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Extremal Length of an Infinite Network Which is not Necessarily Locally Finite

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Introduction

In the preceding paper [2], we introduced a generalized extremal length of an infinite network N which is locally finite, i.e., each node has only a finite number of incident arcs, and investigated the generalized reciprocal relation between the extremal distance $EL_p(A, B)$ (resp. $EL_p(A, \infty)$) and the extremal width $EW_q(A, B)$ (resp. $EW_q(A, \infty)$) relative to mutually disjoint nonempty finite subsets A and B of nodes (resp. a finite subset A of nodes and the ideal boundary ∞ of the network N). In this paper we shall be concerned with the same problem on an infinite network which is not necessarily locally finite. It will be shown in §2 that the generalized reciprocal relation between $EL_p(A, B)$ and $EW_q(A, B)$ still holds in the case where N is not necessarily locally finite. However, the generalized reciprocal relation between $EL_p(A, \infty)$ and $EW_q(A, \infty)$ does not hold, in general, in the present case. In §3 we shall introduce a p-almost locally finite network, for which the generalized reciprocal relation holds. We shall also study the stability of $\{EL_p(A, X-X_n)\}$ and $\{EW_q(A, X-X_n)\}$ with respect to an exhaustion $\{<X_n, Y_n>\}$ of N in the case where N is a p-almost locally finite network.

§1. Preliminaries

Let X be a finite or countably infinite set of nodes, let Y be a finite or countably infinite set of arcs and let K be a function on $X \times Y$ satisfying the following conditions:

(1.1) The range of K is $\{-1, 0, 1\}$.

(1.2) For each $y \in Y$, $e(y) \equiv \{x \in X; K(x, y) \neq 0\}$ consists of exactly two nodes x_1, x_2 and $K(x_1, y)K(x_2, y) = -1$.

(1.3) For any $x, x' \in X$, there are $x_1, \dots, x_n \in X$ and $y_1, \dots, y_{n+1} \in Y$ such that $e(y_j) = \{x_{j-1}, x_j\}$, $j = 1, \dots, n+1$ with $x_0 = x$ and $x_{n+1} = x'$.

For each $x \in X$, the set

$$Y(x) = \{y \in Y; K(x, y) \neq 0\}$$

is nonempty by (1.3). Let us put

$$X_\infty = \{x \in X;\ Y(x) \text{ is an infinite set}\}.$$

Let r be a strictly positive real function on Y. Then $N=\{X, Y, K, r\}$ is called a network. We say that N is a finite network if Y is a finite set, that N is an infinite network if Y is an infinite set and that N is locally finite if $X_\infty=\emptyset$ (the empty set). We remark that A. H. Zemanian [4] studied an electrical problem on an infinite network which is not necessarily locally finite.

Let X' and Y' be subsets of X and Y respectively and K' and r' be the restrictions of K and r onto $X' \times Y'$ and Y' respectively. Then $N'=\{X', Y', K', r'\}$ is called a subnetwork of N if (1.2) and (1.3) are fulfilled replacing X, Y, K by X', Y', K' respectively. In order to emphasize the sets of nodes and arcs of N', we often write $N'=\langle X', Y'\rangle$.

A sequence $\{N_n\}$ ($N_n=\langle X_n, Y_n\rangle$) of finite subnetworks of N is called an exhaustion of N if the following conditions are fulfilled:

(1.4) For each n, $X_n \subset X_{n+1}$ and $Y_n \subset Y_{n+1}$, and furthermore at least one of these inclusions is strict.

$$X = \bigcup_{n=1}^{\infty} X_n \quad \text{and} \quad Y = \bigcup_{n=1}^{\infty} Y_n. \tag{1.5}$$

$$Y(x) \subset Y_{n+1} \qquad \text{for all} \quad x \in X_n - X_\infty. \tag{1.6}$$

Let p and q be conjugate exponents, i.e.,

$$1/p + 1/q = 1 \quad \text{and} \quad 1 \leqq p \leqq \infty. \tag{1.7}$$

Let $L(X)$ and $L(Y)$ be the sets of all real functions on X and Y respectively. For $u \in L(X)$ and $w \in L(Y)$, we set

$$D_p(u) = D_p(u; N) = \sum_{y \in Y} r(y)^{1-p} \Big|\sum_{x \in X} K(x, y)u(x)\Big|^p \qquad (1 \leqq p < \infty), \tag{1.8}$$

$$D_\infty(u) = D_\infty(u; N) = \sup_{y \in Y} r(y)^{-1} \Big|\sum_{x \in X} K(x, y)u(x)\Big|, \tag{1.8'}$$

$$H_p(w) = H_p(w; N) = \sum_{y \in Y} r(y)|w(y)|^p \qquad (1 \leqq p < \infty), \tag{1.9}$$

$$H_\infty(w) = H_\infty(w; N) = \sup_{y \in Y} |w(y)|. \tag{1.9'}$$

For $w, w' \in L(Y)$, we define $((w, w'))_N$ by

$$((w, w'))_N = \sum_{y \in Y} r(y)w(y)w'(y) \tag{1.10}$$

whenever the sum on the right can be defined without any ambiguity. Denote by

$L_0(Y)$ the set of all $w \in L(Y)$ such that $\{y \in Y;\ w(y) \neq 0\}$ (the support of w) is a finite set. Let us put

$$L^+(Y) = \{w \in L(Y);\ w(y) \geqq 0 \text{ on } Y\},$$

$$L_p(Y; r) = \{w \in L(Y);\ H_p(w; N) < \infty\},$$

$$L_p^+(Y; r) = \{w \in L^+(Y);\ H_p(w; N) < \infty\}.$$

Note that $L_p(Y; r)$ $(1 < p < \infty)$ is a reflexive Banach space with respect to the norm $[H_p(w; N)]^{1/p}$.

For a nonempty subset A of X, we set

$$\mathbf{D}_p^A(N) = \{u \in L(X);\ D_p(u; N) < \infty \text{ and } u = 0 \text{ on } A\}.$$

The following results can be proved in the same manner as in [3; Lemmas 1 and 2]:

LEMMA 1.1. *Let F be a nonempty finite subset of X and put $\|u\|_p = [D_p(u; N)]^{1/p}$ if $1 \leqq p < \infty$ and $\|u\|_\infty = D_\infty(u; N)$. Then there exists a constant M depending only on A, F and p such that*

$$\sum_{x \in F} |u(x)| \leqq M \|u\|_p$$

for all $u \in \mathbf{D}_p^A(N)$.

LEMMA 1.2. *Let T be a normal contraction of the real line R and $u \in \mathbf{D}_p^A(N)$. Then $Tu \in \mathbf{D}_p^A(N)$ and $D_p(Tu; N) \leqq D_p(u; N)$.*

§2. Extremal distance and extremal width

We can define paths P on N and their index functions p_P exactly as in the case where N is locally finite (see [2]). We say that a subset Q of Y is a cut on N if there exist mutually disjoint nonempty subsets X' and X'' of X such that $X = X' \cup X''$ and Q is equal to the set $(X', X'')_N \equiv \{y \in Y;\ e(y) = \{x', x''\}$ for some $x' \in X'$ and $x'' \in X''\}$. For mutually disjoint subsets A and B of X, a cut $Q = (X', X'')_N$ is called a cut between A and B in N if $A \subset X'$ and $B \subset X''$. Denote by $\mathbf{P}_{A,B}(N)$ the set of all paths from A to B in N and by $\mathbf{Q}_{A,B}(N)$ the set of all cuts between A and B in N. For a nonempty finite subset A of X, a cut $Q = (X', X'')_N$ is called a cut between A and the ideal boundary ∞ of N if $A \subset X'$ and X' is a finite set. Denote by $\mathbf{P}_{A,\infty}(N)$ the set of all paths from A to the ideal boundary ∞ of N and by $\mathbf{Q}_{A,\infty}(N)$ the set of all cuts between A and the ideal boundary ∞ of N. For $Q = (X', X'')_N \in \mathbf{Q}_{A,B}(N)$ or $\mathbf{Q}_{A,\infty}(N)$, we define a characteristic function $u = u_Q \in L(X)$ and an index function $s_Q \in L(Y)$ by $u(x) = 0$ if $x \in X'$, $u(x) = 1$ if $x \in X''$

and $s_Q(y)=\sum_{x\in X}K(x, y)u(x)$. For a path P and a cut Q on N, we define functions ϕ_P and ψ_Q on Y by

$$\phi_P(Y) = |p_P(y)| \quad \text{and} \quad \psi_Q(y) = r(y)^{-1}|s_Q(y)|. \tag{2.1}$$

Note that $\mathbf{P}_{A,\infty}(N)=\emptyset$ if X is a finite set and that $\mathbf{Q}_{A,\infty}(N)\neq\emptyset$ if and only if $A\neq X$.

The extremal distance $EL_p(A, B; N)$ (resp. $EL_p(A, \infty; N)$) of order p of N relative to A and B (resp. A and ∞) and the extremal width $EW_q(A, B; N)$ (resp. $EW_q(A, \infty; N)$) of order q of N relative to A and B (resp. A and ∞) are defined in a way analogous to that in [2], i.e.,

$$EL_p(A, B; N)^{-1} = \inf\{H_p(W; N);\ W\in E_p(\mathbf{P}_{A,B}(N))\}, \tag{2.2}$$

where

$$E_p(\mathbf{P}_{A,B}(N)) = \{W\in L_p^+(Y; r);\ ((W, \phi_P))_N \geqq 1 \quad \text{for all} \quad P\in \mathbf{P}_{A,B}(N)\};$$

$$EW_q(A, B; N)^{-1} = \inf\{H_q(W; N);\ W\in E_q^*(\mathbf{Q}_{A,B}(N))\}, \tag{2.3}$$

where

$$E_q^*(\mathbf{Q}_{A,B}(N)) = \{W\in L_q^+(Y; r);\ ((\psi_Q, W))_N \geqq 1 \quad \text{for all} \quad Q\in \mathbf{Q}_{A,B}(N)\}.$$

We use the convention in this paper that the infimum of a real function on the empty set is equal to ∞. $EL_p(A, \infty; N)$ and $EW_q(A, \infty; N)$ are defined by (2.2) and (2.3), $\mathbf{P}_{A,B}(N)$ and $\mathbf{Q}_{A,B}(N)$ being replaced by $\mathbf{P}_{A,\infty}(N)$ and $\mathbf{Q}_{A,\infty}(N)$ respectively.

REMARK 2.1. In case $1<p<\infty$, we have

$$EW_q(A, B; N)^{-1} = \inf\{H_p(W; N);\ W\in E_p^{**}(\mathbf{Q}_{A,B}(N))\}, \tag{2.3'}$$

where

$$E_p^{**}(\mathbf{Q}_{A,B}(N)) = \{W\in L_p^+(Y; r);\ \sum_Q W(y)^{p-1} \geqq 1 \quad \text{for all} \quad Q\in \mathbf{Q}_{A,B}(N)\}.$$

In [2], we used the inverse of the value on the right of (2.3′) as the definition of extremal width of order p of N relative to A and B and denoted it by $EW_p(A, B)$.

Hereafter in this section we always assume that A and B are mutually disjoint nonempty subsets of X.

By the same argument as in the proof of Theorem 2.1 in [2] we obtain

THEOREM 2.1. *$EL_p(A, B; N)$ is equal to the reciprocal of the value $d_p(A, B; N)$ of the following extremum problem*:

(2.4) *Find* $d_p(A, B; N) = \inf\{D_p(u; N); u \in \mathbf{D}_p^A(N)$ *and* $u = 1$ *on* $B\}$.

THEOREM 2.2. *Let* $\{N_n\}$ $(N_n = \langle X_n, Y_n \rangle)$ *be an exhaustion of* N *such that* $A \cap X_1 \neq \emptyset$ *and* $B \cap X_1 \neq \emptyset$ *and put* $A_n = A \cap X_n$ *and* $B_n = B \cap X_n$. *Then* $EL_p(A_n, B_n; N_n) \to EL_p(A, B; N)$ *as* $n \to \infty$.

PROOF. We set $a_n = EL_p(A_n, B_n; N_n)^{-1}$ and $a = EL_p(A, B; N)^{-1}$. Since $\mathbf{P}_{A_n, B_n}(N_n) \subset \mathbf{P}_{A_{n+1}, B_{n+1}}(N_{n+1}) \subset \mathbf{P}_{A,B}(N)$, $a_n \leqq a_{n+1} \leqq a$, and hence $\lim_{n\to\infty} a_n \leqq a$. To prove the converse inequality, we may assume that $\lim_{n\to\infty} a_n < \infty$. For each n, we can find $W_n \in E_p(\mathbf{P}_{A_n, B_n}(N_n))$ such that $H_p(W_n; N_n) < a_n + n^{-1}$. Define $\overline{W}_n \in L(Y)$ by $\overline{W}_n = W_n$ on Y_n and $\overline{W}_n = 0$ on $Y - Y_n$. Since $\{H_p(\overline{W}_n; N)\}$ is bounded and Y is a countable set, there exists a pointwise convergent subsequence of $\{\overline{W}_n\}$. Denote it again by $\{\overline{W}_n\}$ and let $\overline{W}$ be its limit. Since $\mathbf{P}_{A,B}(N)$ is the union of $\{\mathbf{P}_{A_n, B_n}(N_n); n = 1, 2, \ldots\}$, we have $\overline{W} \in E_p(\mathbf{P}_{A,B}(N))$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (a_n + n^{-1}) \geqq \varliminf_{n\to\infty} H_p(\overline{W}_n; N) \geqq H_p(\overline{W}; N) \geqq a.$$

Our theorem is now proved.

We say that $w \in L(Y)$ is a flow from A to B of strength $I(w)$ in N if

(2.5) $$\sum_{y\in Y} |K(x, y)|\,|w(y)| < \infty \qquad \text{for each} \quad x \in X_\infty,$$

(2.6) $$\sum_{x\in A\cup B} \Big|\sum_{y\in Y} K(x, y)w(y)\Big| < \infty,$$

(2.7) $$\sum_{y\in Y} K(x, y)w(y) = 0 \qquad \text{for all} \quad x \in X - A - B,$$

(2.8) $$I(w) = -\sum_{x\in A} \sum_{y\in Y} K(x, y)w(y) = \sum_{x\in B} \sum_{y\in Y} K(x, y)w(y).$$

Denote by $G(A, B; N)$ the set of all flows from A to B in N which are in $L_0(Y)$. We remark that A and B need not be finite sets.

We have

THEOREM 2.3. $EW_q(A, B; N)$ *is equal to the reciprocal of the value* $d_q^*(A, B; N)$ *of the following extremum problem*:

(2.9) *Find* $d_q^*(A, B; N) = \inf\{H_q(w; N); w \in G(A, B; N)$ *and* $I(w) = 1\}$.

For the proof of this theorem, we shall consider the following two infinite linear programming problems with respect to $W \in L^+(Y)$:

(2.10) (Max-flow problem) Find

$$M(W; G(A, B; N)) = \sup\{I(w); w \in G(A, B; N) \text{ and } |w| \leqq W \text{ on } Y\}.$$

(2.11) (Min-cut problem) Find

$$M^*(W; \mathbf{Q}_{A,B}(N)) = \inf\{((\psi_Q, W))_N; Q \in \mathbf{Q}_{A,B}(N)\}.$$

We have

LEMMA 2.1. $M(W; G(A, B; N)) = M^*(W; \mathbf{Q}_{A,B}(N))$.

PROOF. Our equality is a well-known result which states that max-flow equals min-cut if N is a finite network, so that we may consider the case where N is an infinite network. We can prove the inequality $M(W; G(A, B; N)) \leqq M^*(W; \mathbf{Q}_{A,B}(N))$ as in the case where N is locally finite (see [3; Theorem 6]). To prove the converse inequality, we consider an exhaustion $\{N_n\}$ ($N_n = \langle X_n, Y_n \rangle$) of N such that $A \cap X_1 \neq \emptyset$ and $B \cap X_1 \neq \emptyset$. Define $W_n \in L(Y)$ by $W_n = W$ on Y_n and $W_n = 0$ on $Y - Y_n$. Then we have $M(W_n; G(A_n, B_n; N_n)) = M^*(W_n; \mathbf{Q}_{A_n,B_n}(N_n))$ with $A_n = A \cap X_n$ and $B_n = B \cap X_n$. It is easily seen that $M(W_n; G(A, B; N)) = M(W_n; G(A_n, B_n; N_n))$ and $M^*(W_n; \mathbf{Q}_{A,B}(N)) = M^*(W_n; \mathbf{Q}_{A_n,B_n}(N_n))$. Therefore we have

$$M(W; G(A, B; N)) \geqq M(W_n; G(A, B; N)) = M^*(W_n; \mathbf{Q}_{A,B}(N)).$$

The rest of the proof can be carried out by the same reasoning as in the proof of Theorem 6 in [3].

PROOF OF THEOREM 2.3. We set $EW_q = EW_q(A, B; N)$ and $d_q^* = d_q^*(A, B; N)$. The proof of the inequality $EW_q^{-1} \leqq d_q^*$ is the same as in the proof of Theorem 4.1 in [2]. To prove the converse inequality, let $W \in E_q^*(\mathbf{Q}_{A,B}(N))$, i.e., $W \in L_q^+(Y; r)$ and $M^*(W; \mathbf{Q}_{A,B}(N)) \geqq 1$. For each $0 < t < M(W; G(A, B; N))$, we can find $w \in G(A, B; N)$ such that $|w| \leqq W$ on Y and $I(w) > t$. Thus $d_q^* \leqq H_q(w/I(w)) \leqq H_q(W/t)$. By letting $t \to M(W; G(A, B; N))$, we have $d_q^* \leqq H_q(W)$ since $M(W; G(A, B; N)) = M^*(W; \mathbf{Q}_{A,B}(N)) \geqq 1$ by Lemma 2.1. Hence $d_q^* \leqq EW_q^{-1}$.

THEOREM 2.4. *Let $\{N_n\}$ ($N_n = \langle X_n, Y_n \rangle$) be an exhaustion of N such that $A \cap X_1 \neq \emptyset$ and $B \cap X_1 \neq \emptyset$ and put $A_n = A \cap X_n$, $B_n = B \cap X_n$. Then $EW_q(A_n, B_n; N_n) \to EW_q(A, B; N)$ as $n \to \infty$.*

PROOF. On account of Theorem 2.3, it suffices to show that $d_q^*(A_n, B_n; N_n) \to d_q^*(A, B; N)$ as $n \to \infty$. Notice that

$$d_q^*(A_n, B_n; N_n) = \inf\{H_q(w; N); w \in G_n(A, B; N) \text{ and } I(w) = 1\},$$

where $G_n(A, B; N) = \{w \in G(A, B; N); w = 0 \text{ on } Y - Y_n\}$. Our assertion follows from the relations

$$G_n(A, B; N) \subset G_{n+1}(A, B; N),$$

$$G(A, B; N) = \bigcup_{n=1}^{\infty} G_n(A, B; N).$$

THEOREM 2.5. *$EL_p(A, B; N) = EW_q(A, B; N)^{1-p}$ for all p, $1<p<\infty$.*

PROOF. We proved in [2; Theorem 5.2] that $EL_p(A, B; N) = EW_q(A, B; N)^{1-p}$ in the case where N is a locally finite infinite network. The reasoning of its proof is still effective in the case where N is a finite network. We consider the case where N may not be locally finite. Let $\{N_n\}$ ($N_n = <X_n, Y_n>$) be an exhaustion of N such that $A \cap X_1 \neq \emptyset$ and $B \cap X_1 \neq \emptyset$ and put $A_n = A \cap X_n$ and $B_n = B \cap X_n$. Then $EL_p(A_n, B_n; N_n) = EW_q(A_n, B_n; N_n)^{1-p}$. Now our assertion follows readily from Theorems 2.2 and 2.4.

REMARK 2.2. R. J. Duffin [1] proved that $EL_2(A, B; N)^{-1} = EW_2(A, B; N)$ in the case where N is a finite network. Theorem 2.5 is a generalization of Theorem 12 in [3] and Theorem 5.2 in [2].

LEMMA 2.2. *Let Λ be a nonempty subset of $L^+(Y)$. Then the following two values are reciprocal to each other:*

(2.12) $\inf\{H_\infty(W; N);\ W \in L^+_\infty(Y; r)$ *and* $((W, f))_N \geqq 1$ *for all* $f \in \Lambda\}$.

(2.13) $\inf\{H_1(f; N); f \in \Lambda\}$.

PROOF. Denote by a and b the values of (2.12) and (2.13) respectively. We show that $b^{-1} \leqq a$. We may assume that $a<\infty$ and $b<\infty$. Let $W \in L^+_\infty(Y; r)$ satisfy that $((W, f))_N \geqq 1$ for all $f \in \Lambda$. Since $((W, f))_N \leqq H_\infty(W) H_1(f)$, we have $1 \leqq b H_\infty(W)$, so that $1 \leqq ab$. To establish the converse inequality, we may assume that $0<b$. For each t with $b^{-1}<t<\infty$, we define $W \in L(Y)$ by $W=t$ on Y. Then $W \in L^+_\infty(Y; r)$ and $((W, f))_N = t H_1(f) \geqq bt \geqq 1$ for all $f \in \Lambda$. Therefore $a \leqq H_\infty(W) = t$, and hence $a \leqq b^{-1}$. This completes the proof.

COROLLARY 1. *The following relations hold*:

(2.14) $$EL_\infty(A, B; N) = \inf\{H_1(\phi_P; N); P \in \mathbf{P}_{A,B}(N)\},$$

(2.15) $$EW_\infty(A, B; N) = \inf\{H_1(\psi_Q; N); Q \in \mathbf{Q}_{A,B}(N)\}.$$

COROLLARY 2. *Let A be a nonempty finite subset of X. Then*

(2.16) $$EL_\infty(A, \infty; N) = \inf\{H_1(\phi_P; N); P \in \mathbf{P}_{A,\infty}(N)\},$$

(2.17) $$EW_\infty(A, \infty; N) = \inf\{H_1(\psi_Q; N); Q \in \mathbf{Q}_{A,\infty}(N)\}.$$

REMARK 2.3. (2.14) shows that the extremal distance of order ∞ is the value of the shortest path problem which is well-known in the finite network theory.

PROPOSITION 2.1. *Let $N'=\langle X', Y'\rangle$ be a finite subnetwork of N and let A' and B' be mutually disjoint nonempty subsets of X'. Then $EL_p(A', B'; N')=EW_q(A', B'; N')^{-1}$ for $p=1, \infty$.*

PROOF. We set $d_p=d_p(A', B'; N')$ and $d_q^*=d_q^*(A', B'; N')$. On account of Theorems 2.1 and 2.3, we have to show $d_p^{-1}=d_q^*$. It is easy to see that $d_p^{-1}\leqq d_q^*$ (cf. the proof of [2; Theorem 5.1]). We prove the converse inequality first for $p=1$. There exists $\bar{Q}\in \mathbf{Q}_{A',B'}(N')$ such that $H_1(\psi_{\bar{Q}}; N')=EW_\infty(A', B'; N')=(d_\infty^*)^{-1}$; let $\bar{u}=u_{\bar{Q}}$ be the characteristic function of $\bar{Q}$. Then $\bar{u}=0$ on A', $\bar{u}=1$ on B', so that, $d_1\leqq D_1(\bar{u}; N')=H_1(\psi_{\bar{Q}}; N')=(d_\infty^*)^{-1}$. Next we show the converse inequality for $p=\infty$. There exists $\bar{P}\in \mathbf{P}_{A',B'}(N')$ such that $H_1(\phi_{\bar{P}}; N')=EL_\infty(A', B'; N')=d_\infty^{-1}$; let $\bar{w}$ be the index of $\bar{P}$. Then $\bar{w}\in G(A', B'; N')$, $I(\bar{w})=1$ and $H_1(\bar{w}; N')=d_\infty^{-1}$, which yield $d_1^*\leqq d_\infty^{-1}$. This completes the proof.

With the aid of Theorems 2.2 and 2.4 and Proposition 2.1, we obtain

THEOREM 2.6. *$EL_p(A, B; N) = EW_q(A, B; N)^{-1}$ for $p = 1, \infty$.*

§3. A p-almost locally finite network

Next we shall be concerned with a generalized reciprocal relation between $EL_p(A, \infty; N)$ and $EW_q(A, \infty; N)$, which means the equality $EL_p(A, \infty; N)=EW_q(A, \infty; N)^{1-p}$ $(1<p<\infty)$ and the equality $EL_p(A, \infty; N)=EW_q(A, \infty; N)^{-1}$ $(p=1, \infty)$. The following examples show that the generalized reciprocal relation does not hold in general:

EXAMPLE 3.1. Let us take $X=\{x_n; n=0, 1, 2,...\}$, $Y=\{y_n, y'_n; n=1, 2,...\}$ and define $K(x, y)$ by

$$K(x_n, y_n) = K(x_0, y'_n) = 1,\ K(x_n, y'_n) = K(x_{n+1}, y_n) = -1$$

for $n=1, 2,...$ and

$$K(x, y) = 0 \quad \text{for other pairs} \quad (x, y).$$

Let $r=1$ on Y. Then $N=\{X, Y, K, r\}$ is an infinite network and $X_\infty=\{x_0\}$. Let $A=\{x_0\}$. Then we have $EL_p(A, \infty; N)=\infty$ $(1<p\leqq\infty)$, $EL_1(A, \infty; N)=1$, $EW_q(A, \infty; N)=\infty$ $(1<q\leqq\infty)$ and $EW_1(A, \infty; N)=0$.

EXAMPLE 3.2. Let X, Y, K and A be the same as above. We define $r\in L(Y)$ by $r(y_n)=1$ and $r(y'_n)=n^{-1}$ for $n=1, 2,...$. Then we have

$$EL_\infty(A, \infty; N) = EW_1(A, \infty; N) = \infty.$$

We say that N is *p-almost locally finite* if

(3.1) $$\sum_{y\in Y(x)} r(y)^{1-p}<\infty \qquad \text{for all} \quad x\in X\ (1\leqq p<\infty),$$

(3.1') $\{y\in Y(x);\ r(y)^{-1}\geqq\varepsilon\}$ is a finite set for all $x\in X$ and all $\varepsilon>0$ $(p=\infty)$.

Note that "1-almost locally finite" means locally finite. In this section we always assume that A is a nonempty finite subset of X and $A\neq X$.

For each subnetwork $N'=<X', Y'>$ of N, we put

$$C(N') = \bigcup_{x\in X'} Y(x) - Y',$$

$$\alpha_p(N') = \sum_{y\in C(N')} r(y)^{1-p} \qquad \text{if} \quad 1\leqq p<\infty,$$

$$\alpha_\infty(N') = \sup_{y\in C(N')} r(y)^{-1}.$$

First we prove

Lemma 3.1. *Let $1<p\leqq\infty$ and assume that N is p-almost locally finite but not locally finite. Then for each $\varepsilon>0$, there exists a locally finite subnetwork $N'=<X', Y'>$ of N such that $A\subset X'$ and $\alpha_p(N')<\varepsilon$.*

Proof. Since A is a finite set, there exists a finite subnetwork $<X'_0, Y'_0>$ of N such that $A\subset X'_0$. Choose $\bar{Y}\subset Y-Y'_0$ so that $Y(x)-\bar{Y}$ is a finite set for each $x\in X_\infty$ and $\sum_{y\in\bar{Y}} r(y)^{1-p}<\varepsilon$ if $1<p<\infty$ and $\sup_{y\in\bar{Y}} r(y)^{-1}<\varepsilon$ if $p=\infty$. We call a path admissible if it starts from a node in X'_0 and its arcs are all in $Y-\bar{Y}$, and denote by Γ the set of all admissible paths. Denote by $C_X(P)$ the terminal nodes of $P\in\Gamma$, and by $C_Y(P)$ the set of arcs of $P\in\Gamma$. We define $X'=\bigcup_{P\in\Gamma} C_X(P)$ and $Y'=\bigcup_{P\in\Gamma} C_Y(P)$. It is easy to see that $N'=<X', Y'>$ is a locally finite subnetwork of N. Evidently $X'_0\subset X'$ and $Y'\subset Y-\bar{Y}$. From the definition of N', it follows that $C(N')\subset\bar{Y}$. Hence $\alpha_p(N')<\varepsilon$.

For a subnetwork $N'=<X', Y'>$ of N such that $A\subset X'$, we always have

(3.2) $$EL_p(A, \infty; N)\leqq EL_p(A, \infty; N'),$$

(3.3) $$EW_q(A, \infty; N')\leqq EW_q(A, \infty; N).$$

Remark 3.1. If $\mathbf{P}_{A,\infty}(N)=\emptyset$, then $W=0\in E_p(\mathbf{P}_{A,\infty}(N))$, so that $EL_p(A, \infty; N)=\infty$.

We shall prove

Lemma 3.2. *Let $\varepsilon>0$ and $N'=<X', Y'>$ be a subnetwork of N such that*

$A \subset X'$ and $\alpha_p(N') < \varepsilon$. Then

$$EL_p(A, \infty; N)^{-1} \leqq EL_p(A, \infty; N')^{-1} + \varepsilon. \tag{3.4}$$

PROOF. In order to prove (3.4), we may assume that $EL_p(A, \infty; N')^{-1} < \infty$, i.e., $E_p(\mathbf{P}_{A,\infty}(N')) \neq \emptyset$. For $W' \in E_p(\mathbf{P}_{A,\infty}(N'))$, we define $W \in L(Y)$ by $W = W'$ on Y', $W = r^{-1}$ on $C(N')$ and $W = 0$ on $Y - Y' - C(N')$; then $W \in E_p(\mathbf{P}_{A,\infty}(N))$ and

$$EL_p(A, \infty; N)^{-1} \leqq H_p(W; N) \leqq H_p(W'; N') + \alpha_p(N')$$
$$< H_p(W'; N') + \varepsilon,$$

which leads to (3.4).

LEMMA 3.3. *For each $\varepsilon > 0$, there is $\eta > 0$ such that if $N' = <X', Y'>$ is a subnetwork of N satisfying $A \subset X'$ and $\alpha_p(N') < \eta$, then*

$$EW_q(A, \infty; N')^{-1} - \varepsilon \leqq EW_q(A, \infty; N)^{-1}. \tag{3.5}$$

PROOF. We may assume that $a = EW_q(A, \infty; N)^{-1} < \infty$. Let $1 < q < \infty$ and let $\eta > 0$ with $\eta^{1/p} a^{1/q} < 1$. To prove our assertion it suffices to show that $\alpha_p(N') < \eta$ implies

$$EW_q(A, \infty; N')^{-1} \leqq a[1 - \eta^{1/p} a^{1/q}]^{-q}. \tag{3.5'}$$

For each $0 < t < 1$ such that $\eta^{1/p}(a+t)^{1/q} < 1$, there exists $W \in E_q^*(\mathbf{Q}_{A,\infty}(N))$ such that $H_q(W; N) < a + t$. For each $Q' \in \mathbf{Q}_{A,\infty}(N')$, let $Q'(A)$ and $Q'(\infty)$ be the subsets of X' which determine Q' and define $Q(A)$ and $Q(\infty)$ by $Q(A) = Q'(A)$ and $Q(\infty) = Q'(\infty) \cup (X - X')$. Then $Q = (Q(A), Q(\infty))_N \in \mathbf{Q}_{A,\infty}(N)$ and $Q \subset Q' \cup C(N')$. Hence we have

$$1 \leqq ((\psi_Q, W))_N \leqq ((\psi_{Q'}, W))_{N'} + \sum_{y \in C(N')} W(y)$$
$$\leqq ((\psi_{Q'}, W))_{N'} + [\sum_{y \in C(N')} r(y)^{1-p}]^{1/p} [H_q(W; N)]^{1/q}$$
$$\leqq ((\psi_{Q'}, W))_{N'} + \eta^{1/p}(a + t)^{1/q}$$

by Hölder's inequality. Writing $b = 1 - \eta^{1/p}(a+t)^{1/q} > 0$, we see that $W/b \in E_q^*(\mathbf{Q}_{A,\infty}(N'))$. It follows that

$$EW_q(A, \infty; N')^{-1} \leqq H_q(W/b; N') \leqq H_q(W/b; N)$$
$$< (a + t)[1 - \eta^{1/p}(a + t)^{1/q}]^{-q}.$$

By letting $t \to 0$, we obtain (3.5'). In case $q = 1, \infty$, we can similarly prove $EW_q(A, \infty; N')^{-1} \leqq a[1 - \eta a]^{-1}$.

By Lemmas 3.1, 3.2 and 3.3, we have

THEOREM 3.1. *Assume that N is p-almost locally finite. Then, for each $\varepsilon>0$, there exists a locally finite subnetwork $N'=<X', Y'>$ of N such that $A\subset X'$ and* (3.4) *and* (3.5) *hold.*

Similarly, we obtain

LEMMA 3.4. *Assume that N is p-almost locally finite. Let Ω be a collection of subsets of $X-A$ such that* $\inf\{EW_q(A, B; N); B\in\Omega\}>0$. *Then, for each $\varepsilon>0$, there is a locally finite subnetwork $N'=<X', Y'>$ of N such that $A\subset X'$, $\alpha_p(N')<\varepsilon$ and*

$$EW_q(A, B\cap X'; N')^{-1}-\varepsilon\leqq EW_q(A, B; N)^{-1} \tag{3.6}$$

for all $B\in\Omega$.

Next we shall study the stability of $\{EL_p(A, X-X_n; N)\}$ and $\{EW_q(A, X-X_n; N)\}$ with respect to an exhaustion $\{<X_n, Y_n>\}$ of N such that $A\subset X_1$, which mean $\lim_{n\to\infty} EL_p(A, X-X_n; N)=EL_p(A, \infty; N)$ and $\lim_{n\to\infty} EW_q(A, X-X_n; N)=EW_q(A, \infty; N)$. These stability were affirmatively solved in the case that N is locally finite and $1<p<\infty$ (cf. Theorems 2.2 and 4.2 in [2]). Since the proofs of Theorems 2.2 and 4.2 in [2] are still effective for $p=1, \infty$ in case N is locally finite, we obtain

PROPOSITION 3.1. *Let $1\leqq p\leqq\infty$ and let N be locally finite. Then $EL_p(A, X-X_n; N)\to EL_p(A, \infty; N)$ as $n\to\infty$ and $EW_q(A, X-X_n; N)\to EW_q(A, \infty; N)$ as $n\to\infty$.*

REMARK 3.2. In case $X-X_n=\emptyset$ for some n, we see that $\mathbf{P}_{A,X-X_n}(N)=\emptyset$, and hence $EL_p(A, X-X_n; N)=\infty$.

We have

LEMMA 3.5. *Let $\{\varepsilon_k\}$ be a sequence of positive numbers such that $\varepsilon_k\to 0$ as $k\to\infty$. Assume that for each ε_k there exists a finite subnetwork $N_{\varepsilon_k}=<X_{\varepsilon_k}, Y_{\varepsilon_k}>$ of N such that $A\subset X_{\varepsilon_k}$ and $\alpha_p(N_{\varepsilon_k})<\varepsilon_k$. Then for every exhaustion $\{<X_n, Y_n>\}$ of N, $EL_p(A, X-X_n; N)\to\infty$ as $n\to\infty$.*

PROOF. For a fixed k, there exists n_0 such that $X_{\varepsilon_k}\subset X_n$ and $Y_{\varepsilon_k}\subset Y_n$ for all $n\geqq n_0$. Notice that

$$EL_p(A, X-X_n; N)^{-1}\leqq EL_p(A, X-X_{\varepsilon_k}; N)^{-1}=d_p(A, X-X_{\varepsilon_k}; N)$$

for all $n\geqq n_0$ by Theorem 2.1. Let us define $u\in L(X)$ by $u=0$ on X_{ε_k} and $u=1$

on $X-X_{\varepsilon_k}$. Then

$$d_p(A,\ X - X_{\varepsilon_k};\ N) \leqq D_p(u;\ N) \leqq \alpha_p(N_{\varepsilon_k}) < \varepsilon_k,$$

so that $EL_p(A, X-X_n; N) > \varepsilon_k^{-1}$ for all $n \geqq n_0$. Thus $EL_p(A, X-X_n; N) \to \infty$ as $n \to \infty$.

THEOREM 3.2. *If N is p-almost locally finite, then $EL_p(A, X-X_n; N) \to EL_p(A, \infty; N)$ as $n \to \infty$.*

PROOF. By Proposition 3.1 we may suppose $p>1$. It is easily seen that

$$EL_p(A,\ X - X_n;\ N) \leqq EL_p(A,\ X - X_{n+1};\ N) \leqq EL_p(A,\ \infty;\ N),$$

so that $\lim_{n\to\infty} EL_p(A, X-X_n; N) \leqq EL_p(A, \infty; N)$. To establish the converse inequality, we may assume that $X_n \neq X$ for all n; otherwise there exists n_0 with $X_{n_0} = X$, so that $EL_p(A, X-X_n; N) = \infty$ for all $n \geqq n_0$, and hence the converse inequality follows. For each $\varepsilon>0$, there exists a locally finite subnetwork $N_\varepsilon = <X_\varepsilon, Y_\varepsilon>$ of N such that $A \subset X_\varepsilon$ and $\alpha_p(N_\varepsilon) < \varepsilon$, by Lemma 3.1. By Lemma 3.5 we may assume that there exists $\varepsilon_0 > 0$ such that N_ε is an infinite subnetwork for each ε with $0<\varepsilon<\varepsilon_0$. Let $0<\varepsilon<\varepsilon_0$ and let $\{<X'_n, Y'_n>\}$ be an exhaustion of N_ε such that $A \subset X'_1$. Then we have

$$\lim_{n\to\infty} EL_p(A,\ X_\varepsilon - X'_n;\ N_\varepsilon) = EL_p(A,\ \infty;\ N_\varepsilon)$$

by Proposition 3.1. We show that

$$EL_p(A,\ X - X'_n;\ N)^{-1} \leqq EL_p(A,\ X_\varepsilon - X'_n;\ N_\varepsilon)^{-1} + \varepsilon. \tag{3.7}$$

For $W' \in E_p(\mathbf{P}_{A, X_\varepsilon - X'_n}(N_\varepsilon))$, we define $W \in L(Y)$ by $W = W'$ on Y_ε, $W = r^{-1}$ on $C(N_\varepsilon)$ and $W=0$ on $Y - Y_\varepsilon - C(N_\varepsilon)$. Then $W \in E_p(\mathbf{P}_{A, X - X'_n}(N))$ and

$$EL_p(A,\ X - X'_n;\ N)^{-1} \leqq H_p(W;\ N) < H_p(W';\ N_\varepsilon) + \varepsilon,$$

which leads to (3.7). For each n, there exists $m(n)$ such that $X'_n \subset X_m$ and $Y'_n \subset Y_m$ for all $m \geqq m(n)$. Notice that $EL_p(A, X-X_m; N) \geqq EL_p(A, X-X'_n; N)$ for all $m \geqq m(n)$. It follows that

$$\begin{aligned}
\lim_{m\to\infty} EL_p(A,\ X - X_m;\ N)^{-1} &\leqq \lim_{n\to\infty} EL_p(A,\ X - X'_n;\ N)^{-1} \\
&\leqq \lim_{n\to\infty} EL_p(A,\ X_\varepsilon - X'_n;\ N_\varepsilon)^{-1} + \varepsilon \\
&= EL_p(A,\ \infty;\ N_\varepsilon)^{-1} + \varepsilon \\
&\leqq EL_p(A,\ \infty;\ N)^{-1} + \varepsilon.
\end{aligned}$$

Thus we have $\lim_{n\to\infty} EL_p(A, X-X_n; N) \geqq EL_p(A, \infty; N)$. This completes the proof.

THEOREM 3.3. *If N is p-almost locally finite, then $EW_q(A, X-X_n; N) \to EW_q(A, \infty; N)$ as $n\to\infty$.*

PROOF. By Proposition 3.1 we may assume that N is not locally finite, so that $1\leqq q<\infty$. We may assume that $X_n \neq X$ for all n; otherwise, there exists n_0 with $X_{n_0}=X$, so that $EW_q(A, X-X_n; N)=EW_q(A, \infty; N)$ for all $n\geqq n_0$. We easily obtain

$$EW_q(A, X-X_n; N) \geqq EW_q(A, X-X_{n+1}; N) \geqq EW_q(A, \infty; N),$$

so that $\delta=\lim_{n\to\infty} EW_q(A, X-X_n; N) \geqq EW_q(A, \infty; N)$. To establish the converse inequality, we assume that $\delta>0$. Let $\Omega=\{B;\, B\subset X-A \text{ and } EW_q(A, B; N)\geqq\delta\}$. Then for each $\varepsilon>0$, there exists a locally finite subnetwork $N_\varepsilon=<X_\varepsilon, Y_\varepsilon>$ of N such that $A\subset X_\varepsilon$, $\alpha_p(N_\varepsilon)<\varepsilon$ and

$$EW_q(A, B\cap X_\varepsilon; N_\varepsilon)^{-1}-\varepsilon \leqq EW_q(A, B; N)^{-1}$$

for all $B\in\Omega$ by Lemma 3.4. We show that there exists $\varepsilon_0>0$ such that N_ε is an infinite network for all ε, $0<\varepsilon<\varepsilon_0$. Suppose the contrary. By Theorems 2.5 and 2.6 and Lemma 3.5, we have $EW_q(A, X-X_n; N)\to 0$ as $n\to\infty$, which contradicts $0<\delta$. Let $0<\varepsilon<\varepsilon_0$ and let $\{<X'_n, Y'_n>\}$ be an exhaustion of N_ε such that $A\subset X'_1$. Then we have

$$\lim_{n\to\infty} EW_q(A, X_\varepsilon-X'_n; N_\varepsilon) = EW_q(A, \infty; N_\varepsilon)$$

by Proposition 3.1. For each n, there exists $m(n)$ such that $X'_n\subset X_m$ and $Y'_n\subset Y_m$ for all $m\geqq m(n)$. Noting that $EW_q(A, X-X_m; N)\leqq EW_q(A, X-X'_n; N)$ and $X-X'_n\in\Omega$, $n=1, 2,\dots$, we obtain

$$\begin{aligned}\lim_{m\to\infty} EW_q(A, X-X_m; N)^{-1} &\geqq \lim_{n\to\infty} EW_q(A, X-X'_n; N)^{-1}\\ &\geqq \lim_{n\to\infty} EW_q(A, X_\varepsilon-X'_n; N_\varepsilon)^{-1}-\varepsilon\\ &= EW_q(A, \infty; N_\varepsilon)^{-1}-\varepsilon\\ &\geqq EW_q(A, \infty; N)^{-1}-\varepsilon.\end{aligned}$$

Thus we have $\lim_{n\to\infty} EW_q(A, X-X_n; N)\leqq EW_q(A, \infty; N)$. This completes the proof.

REMARK 3.3. If N is not p-almost locally finite, the above stability for the extremal distance and extremal width do not hold in general. These are verified by the network N and the set A in Example 3.1.

Now we prove our main result.

THEOREM 3.4. *Let* $1 \leqq p \leqq \infty$ *and assume that* N *is* p*-almost locally finite and* X *is an infinite set. Then* $EL_p(A, \infty; N) = EW_q(A, \infty; N)^{1-p}$ *for* p, $1 < p < \infty$ *and* $EL_p(A, \infty; N) = EW_q(A, \infty; N)^{-1}$ *for* $p = 1, \infty$.

PROOF. Let $1 < p < \infty$ and let $\{<X_n, Y_n>\}$ be an exhaustion of N such that $A \subset X_1$. Since X is an infinite set, $X - X_n \neq \emptyset$ for all n, so that

$$EL_p(A, X - X_n; N) = EW_q(A, X - X_n; N)^{1-p}$$

for all n by Theorem 2.5. By letting $n \to \infty$, we obtain the desired equality by Theorems 3.2 and 3.3. In case $p = 1, \infty$, our assertion follows from Theorems 2.6, 3.2 and 3.3.

REMARK 3.4. The condition that X is an infinite set is essential in the above theorem (except the case $p = 1$) because of the following example:

Let us take $X = \{x_0, x_1\}$, $Y = \{y_1, y_2, \ldots\}$ and define $K(x, y)$ by

$$K(x_0, y_n) = 1 \quad \text{and} \quad K(x_1, y_n) = -1$$

for all n. Let $r(y_j) = 2^j$ for $y_j \in Y$. Then $N = \{X, Y, K, r\}$ is an infinite network that is p-almost locally finite ($1 < p \leqq \infty$). Let $A = \{x_0\}$. Then we have $EL_p(A, \infty; N) = \infty$ and $EW_q(A, \infty; N) \geqq 1/2$.

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