphic function in $\tilde{M}$ and $\delta\omega \in \mathrm{Hom}(\pi_1(M), \boldsymbol{C}^*)$ (see (1.5)) is represented by

$$\delta\omega(\gamma) = \frac{\vartheta(\gamma\tilde{x})}{\vartheta(\tilde{x})} = \exp\left(\int_\gamma \omega\right) \in \boldsymbol{C}^*$$

for $\gamma \in \pi_1(M)$. Therefore ϑ is a theta function with constant multiplicator and it is clear by the construction that the divisor (ϑ) of ϑ is equal to D. Hence D is homologous to zero (cf. [20, Chap. V, p. 101]). Conversely, if there are integers ν_i such that the divisor $D = \sum \nu_i \Sigma_i$ is homologous to zero, then there is a unique theta function ϑ, up to constant multiple, such that $(\vartheta) = D$ and $|\vartheta|$ is single-valued on M (cf. [20, Chap. V, p. 101]). Setting $\omega = d\log\vartheta$, we have

$$\omega \in H^0(M, \mathfrak{A}_M(\log\Sigma)),$$

$$\mathrm{res}(\omega, \Sigma_i) = \nu_i.$$

(c) Let V be a projective algebraic manifold, $\mathcal{O}_V$ (resp. Ω_V^1) the sheaf of germs of holomorphic functions (resp. 1-forms) over V. Let Σ be a hypersurface in V and $\Omega_V^1(\log\Sigma)$ denote the sheaf of germs of the form $\sum_{j=1}^l \alpha_j\omega_j$ with $\alpha_j \in \mathcal{O}_{V,x}$, $\omega_j \in \mathfrak{A}_{V,x}(\log\Sigma)$ and $l = 1, 2, \ldots$, where $x \in M$, which is called the sheaf of germs of logarithmic 1-forms along Σ (cf. Deligne [4] and Iitaka [10]). One notes that

$$H^0(V, \Omega_V^1) \subset H^0(V, \mathfrak{A}_V(\log\Sigma)) \subset H^0(V, \Omega_V^1(\log\Sigma)).$$

Let $\pi: (V^*, \Sigma^*) \to (V, \Sigma)$ be a desingularization of Σ satisfying
(1) π is a composite of monoidal transformations and $\Sigma^* = \pi^{-1}(\Sigma)$,
(2) $\pi|_{V^*-\Sigma^*}: V^* - \Sigma^* \to V - \Sigma$ is biholomorphic,
(3) Σ^* has only simple normal crossings.

LEMMA 1.1. *The mapping π induces an isomorphism*

$$\pi^*: H^0(V, \mathfrak{A}_V(\log\Sigma)) \longrightarrow H^0(V^*, \mathfrak{A}_{V^*}(\log\Sigma^*)).$$

PROOF. It is clear that π^* is injective. Let $\lambda: V \to V^*$ be the inverse of π which is a meromorphic mapping, and S the singular locus of λ. Then codim $S \geqq 2$ and $\lambda|_{V-S}: V - S \to V^*$ is holomorphic. Let $\omega^* \in H^0(V^*, \mathfrak{A}_{V^*}(\log\Sigma^*))$ and set $\omega' = (\lambda|_{V-S})^* \omega^* \in H^0(V - S, \mathfrak{A}_V(\log\Sigma))$. It is enough to prove that ω' can be extended in a neighborhood of each point of S as a section of $\mathfrak{A}_V(\log\Sigma)$. Let $x_0 \in S$, take a simply connected open neighborhood U of x_0 and $x_1 \in U - (S \cup \Sigma)$. Noting that $U - S$ is simply connected, we set

$$g(x) = \exp\left(\int_{x_1}^{x} \omega'\right)$$

for $x \in U - S$, which is a meromorphic function in $U - S$. The zeros and poles of g is contained in Σ. Since $\operatorname{codim} S \geqq 2$, g has an extension over U as a meromorphic function. Denote it by the same g. Then $d\log g \in H^0(U, \mathfrak{A}_V(\log \Sigma))$ and $d\log g|_{U-S} = \omega'|_{U-S}$. Q.E.D.

Deligne [5] proved that every global section of $\Omega^1_{V^*}(\log \Sigma^*)$ is d-closed. By Iitaka [10, sections $2 \sim 4$] we have

(1) $H^0(V^*, \Omega^1_{V^*}(\log \Sigma^*) \overset{\pi^*}{\cong} H^0(V, \Omega^1_V(\log \Sigma))$,

(2) there is a basis $\{\omega_j^*\}$ of the $\boldsymbol{C}$-vector space $H^0(V^*, \Omega^1_{V^*}(\log \Sigma^*))$ such that every $\omega_j^* \in H^0(V^*, \mathfrak{A}_{V^*}(\log \Sigma^*))$.

Combining these with Lemma 1.1, we obtain

PROPOSITION 1.2. *Let the notation be as above. Then there exists a basis* $\{\omega_j\}$ *of the* $\boldsymbol{C}$*-vector space* $H^0(V, \Omega^1_V(\log \Sigma))$ *such that every* $\omega_j \in H^0(V, \mathfrak{A}_V(\log \Sigma))$, *i.e.,* $H^0(V, \mathfrak{A}_V(\log \Sigma))$ *generates* $H^0(V, \Omega^1_V(\log \Sigma))$ *over* $\boldsymbol{C}$.

2. Generalization of Nevanlinna's "lemma on the logarithmic derivative"

Let M be a compact Kähler manifold, $\Sigma = \cup \Sigma_i$ hypersurface in M with irreducible components Σ_i and $\omega \in H^0(M, \mathfrak{A}_M(\log \Sigma))$. Set

$$D_1 = \sum_{\operatorname{res}(\omega, \Sigma_i) > 0} \operatorname{res}(\omega, \Sigma_i)\Sigma_i,$$

$$D_2 = \sum_{\operatorname{res}(\omega, \Sigma_i) < 0} - \operatorname{res}(\omega, \Sigma_i)\Sigma_i.$$

Let $[D_j] \to M$ be the line bundles over M determined by D_j and $\sigma_j \in \Gamma(M, [D_j])$ such that $(\sigma_j) = D_j$.

Let $\{U_l\}$ be a finite open covering of M such that every U_l is simply connected and each restriction $[D_j]|_{U_l}$ of $[D_j]$ is trivial. Then each σ_j is given by a system $\{\sigma_{jl}\}$ of holomorphic functions σ_{jl} in U_l. In each U_l, ω is written as

$$\omega = d\log \sigma_{1l} - d\log \sigma_{2l} + \eta'_l, \tag{2.1}$$

where η'_l is a holomorphic closed 1-form in U_l. We take the integral of (2.1) in U_l which is determined by modulo $2\pi i$:

$$\int_y^x \omega \equiv \log \sigma_{1l}(x) - \log \sigma_{1l}(y) + \log \frac{1}{\sigma_{2l}(x)} - \log \frac{1}{\sigma_{2l}(y)} + \int_y^x \eta'_l$$

(mod. $2\pi i$) for $x, y \in U_l$. Take a metric $\{\rho_{jl}\}$ in each $[D_j]$ so that

$$\|\sigma_j(x)\|^2 = |\sigma_{jl}(x)|^2/\rho_{jl}(x) \leqq 1.$$

Set

$$\eta_l = \eta'_l + \frac{1}{2}\{d \log \rho_{1l} - d \log \rho_{2l}\}.$$

LEMMA 2.1. *Let the notation be as above. Then*

$$\begin{aligned}\mathrm{Re}\int_y^x \omega &= \log \|\sigma_1(x)\| - \log \|\sigma_1(y)\| \\ &\quad + \log \frac{1}{\|\sigma_2(x)\|} - \log \frac{1}{\|\sigma_2(y)\|} + \mathrm{Re}\int_y^x \eta_l\end{aligned}$$

for $x, y \in U_l$, where $\mathrm{Re}(\cdot)$ *denotes the real part of* $(\cdot)$.

We fix once and henceforth a Kähler metric h and the associated form Ω on M.

Let $f: \boldsymbol{C} \to M$ be a holomorphic curve. To avoid tiresome arguments, as mentioned in section 1(a), we always assume that $f(0) \notin \Sigma$. Let $T_f(r)$ be the characteristic function of f relative to Ω. Let $\omega \in H^0(M, \mathfrak{A}_M(\log \Sigma))$ and set

$$f^*\omega = \zeta(z)dz.$$

Then $\zeta(z)$ is a meromorphic function with poles of order one and their residues are integers. Set

$$G(z) \equiv \int_0^z f^*\omega \qquad (\mathrm{mod.}\ 2\pi i),$$

$$g(z) = \exp(G(z)).$$

LEMMA 2.2. *There are constants $K>0$, A and B such that*

$$T(r, g) \leqq K\left\{\left(\frac{r}{2\pi}\right)^{1/2}\left(r\frac{d}{dr}T_f(r) + A\right)^{1/2} + T_f(r)\right\} + B.$$

PROOF. Let $z = re^{i\theta}$ and $L = \{te^{i\theta}; 0 \leqq t \leqq r\}$. Take points $0 = z_0, z_1, \dots, z_k = z$ on L so that $|z_{i-1}| < |z_i|$, $f(z_i) \notin \Sigma$ and the image of each closed line segment $L[z_{i-1}, z_i]$ of L between z_{i-1} and z_i by f is contained in some $U_{l_i} \in \{U_l\}$, where $\{U_l\}$ is the covering of M taken above. It follows from Lemma 2.1 that

$$\begin{aligned}\mathrm{Re}\, G(z) &= \sum_{i=1}^k \mathrm{Re}(G(z_i) - G(z_{i-1})) \\ &= \sum_{i=1}^k \Big\{\log \|\sigma_1(f(z_i))\| - \log \|\sigma_1(f(z_{i-1}))\|\end{aligned}$$

$$+\log\frac{1}{\|\sigma_2(f(z_i))\|}-\log\frac{1}{\|\sigma_2(f(z_{i-1}))\|}$$

$$+\operatorname{Re}\int_{L[z_{i-1},z_i]}f^*\eta_{l_i}\Big\}$$

$$\leqq\log\frac{1}{\|\sigma_2(f(z))\|}+\log\frac{\|\sigma_1(f(0))\|}{\|\sigma_2(f(0))\|}$$

$$+\sum_{i=1}^{k}\operatorname{Re}\int_{L[z_{i-1},z_i]}f^*\eta_{l_i}.$$

Here the last inequality follows from $\|\sigma_1\|\leqq 1$. We may assume that each closed 1-form η_l is defined in a neighborhood of the closure $\bar{U}_l$ of U_l. Since M is compact, there is a positive constant K such that

$$|\eta_l(X)|\leqq K\sqrt{h(X,\bar{X})}$$

for any vector field X in every U_l. Set $f^*h=s(z)dz\otimes d\bar{z}$ and $f^*\Omega=s(z)(i/2)dz\wedge d\bar{z}$. Then

$$\sum_{i=1}^{k}\operatorname{Re}\int_{L[z_{i-1},z_i]}f^*\eta_{l_i}\leqq K\int_0^r\sqrt{s(te^{i\theta})}dt.$$

Using the notation, $\operatorname{Re}^+G(z)=\max\{\operatorname{Re}G(z),0\}$, we get

$$\operatorname{Re}^+G(z)\leqq K\int_0^r\sqrt{s(te^{i\theta})}dt+\log\frac{1}{\|\sigma_2(f(re^{i\theta}))\|}+\log^+\frac{\|\sigma_2(f(0))\|}{\|\sigma_1(f(0))\|}.$$

Hence we have

$$\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}^+G(re^{i\theta})d\theta \tag{2.2}$$

$$\leqq K\frac{1}{2\pi}\int_0^{2\pi}d\theta\int_0^r\sqrt{s(te^{i\theta})}\,dt+m_f(r,D_2)+\log^+\frac{\|\sigma_2(f(0))\|}{\|\sigma_1(f(0))\|}$$

$$\leqq K\left(\frac{r}{2\pi}\right)^{1/2}\left(\int_0^{2\pi}d\theta\int_0^r s(te^{i\theta})dt\right)^{1/2}+m_f(r,D_2)$$

$$+\log^+\frac{\|\sigma_2(f(0))\|}{\|\sigma_1(f(0))\|}\qquad\text{(by Schwarz's inequality)}$$

$$\leqq K\left(\frac{r}{2\pi}\right)^{1/2}\left(\int_0^{2\pi}d\theta\int_0^r ts(te^{i\theta})\,dt+\int_0^{2\pi}d\theta\int_0^1 s(te^{i\theta})\,dt\right)^{1/2}$$

$$+m_f(r,D_2)+\log^+\frac{\|\sigma_2(f(0))\|}{\|\sigma_1(f(0))\|}.$$

Set

$$A=\int_0^{2\pi}d\theta\int_0^1 s(te^{i\theta})\,dt,$$

$$B=\log^+\frac{\|\sigma_2(f(0))\|}{\|\sigma_1(f(0))\|}.$$

Since $\int_{\Delta(r)} f^*\Omega=r\frac{d}{dr}T_f(r)$, it follows from (2.2) that

$$\begin{aligned}(2.3)\qquad m(r, g) &= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}^+ G(re^{i\theta})d\theta \\ &\leqq K\Big(\frac{r}{2\pi}\Big)^{1/2}\Big(r\frac{d}{dr}T_f(r)+A\Big)^{1/2}+m_f(r, D_2)+B.\end{aligned}$$

It is clear that

$$(2.4)\qquad N(r, g)\leqq N_f(r, D_2).$$

Choosing greater K and B if necessary, we have by (1.2), (1.3), (2.3) and (2.4)

$$T(r, g)\leqq K\Big\{\Big(\frac{r}{2\pi}\Big)^{1/2}\Big(r\frac{d}{dr}T_f(r)+A\Big)^{1/2}+T_f(r)+B\Big\}.$$

Q. E. D.

LEMMA 2.3. *Let $\omega\in H^0(M, \mathfrak{A}_M(\log\Sigma))$ and set $f^*\omega=\zeta(z)dz$. Then*

$$(2.5)\qquad m(r, \zeta)\leqq O(\log^+ T_f(r))+O(\log^+ r)+O(1)$$

for all r if f is of finite order, and otherwise except for r belonging to a union of intervals whose total linear measure is finite.

PROOF. Setting $g(z)=\exp\Big(\int_0^z f^*\omega\Big)$, we have

$$(2.6)\qquad \zeta(z)=\frac{g'(z)}{g(z)}.$$

Suppose that f is of finite order. Then $T_f(r)=O(r^\mu)$ with $\mu>0$ and $dT_f(r)/dr=O(r^{\mu-1})$. Combining these with (2.6), Lemma 2.2 and (1.1), we obtain (2.5).

In the case where f is of infinite order, by (2.6), Lemma 2.2 and (1.1) we have

$$\begin{aligned}(2.7)\qquad m(r, \zeta) &= O\Big(\log^+\frac{d}{dr}T_f(r)\Big)+O(\log^+ T_f(r)) \\ &\quad + O(\log^+ r)+O(1)\end{aligned}$$

for $r\notin E'$, where E' is a union of intervals whose total linear measure is finite.

It is easily verified that

$$\frac{d}{dr}T_f(r) \leqq \{T_f(r)\}^2 \tag{2.8}$$

for $r \notin E''$, where E'' is a set similar to E'.

It follows from (2.7) and (2.8) that

$$m(r, \zeta) = O(\log^+ T_f(r)) + O(\log^+ r) + O(1)$$

for $r \notin E' \cup E''$. Q.E.D.

REMARK. Let M be the Riemann sphere $\boldsymbol{P}^1$ with inhomogeneous coordinate w and $\Sigma = \{0, \infty\}$. Then $\omega = dw/w \in H^0(\boldsymbol{P}^1, \mathfrak{A}_{\boldsymbol{P}^1}(\log \Sigma))$. In this case Lemma 2.3 is nothing but Nevanlinna's lemma on the logarithmic derivative (cf. section 1(a)).

For a given holomorphic curve $f: \boldsymbol{C} \to M$, let us denote by $S_f(r)$ a quantity satisfying

$$S_f(r) = O(\log^+ T_f(r)) + O(\log^+ r) + O(1) \tag{2.9}$$

for all r if f is of finite order, and otherwise except for r belonging to a union of intervals whose total linear measure is finite.

COROLLARY 2.4. *Let ω and ζ be as above. Then*

$$T(r, \zeta) \leqq \bar{N}_f(r, D) + S_f(r),$$

where $D = \sum |\mathrm{res}(\omega, \Sigma_i)| \Sigma_i$.

PROOF. Since every pole z of ζ is of order one and since z can be a pole of ζ only if $f(z) \in \mathrm{Supp}\, D$,

$$N(r, \zeta) \leqq \bar{N}_f(r, D).$$

Hence our assertion follows from Lemma 2.3. Q.E.D.

Let M be a projective algebraic manifold V and Σ a hypersurface in V. Let $f: \boldsymbol{C} \to V$ be a holomorphic curve and set $f^*\omega = \zeta(z)dz$ for $\omega \in H^0(V, \Omega_V^1(\log \Sigma))$.

COROLLARY 2.5. $T(r, \zeta) \leqq \bar{N}_f(r, \Sigma) + S_f(r)$.

PROOF. By Proposition 1.2 there are $\omega_j \in H^0(V, \mathfrak{A}_V(\log \Sigma))$ and $c_j \in \boldsymbol{C}$, $j = 1, \dots, q$, such that

$$\omega = c_1\omega_1 + \cdots + c_q\omega_q.$$

Setting $f^*\omega_j = \zeta_j dz$, we get

$$\zeta = c_1\zeta_1 + \cdots + c_q\zeta_q.$$

We infer from Lemma 2.3 that

$$m(r, \zeta) \leqq \sum_{j=1}^{q} m(r, \zeta_j) + O(1) = S_f(r).$$

Since any pole of ζ is of order one and $z \in \boldsymbol{C}$ may be a pole of ζ only if $f(z) \in \Sigma$,

$$N(r, \zeta) \leqq \overline{N}_f(r, \Sigma).$$

Hence

$$\begin{aligned} T(r, \zeta) &= N(r, \zeta) + m(r, \zeta) \\ &\leqq \overline{N}_f(r, \Sigma) + S_f(r). \end{aligned} \qquad \text{Q.E.D.}$$

3. Main Theorem

Let X be a smooth irreducible quasiprojective variety of dimension n (cf. [18]) and $\{\omega_i\}_{i=1,\dots,n+1}$ a system of $n+1$ regular rational closed 1-forms on X such that

$$(3.1) \qquad \begin{cases} \text{the regular rational } n\text{-forms} \\ \quad \omega_1 \wedge \cdots \wedge \check{\omega}_i \wedge \cdots \wedge \omega_{n+1}{}^{*)}, \qquad i = 1,\dots, n+1, \\ \text{are linearly independent (over } \boldsymbol{C}). \end{cases}$$

DEFINITION. We say that a holomorphic mapping $f: U \to X$ from a connected open set U in $\boldsymbol{C}$ into X is degenerate with respect to $\{\omega_i\}$ if $f(U)$ is contained in a subvariety

$$\{\sum_{i=1}^{n+1} \lambda_i \omega_1 \wedge \cdots \wedge \check{\omega}_i \wedge \cdots \wedge \omega_{n+1} = 0\}$$

with $(\lambda_1,\dots, \lambda_{n+1}) \in \boldsymbol{C}^{n+1} - \{O\}$. If it is not the case, f is said to be non-degenerate with respect to $\{\omega_i\}$.

Now we recall Ochiai's theorem [17, Theorem A]:

OCHIAI'S THEOREM. *Suppose that there exists a system $\{\omega_i\}_{i=1,\dots,n+1}$ of $n+1$ regular rational closed 1-forms on X satisfying* (3.1). *Let $f: U \to X$ be a holomorphic mapping from a connected open set U in $\boldsymbol{C}$ into X which is non-*

*) The symbol "$\check{\omega}_i$" means that ω_i shall be omitted.

degenerate with respect to $\{\omega_i\}$. *Then for every rational function* $\phi \in \mathscr{R}(X)$ *such that* $f^*\phi$ *is defined, the meromorphic function* $f^*\phi$ *is algebraic over the field generated by* $\{\zeta_i^{(k)}\}_{1 \leq i \leq n+1, 0 \leq k \leq n-1}$, *where* $\zeta_i^{(k)}$ *is the k-th derivative of* ζ_i *defined by* $f^*\omega_i = \zeta_i dz$.

In this section we denote by V an n-dimensional projective algebraic manifold and by Σ a hypersurface in V. Let $\{\omega_i\}_{i=1,\dots,n+1}$ be a system of $n+1$ rational closed 1-forms $\omega_i \in H^0(V, \Omega_V^1(\log \Sigma))$ satisfying (3.1) (i.e., $\{\omega_i|_{V-\Sigma}\}$ satisfies (3.1) in $V-\Sigma$).

Let $f: \boldsymbol{C} \to V$ be a holomorphic curve. Then f is said to be degenerate (resp. non-degenerate) with respect to $\{\omega_i\}$ if the restriction $f|_{\boldsymbol{C}-f^{-1}(\Sigma)}: \boldsymbol{C}-f^{-1}(\Sigma) \to V-\Sigma$ is degenerate (resp. non-degenerate) with respect to $\{\omega_i|_{V-\Sigma}\}$.

Let $T_f(r)$ be the characteristic function of f relative to a Kähler form on V.

Main Theorem. *Suppose that there exists a system* $\{\omega_i\}_{i=1,\dots,n+1}$ *of* $n+1$ *rational closed 1-forms* $\omega_i \in H^0(V, \Omega_V^1(\log \Sigma))$ *satisfying* (3.1). *Let* $f: \boldsymbol{C} \to V$ *be a holomorphic curve which is non-degenerate with respect to* $\{\omega_i\}$. *Then there is a positive constant* κ *such that*

$$\kappa T_f(r) \leqq \overline{N}_f(r, \Sigma) + S_f(r), \tag{I}$$

where $S_f(r)$ *is defined by* (2.9).

Proof. We set $f^*\omega_i = \zeta_i dz$. Using inductively (1.1), (1.2), (1.3) and Corollary 2.5, we see that

$$T(r, \zeta_i^{(k)}) \leqq (k+1)\overline{N}_f(r, \Sigma) + S_f(r) \tag{3.2}$$

for all i and k. Let $\{\phi_j\}_{j=1,\dots,l}$ be a system of generators of the rational function field $\mathscr{R}(V)$ over $\boldsymbol{C}$ such that all $f^*\phi_j$ are defined. Then by Ochiai's theorem there are algebraic relations

$$(f^*\phi_j)^{m_j} + R_{j1}(\zeta_i^{(k)})(f^*\phi_j)^{m_j-1} + \cdots + R_{jm_j}(\zeta_i^{(k)}) = 0 \tag{3.3}$$

for $j=1,\dots, l$, where $R_{j\nu}(\zeta_i^{(k)})$ are rational functions in $\zeta_i^{(k)}$, $i=1,\dots, n+1$, $k=0, 1,\dots, n-1$. We infer from (3.2), (3.3) and Valiron [19] that there is a positive constant K such that

$$T(r, f^*\phi_j) \leqq K\overline{N}_f(r, \Sigma) + S_f(r)$$

for all j. Thus we see that

$$\tilde{T}_f(r) = \max_j \{T(r, f^*\phi_j)\} \leqq K\overline{N}_f(r, \Sigma) + S_f(r). \tag{3.4}$$

The inequalities (1.4) and (3.4) yield (I). Q.E.D.

REMARK 1. It seems that κ is independent of each holomorphic curve f.

REMARK 2. Let us consider the case where V is a closed Riemann surface with genus g.

If $g=0$, then $V=\boldsymbol{P}^1$ with inhomogeneous coordinate w. Condition (3.1) of Main Theorem implies that Σ contains at least three points. If $\Sigma=\{0, 1, \infty\}$, then $\{dw/w, dw/(w-1)\}$ forms a basis of $H^0(\boldsymbol{P}^1, \Omega^1_{\boldsymbol{P}^1}(\log \Sigma))$ and so condition (3.1) is fulfilled. Conversely, it is well known that Σ contains at least three points if (I) holds for any f.

If $g \geqq 2$, then condition (3.1) is satisfied with $\Sigma=\emptyset$. This implies the well known fact that any holomorphic mapping from $\boldsymbol{C}$ into a closed Riemann surface with genus greater than one is a constant mapping.

If $g=1$, then V is an elliptic curve and condition (3.1) asserts that Σ contains at least two points. But it is known that (I) holds if Σ contains only one point (cf., for example, [9, Example (6.15) and section 7] or [15, Theorem B, p. 28]).

The higher dimensional case will be discussed in the next section.

4. Examples and applications

(a) First we examine condition (3.1) of Main Theorem in the classical case where $V=\boldsymbol{P}^n$ and Σ is a union of hyperplanes $\Sigma_i=\{F_i=0\}$, $i=0, 1,\dots, l$. Suppose that there exists a system of $n+1$ rational closed 1-forms in $H^0(\boldsymbol{P}^n, \Omega^1_{\boldsymbol{P}^n}(\log \Sigma))$ satisfying (3.1). Since $\pi_1(\boldsymbol{P}^n)=0$,

$$(4.1) \qquad H^0(\boldsymbol{P}^n, \mathfrak{M}^*_{\boldsymbol{P}^n}(\Sigma)) \xrightarrow{d\log} H^0(\boldsymbol{P}^n, \mathfrak{U}_{\boldsymbol{P}^n}(\log \Sigma)) \longrightarrow 0$$

(cf. (1.5)). The system $\{\omega_i=d\log(F_i/F_0)\}_{i=1,\dots,l}$ forms a basis of $H^0(\boldsymbol{P}^n, \Omega^1_{\boldsymbol{P}^n}(\log \Sigma))$. Hence Σ contains at least $n+2$ hyperplanes. Assume that Σ consists of $n+2$ hyperplanes $\Sigma_i=\{F_i=0\}$, $i=0, 1,\dots, n+1$. Then the system $\{\omega_i= d\log(F_i/F_0)\}_{i=1,\dots,n+1}$ satisfies (3.1). Therefore $\omega_1\wedge\cdots\wedge\omega_n\not\equiv 0$. This implies that $\Sigma_0,\dots, \Sigma_n$ are in general position. We may use F_i, $i=0, 1,\dots, n$, as homogeneous coordinates in $\boldsymbol{P}^n$. Set $w_i=F_i$, $i=0, 1,\dots, n$, and

$$F_{n+1} = c_0w_0 + c_1w_1 + \cdots + c_nw_n,$$

where $(c_0, c_1,\dots, c_n)\in \boldsymbol{C}^{n+1}-\{O\}$. Using inhomogeneous coordinate system $(x_1,\dots, x_n)$ with $x_i=w_i/w_0$, we have

$$\omega_1\wedge\cdots\wedge\omega_n = \frac{1}{x_1\cdots x_n}dx_1\wedge\cdots\wedge dx_n,$$

$$\omega_1\wedge\cdots\wedge\check{\omega}_i\wedge\cdots\wedge\omega_{n+1} = \frac{(-1)^{n+1-i}c_i}{x_1\cdots\check{x}_i\cdots x_n(c_0+c_1x_1+\cdots+c_nx_n)}$$

$$\cdot dx_1\wedge\cdots\wedge dx_n$$

for $i=1,\dots, n$. Therefore $c_i \neq 0$ for $i=1,\dots, n$. Since the system $\{c_0+c_1x_1+\cdots+c_nx_n, x_1,\dots, x_n\}$ must be linearly independent, $c_0 \neq 0$. Thus $\Sigma_0, \Sigma_1,\dots, \Sigma_{n+1}$ are in general position. Conversely, let $\Sigma_i=\{F_i=0\}$, $i=0, 1,\dots, n+1$, be $n+2$ hyperplanes in general position. Then it is clear from the above argument that the system $\{\omega_i=d\log(F_i/F_0)\}_{i=1,\dots,n+1}$ satisfies condition (3.1), and moreover that a holomorphic curve $f: \boldsymbol{C}\to\boldsymbol{P}^n$ is degenerate with respect to $\{\omega_i\}$ if and only if $f(\boldsymbol{C})$ is contained in a hyperplane.

We summarize the above results.

PROPOSITION 4.1. *Let Σ be a union of $n+2$ hyperplanes Σ_i, $i=0, 1,\dots, n+1$, in $\boldsymbol{P}^n$.*

(i) *There exists a system of $n+1$ rational closed 1-forms in $H^0(\boldsymbol{P}^n, \Omega^1_{\boldsymbol{P}^n}(\log\Sigma))$ satisfying condition (3.1) if and only if Σ_i's are in general position.*

(ii) *Let $\{\omega_i\}_{i=1,\dots,n+1}\subset H^0(\boldsymbol{P}^n, \Omega^1_{\boldsymbol{P}^n}(\log\Sigma))$ be a system satisfying (3.1). Then a holomorphic curve $f: \boldsymbol{C}\to\boldsymbol{P}^n$ is degenerate with respect to $\{\omega_i\}$ if and only if $f(\boldsymbol{C})$ is contained in a hyperplane.*

Combining Proposition 4.1 with Main Theorem, we obtain Borel's theorem:

COROLLARY (Borel's theorem). *Let $f: \boldsymbol{C}\to\boldsymbol{P}^n$ be a holomorphic curve omitting $n+2$ hyperplanes in general position. Then the image $f(\boldsymbol{C})$ is contained in a hyperplane.*

(b)*) Let $V=\boldsymbol{P}^n$ and Σ be a union of hyperplanes. In case $n=2$, the condition that there exists a system $\{\omega_1, \omega_2, \omega_3\}$ in $H^0(\boldsymbol{P}^2, \Omega^1_{\boldsymbol{P}^2}(\log\Sigma))$ satisfying (3.1) is equivalent to that Σ contains 4 hyperplanes in general position. This equivalence fails for $n\geqq 3$. In fact, let $(w_0, w_1,\dots, w_n)$ be a homogeneous coordinate system in $\boldsymbol{P}^n$ with $n\geqq 3$ and set

$$F_i = w_i \qquad \text{for} \quad i = 0, 1,\dots, n,$$

$$F_{n+1} = w_0 + w_1 + w_2,$$

$$F_{n+2} = w_1 + w_2 + \cdots + w_n.$$

We put

$$\Sigma_i = \{F_i = 0\} \qquad \text{for} \quad i = 0, 1,\dots, n+2,$$

$$\Sigma = \bigcup_{i=0}^{n+2} \Sigma_i,$$

$$\omega_i = d(\log F_i/F_0) \qquad \text{for} \quad i = 1,\dots, n,$$

---

*) The examples in (b) were suggested by Professor S. Iitaka.

$$\omega_{n+1} = d\log(F_{n+1}F_{n+2}/F_0^2).$$

Then any $n+2$ hyperplanes in Σ are not in general position, but it is easily seen that the system $\{\omega_i\}_{i=1,\dots,n+1}$ satisfies (3.1).

In this case, a holomorphic curve $f: \boldsymbol{C} \to \boldsymbol{P}^n$ is degenerate with respect to $\{\omega_i\}_{i=1,\dots,n+1}$ if and only if $f(\boldsymbol{C})$ is contained in a quadratic hypersurface defined by

$$\begin{aligned} &\alpha_1 F_1(F_{n+1} + F_{n+2}) + \alpha_2 F_2(F_{n+1} + F_{n+2}) \\ &\quad + \sum_{i=3}^{n} \alpha_i F_i F_{n+1} + \alpha_{n+1} F_{n+1} F_{n+2} = 0, \end{aligned}$$

where $(\alpha_1,\dots,\alpha_{n+1}) \in \boldsymbol{C}^{n+1} - \{O\}$.

(c) We give an example for which $d\log$ in (1.5) is not surjective (cf. (4.1)). Let V be an Abelian variety A of dimension n. Then there are n linearly independent regular rational closed 1-forms $\omega_1,\dots,\omega_n$ on A.

Let D_1 and D_2 be effective divisors on A with no common component, which are homologously equivalent. Then there is a theta function ϑ on A such that the divisor $(\vartheta)_0$ of zeros of ϑ equals D_1, the divisor $(\vartheta)_\infty$ of poles of ϑ equals D_2 and $|\vartheta|$ is single-valued on A (cf. section 1(b)). Set $\Sigma = \mathrm{Supp}(D_1 + D_2)$ and

$$\omega_{n+1} = d\log\vartheta \in H^0(A, \mathfrak{A}_A(\log\Sigma)).$$

PROPOSITION 4.2. *The system* $\{\omega_i\}_{i=1,\dots,n+1} \subset H^0(A, \mathfrak{A}_A(\log\Sigma))$ *given above satisfies condition* (3.1) *if* D_1 *and* D_2 *are ample**).

REMARK. In the case where D_1 and D_2 are linearly equivalent, this proposition was shown in the proof of Theorem 12–1 of the original draft of Ochiai [17]. His method works for the above system $\{\omega_i\}$.

Since this proposition has a meaning different from Ochiai's in the sense that it gives an interesting example for our Main Theorem, and since the proof is not so long, we shall prove it.

PROOF. Assume that there is a non-trivial linear relation

$$\sum_{i=1}^{n+1} \lambda_i \omega_1 \wedge \cdots \wedge \check{\omega}_i \wedge \cdots \wedge \omega_{n+1} = 0. \tag{4.2}$$

Set $\omega_{n+1} = \sum_{i=1}^{n} \xi_i \omega_i$. Then (4.2) implies that

$$\sum_{i=1}^{n} (-1)^{n-i} \lambda_i \xi_i + \lambda_{n+1} = 0.$$

*) A divisor D is called ample if the cohomology class $c_1(D)$ contains a positive definite (1, 1)-form.

We may assume that $\lambda_1 \neq 0$. Then

$$\omega_{n+1} = (-1)^n \frac{\lambda_{n+1}}{\lambda_1}\omega_1 + \sum_{j=2}^{n} \xi_j \left\{\omega_j + (-1)^j \frac{\lambda_j}{\lambda_1}\omega_1\right\}. \tag{4.3}$$

Set

$$\alpha_1 = \frac{(-1)^n}{\lambda_1}\omega_1,$$

$$\alpha_j = \omega_j + (-1)^j \frac{\lambda_j}{\lambda_1}\omega_1, \; j = 2,\dots, n.$$

Then $\alpha_1,\dots, \alpha_n$ are linearly independent and there is a lattice Γ in $\boldsymbol{C}^n$ such that

$$\pi\colon \boldsymbol{C}^n \to \boldsymbol{C}^n/\Gamma = A,$$

$$\pi^*\alpha_j = dz_j, \quad j = 1,\dots, n,$$

where $(z_1,\dots, z_n)$ is the natural coordinate system in $\boldsymbol{C}^n$. It follows from (4.3) that

$$\pi^*(\omega_{n+1})\left(\frac{\partial}{\partial z_1}\right) = \lambda_{n+1}.$$

This implies that

$$\frac{\partial \vartheta}{\partial z_1} = \lambda_{n+1}\vartheta.$$

Therefore ϑ is written as

$$\vartheta(z_1,\dots, z_n) = \vartheta_0(z_2,\dots, z_n)e^{\lambda_{n+1}z_1}, \tag{4.4}$$

where ϑ_0 is a meromorphic function in $z_2,\dots, z_n$.

Let $a(t)$ be the one parameter group of transformations generated by $\partial/\partial z_1$. Then we infer from (4.4) that the zeros and the poles of ϑ are $a(t)$-invariant. Let B be the connected component of $0 \in A$ of the group

$$\{x \in A;\, x + D_j = D_j \quad \text{for} \quad j = 1, 2\}.$$

Then $\dim B \geqq 1$. This is absurd since D_j are ample (cf. Weil [20, Corollary 3, p. 115]). Q.E.D.

(d) Let $M = \boldsymbol{C}^n/\Gamma$ be a complex torus and $\Omega = (i/2\pi)\sum dz_i \wedge d\bar{z}_i$ the natural flat Kähler form on M. Let $f\colon \boldsymbol{C} \to M$ be a holomorphic curve and $T_f(r)$ the characteristic function relative to Ω.

LEMMA 4.3. *Let $f\colon \boldsymbol{C} \to M$ be a non-constant holomorphic curve. Then* *there is a positive constant c such that*

$$T_f(r) \geqq cr^2 \qquad \textit{for all large} \quad r.$$

PROOF. Let $\tilde{f}: \boldsymbol{C} \to \boldsymbol{C}^n$ be a lifting of f and set $\tilde{f} = (f_1, \dots, f_n)$. By definition

$$T_f(r) = \int_0^r \frac{dt}{t} \int_{\Delta(t)} \sum_{i=1}^n |f_i'(z)|^2 \frac{i}{2\pi} dz \wedge d\bar{z},$$

where z is the natural coordinate in $\boldsymbol{C}$. Take a point $z_0 \in \boldsymbol{C}$ so that $\sum |f_i'(z_0)|^2 = C > 0$. Let ϕ be a conformal mapping of $\Delta(t)$ with $t > |z_0|$ defined by

$$z = \phi(w) = t^2 \frac{w + z_0}{t^2 + \bar{z}_0 w}.$$

Then we have

$$\begin{aligned} &\int_{\Delta(t)} \sum_{i=1}^n |f_i'(z)|^2 \frac{i}{2\pi} dz \wedge d\bar{z} \\ &= \int_{\Delta(t)} \sum_{i=1}^n |f_i'(\phi(w))|^2 t^4 \frac{|t^2 + \bar{z}_0 w - w\bar{z}_0 - |z_0|^2|^2}{|t^2 + \bar{z}_0 w|^4} \cdot \frac{i}{2\pi} dw \wedge d\bar{w}. \end{aligned}$$

The subharmonicity of the integrand yields

$$\int_{\Delta(t)} \sum_{i=1}^n |f_i'(z)|^2 \frac{i}{2\pi} dz \wedge \bar{z} \geqq C \frac{(t^2 - |z_0|^2)^2}{t^2} \geqq C \frac{t^2}{4}$$

for $t \geqq \sqrt{2}|z_0|$. Hence

$$T_f(r) \geqq \frac{C}{8}(r^2 - 2|z_0|^2) + T_f(\sqrt{2}|z_0|)$$

for $r \geqq \sqrt{2}|z_0|$. Q.E.D.

In general, a holomorphic curve is called algebraically degenerate if the image is contained in a proper subvariety.

THEOREM 4.1. *Let D_1 and D_2 be effective divisors on M with no common component which are homologously equivalent. Then any holomorphic curve $f: \boldsymbol{C} \to M$ omitting* $\mathrm{Supp}(D_1 + D_2)$ *is necessarily algebraically degenerate.*

PROOF. By Weil [20, Chap. VI], there are an Abelian variety A of positive dimension, a surjective homomorphism $\lambda: M \to A$ and ample divisors Z_1 and Z_2 on A such that $D_j = \lambda^* Z_j$, $j = 1, 2$. Therefore Z_1 and Z_2 have no common component and are homologously equivalent. It is enough to prove that the holomorphic curve $g = \lambda \circ f: \boldsymbol{C} \to A - \mathrm{Supp}(Z_1 + Z_2)$ is algebraically degenerate.

Assume that g is not algebraically degenerate. It follows from Main Theorem and Proposition 4.2 that there is a positive constant κ such that

$$\kappa T_g(r) \leqq \overline{N}_g(r, Z_1 + Z_2) + S_g(r). \tag{4.5}$$

By the assumption, $\overline{N}_g(r, Z_1+Z_2)=0$. From (4.5), (2.9) and Lemma 4.3 we obtain a contradiction

$$0 < \kappa \leq \varliminf_{r\to\infty} S_g(r)/T_g(r) = 0.$$

Q. E. D.

REMARK. This theorem gives a partial answer to the following conjecture due to S. Lang (cf. [7, Problem F]):

If a holomorphic curve $f: \boldsymbol{C}\to\boldsymbol{C}^n/\Gamma$ omits a hypersurface in a complex torus $\boldsymbol{C}^n/\Gamma$, then f is algebraically degenerate.

When f is a group homomorphism, this was proved by Ax [2].

Theorem 4.1 was proved by Ochiai [17, section 5] when D_1 and D_2 are linearly equivalent.

(e) Let V be a projective algebraic manifold and D an effective ample divisor on V (see the footnote at p. 848).

LEMMA 4.4. *Let V and D be as above. Let $f: \boldsymbol{C}\to V$ be a non-constant holomorphic curve. Then the closure $\overline{f(\boldsymbol{C})}$ of $f(\boldsymbol{C})$ intersects* $\operatorname{Supp} D$.

PROOF. Assume that $\overline{f(\boldsymbol{C})} \cap \operatorname{Supp} D=\emptyset$. Take a positive integer ν so that the mapping

$$T: V \ni x \longrightarrow (\sigma_0(x), \dots, \sigma_N(x)) \in \boldsymbol{P}^N$$

is an embedding, where $\{\sigma_0, \dots, \sigma_N\}$ is a basis of $\Gamma(V, [\nu D])$ and $(\sigma_0)=\nu D$. Set

$$s = \frac{|\sigma_0|^2}{|\sigma_0|^2 + \cdots + |\sigma^N|^2},$$

which is a smooth function on V. Since $\overline{f(\boldsymbol{C})} \cap \operatorname{Supp} D=\emptyset$, there is a constant ε such that

$$s \geqq \varepsilon > 0 \quad \text{on} \quad f(\boldsymbol{C}).$$

Then the holomorphic functions $\alpha_j=(\sigma_j\circ f)/(\sigma_0\circ f)$, $j=1,\dots, N$ satisfy $|\alpha_j|^2 \leqq 1/\varepsilon$. Thus α_j are constants. Since T is an embedding, f is constant. This is absurd.

Q. E. D.

The following conjecture was posed by Griffiths (cf. Griffiths [7, Problem F, p. 381] and Kobayashi [12, Problem D. 9, p. 404]):

Let A be an Abelian variety and D an effective ample divisor on A. If a holomorphic curve $f: \boldsymbol{C} \to A$ omits $\operatorname{Supp} D$, then f is constant.

We shall prove this in a special case.

THEOREM 4.2. *Let A be an Abelian surface and D_j, $j=1, 2$, effective ample divisors on A with no common component which are homologously equivalent. Then any holomorphic curve $f: \boldsymbol{C} \to A$ omitting* $\operatorname{Supp}(D_1+D_2)$ *is constant.*

PROOF. By Theorem 4.1 there is a curve X in A such that $f(\boldsymbol{C}) \subset X$. By Lemma 4.4, $X \cap \operatorname{Supp} D_j \neq \emptyset$ for $j=1, 2$. Therefore we have a holomorphic curve

$$f: \boldsymbol{C} \longrightarrow X - \operatorname{Supp}(D_1 + D_2).$$

Since the genus of X is greater than zero, f must be constant (cf., for example, [9, Example (6.15)]). Q.E.D.

REMARK. Green [6] proved the above conjecture in the case where $\operatorname{Supp} D$ contains no non-trivial complex subtorus, by showing that $A - \operatorname{Supp} D$ is completely hyperbolic and hyperbolically embedded in A in the sense of Kobayashi (cf. [12]).

References

[1] L. V. Ahlfors, The theory of meromorphic curves, Acta Soc. Sci. Fennicae Nova Ser. A. **3** (1941), 3–31.

[2] J. Ax, Some topics in differential algebraic geometry II: On the zeros of theta functions, Amer. J. Math. **94** (1972), 1205–1213.

[3] H. Cartan, Sur les zéros des combinaisons linéaires de p fonctions holomorphes données, Mathematica **7** (1933), 5–31.

[4] P. Deligne, Equations différentielles à points singuliers réguliers, Lecture Notes in Math. **163**, Springer, Berlin, 1970.

[5] ———, Théorie de Hodge II, Inst. Hautes Études Sci. Publ. **40** (1971), 5–57.

[6] M. L. Green, Holomorphic maps to complex tori, preprint.

[7] P. Griffiths, Holomorphic mappings: Survey of some results and discussion on open problems, Bull. Amer. Math. Soc. **78** (1972), 374–382.

[8] ———, Differential geometry and complex analysis, Proc. Symp. Pure Math. Amer. Math. Soc. **27** (1975), 43–64.

[9] ——— and J. King, Nevanlinna theory and holomorphic mappings between algebraic varieties, Acta Math. **130** (1973), 145–220.

[10] S. Iitaka, Logarithmic forms of algebraic varieties, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **23** (1976), 525–544.

[11] ————, On logarithmic Kodaira dimension of algebraic varieties, Complex Analysis and Algebraic Geometry, pp. 175–189, Iwanami, Tokyo, 1977.

[12] S. Kobayashi, Intrinsic distances, measures and geometric function theory, Bull. Amer. Math. Soc. **82** (1976), 357–416.

[13] R. Nevanlinna, Le théorème de Picard-Borel et la théorie des fonctions méromorphes, Gautheir-Villars, Paris, 1939.

[14] J. Noguchi, Meromorphic mappings of a covering space over C^m into a projective variety and defect relations, Hiroshima Math. J. **6** (1976), 265–280.

[15] ————, Holomorphic mappings into closed Riemann surfaces, Hiroshima Math. J. **6** (1976), 281–291.

[16] ————, Meromorphic mappings into a compact complex space, Hiroshima Math. J. **7** (1977), 411–425.

[17] T. Ochiai, On holomorphic curves in algebraic varieties with ample irregularity, to appear in Invent. Math.

[18] I. R. Shafarevich, Basic algebraic geometry, Springer, Berlin, 1974.

[19] G. Valiron, Sur la dérivée des fonctions algébroïdes, Bull. Soc. Math. France **59** (1931), 17–39.

[20] A. Weil, Introduction à l'étude des variétés kähleriennes, Hermann, Paris, 1958.

[21] H. and J. Weyl, Meromorphic curves, Ann. of Math. **39** (1938), 516–538.

[22] H. Wu, The equidistribution theory of holomorphic curves, Ann. Math. Studies, no. **64**, Princeton Univ. Press, Princeton, N. J., 1970.

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