HIROSHIMA MATH. J.
7 (1977), 743–768

On Weierstrass Points of Non-hyperelliptic Compact Riemann Surfaces of Genus Three

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(Received May 16, 1977)

The purpose of this paper is, first, to determine the equations of compact Riemann surfaces of genus three, considering these surfaces as coverings of the Riemann sphere. We obtain Theorem 1 which asserts that the equations are given by

$$y^3 - \gamma_2(x)y - \gamma_3(x) = 0.$$

Here $\gamma_2(x)$ is a polynomial of degree 3 or less than 3, and $\gamma_3(x)$ is a polynomial of degree 5 or 4. Both of them depend on Weierstrass points.

Next, we construct a basis of differentials of the first kind for these Riemann surfaces. Using these results, we investigate Weierstrass points of these Riemann surfaces. Our main interest is to determine Riemann surfaces which have exactly 12 Weierstrass points. The number 12 is the smallest one for all the non-hyperelliptic compact Riemann surfaces of genus three. We obtain Theorem 2 which asserts that Riemann surfaces having just 12 Weierstrass points are exactly two and these equations in homogeneous coordinates are given by

$$x^4 + y^4 + z^4 = 0 \tag{1}$$

and

$$x^4 + y^4 + z^4 + 3(x^2y^2 + y^2z^2 + z^2x^2) = 0. \tag{2}$$

§1. Preliminaries

Given any point P on a compact Riemann surface of genus g (≥ 1), there are exactly g orders which can be specified

$$1 = n_1 < n_2 < \cdots < n_3 < 2g$$

such that there does not exist any meromorphic function on the surface whose only singularity is a pole of order n_i ($1 \leq i \leq g$) at P.

These g orders are called the gaps at P. A point whose gap sequence contains an integer greater than g is called a Weierstrass point.

LEMMA 1 (Hurwitz [3]). *For surfaces of genus $g=0$ or $g=1$ there are no Weierstrass points. If $g\geq 2$ there always exist Weierstrass points. The number N of Weierstrass points satisfies the inequality*:

$$2g+2 \leq N \leq (g-1)g(g+1).$$

If N is equal to $2g+2$, then the surfaces are hyperelliptic and vice versa.

LEMMA 2 (Schmidt [8]). *Let $\{u_i\,(1\leq i\leq g)\}$ be a basis of integrals of the first kind on a compact Riemann surface* R *of genus g. Let u be an integral of the first kind on* R. *Put*

$$\varDelta_u = \begin{vmatrix} \dfrac{du_1}{du} & \cdots & \dfrac{du_g}{du} \\ \vdots & & \vdots \\ \dfrac{d^g u_1}{du^g} & \cdots & \dfrac{d^g u_g}{du^g} \end{vmatrix}.$$

Then $\varDelta_u(du)^{g(g+1)/2}$ is a differential form of degree $g(g+1)/2$, i.e., if t is another integral, we have

$$\varDelta_u(du)^{g(g+1)/2} = \varDelta_t(dt)^{g(g+1/)2}.$$

We call this form the Wronskian of R. *Let the divisor of the Wronskian of* R *be*

$$\operatorname{div}\{\varDelta_u(du)^{g(g+1)/2}\} = m_1\mathrm{P}_1 + \cdots + m_r\mathrm{P}_r.$$

Then $\mathrm{P}_1,\ldots,\mathrm{P}_r$ *are all Weierstrass points on* R *and we have*

$$m_1 + \cdots + m_r = (g-1)g(g+1).$$

LEMMA 3. *Let* R *be a non-hyperelliptic Riemann surface of genus* 3. *Let* P *be an arbitrary Weierstrass point on* R. *For the gap sequence of* P *there are following two cases*:

(1) $$n_1 = 1,\ n_2 = 2,\ n_3 = 4,$$

(2) $$n_1 = 1,\ n_2 = 2,\ n_3 = 5$$

and the multiplicity m of P *is one for the case* (1) *and two for the case* (2).

PROOF. Since the multiplicity is given by the formula ([3], p. 408)

$$m = n_1 + n_2 + n_3 - g(g+1)/2 = n_1 + n_2 + n_3 - 6,$$

we have $m=1$ in (1) and $m=2$ in (2).

§ 2. Equations of Riemann surfaces

Let R be a compact Riemann surface of genus three. We assume that R is non-hyperelliptic. We shall give a canonical form of equation of R. By lemmas in § 1, we know that there exist Weierstrass points on R. Let P be one of them. Then there exists a meromorphic function on R which has P as an only singularity of a pole of order three. We denote the function by x. The function x is considered as a mapping of R onto the Riemann x-sphere. We see that R is conformally equivalent to a three-sheeted covering surface over the Riemann x-sphere, and we assume that the point P of R is over the point ∞ at infinity. We denote the point by P_∞.

Let K be the field of meromorphic functions on R. Then K is an algebraic extension of degree 3 over the rational function field $\mathbf{C}(x)$ such that $[K: \mathbf{C}(x)] = \deg(3P_\infty)\ (=3)$. Let y be another function of K which has P_∞ as an only singularity of a pole of order 4 or 5. Then we see that $K = \mathbf{C}(x, y)$ and we have an irreducible equation in x and y:

$$p_0(x)y^3 + p_1(x)y^2 + p_2(x)y + p_3(x) = 0.$$

Here $p_0(x)$, $p_1(x)$, $p_2(x)$ and $p_3(x)$ are polynomials in x. We can assume that $p_0(x)$ is a non-zero constant, since y is integral over $\mathbf{C}[x]$. Hence we assume that $p_0(x)$ is equal to one and moreover we may assume that $p_1(x)$ is equal to zero. We rewrite the above equation

$$y^3 - \gamma_2(x)y - \gamma_3(x) = 0$$

with polynomials γ_2 and γ_3 in x. Now, let t be a local parameter at P_∞. Then we can express the functions x and y in

$$x = 1/t^3 + \cdots,$$

$$y = 1/t^4 \quad \text{or} \quad y = 1/t^5.$$

If $y = 1/t^4$, then we have $y^3 = t^{-12}$ and

$$\gamma_2(x)y = t^{-3n_2-4} + \cdots, \quad \gamma_3(x) = t^{-3n_3} + \cdots.$$

Here $n_2 = \deg \gamma_2(x)$ and $n_3 = \deg \gamma_3(x)$. We obtain that $n_2 \leq 2$ and $n_3 = 4$. If $y = 1/t^5$, then we have $y^3 = t^{-15}$ and

$$\gamma_2(x)y = t^{-3n_2-5} + \cdots, \quad \gamma_3(x) = t^{-3n_3} + \cdots.$$

We obtain that $n_2 \leq 3$ and $n_3 = 5$. Thus we have the following theorem:

THEOREM 1. *If the gap sequence at* P_∞ *is* $\{1, 2, 5\}$, *then the equation of* R *is given by*

$$(*) \qquad y^3 - \gamma_2(x)y - \gamma_3(x) = 0,$$

where

$$(\#) \qquad \gamma_2(x) = a_0x^2 + a_1x + a_2, \quad \gamma_3(x) = x^4 + b_1x^3 + b_2x^2 + b_3x + b_4.$$

If the gap sequence at P_∞ *is* $\{1, 2, 4\}$, *then the equation of* R *is given by* $(*)$ *with*

$$(\#\#) \qquad \gamma_2(x) = a_0x^3 + \cdots + a_3, \quad \gamma_3(x) = x^5 + b_1x^4 + \cdots + b_5.$$

Of course, in both cases, the coefficients a's and b's must satisfy certain relations which come from the fact that R *must be of genus* 3.

Now, by the formula of Riemann-Hurwitz:

$$2g - 2 = n(2g' - 2) + V$$

we have $V=10$. Here V means the total sum of all ramification indices. The Newton polygon shows that the ramification index of P_∞ over $x=\infty$ is two. Hence we have the following five cases:

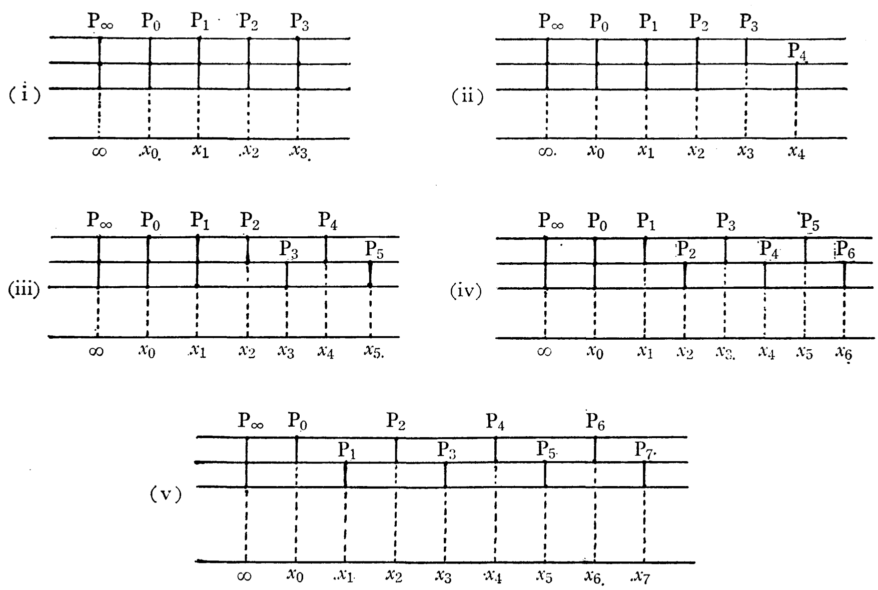

Fig. 1.

It is convenient to normalize the positions of ramification points as follows:

$$\infty,\ x_0 = 0,\ x_1 = 1,\ x_2 = t_1, \ldots, x_7 = t_6.$$

It is worthwhile to express above coefficients of $\gamma_2(x)$ and $\gamma_3(x)$ in terms of $t_1, \ldots, t_5$ and t_6, since the number of parameters t is equal to the dimension of the space of moduli of compact Riemann surfaces of genus three.

First, we shall investigate in case (i). If the gap sequence at P_∞ is $\{1, 2, 5\}$, then we have

$$y^3 - \gamma_2(x)y - \gamma_3(x) = 0$$

with (♯) and the discriminant of the equation is given by

$$\Delta(x) = 4\gamma_2^3(x) - 27\gamma_3^2(x).$$

Then, $x=0$, 1, t_1 and t_2 are zeros of $\Delta(x)$, and y has triple roots at these points. Therefore, we have

$$\begin{aligned} b_4 &= 0, & a_2 &= 0, \\ 1 + b_1 + \cdots + b_4 &= 0, & a_0 + a_1 + a_2 &= 0, \\ t_1^4 + b_1 t_1^3 + \cdots + b_4 &= 0, & a_0 t_1^2 + a_1 t_1 + a_2 &= 0, \\ t_2^4 + b_1 t_2^3 + \cdots + b_4 &= 0, & a_0 t_2^2 + a_1 t_2 + a_2 &= 0. \end{aligned}$$

Hence we have $a_0 = a_1 = a_2 = 0$ and the equation comes to be

$$(*) \qquad y^3 = x(x-1)(x-t_1)(x-t_2).$$

If the gap sequence at P_∞ is $\{1, 2, 4\}$, then we have

$$(**) \qquad y^3 - \gamma_2(x)y - \gamma_3(x) = 0$$

with (♯♯). By the same reasoning as above, we have

$$y^3 = x(x-1)(x-t_1)(x-t_2)(x-c)$$

with a constant c. Here c must be equal to one of these values 0, 1, t_1 and t_2, since otherwise this equation does not represent a Riemann surface of genus three.

Secondly, we shall investigate in case (v). Let the gap sequence at P_∞ is $\{1, 2, 5\}$. Then we have

$$\Delta(x) = -27x(x-1)(x-t_1)\cdots(x-t_6)$$

for the discriminant and we obtain the following relations:

(1) $$4a_2^3 - 27b_4^2 = 0,$$

(2) $$4(a_0 + a_1 + a_2)^3 - 27(1 + b_1 + \cdots + b_4)^2 = 0,$$

(3) $$4(a_0 t_1^2 + a_1 t_1 + a_2)^3 - 27(t_1^4 + b_1 t_1^3 + \cdots + b_4)^2 = 0,$$

$\vdots$

(8) $$4(a_0 t_6^2 + a_1 t_6 + a_2)^3 - 27(t_6^4 + b_1 t_6^3 + \cdots + b_4{}^2) = 0.$$

We consider these relations in more detail. Let $x=\tau$ be a point of $\{0, 1, t_1,\dots, t_6\}$. We must have a point of R whose ramification index is just one over the point τ over the Riemann sphere. Let the double root of y at τ be β and put $X=x-\tau$, $Y=y-\beta$. Then we have

$$\begin{aligned} &Y^3 + 3\beta Y^2 - a_0 X^2 Y - (2a_0\tau + a_1)XY \\ &- \{X^4 + (4\tau + b_1)X^3 + (6\tau^2 + 3b_1\tau + b_2 + a_0\beta)X^2 \\ &+ (4\tau^3 + 3b_1\tau^2 + 2b_2\tau + b_3 + (2a_0\tau + a_1)\beta)X\} = 0. \end{aligned}$$

If $\beta=0$, then the Newton polygon shows that if

$$4\tau^3 + 3b_1\tau^2 + 2b_2\tau + b_3 \neq 0,$$

our case reduces to (iv) and if

$$4\tau^3 + 3b_1\tau^2 + 2b_2\tau + b_3 = 0,$$

our case reduces to (iv) with some restrictions. If $\beta\neq 0$, then the Newton polygon shows that if

$$4\tau^3 + 3b_1\tau^2 + 2b_2\tau + b_3 + (2a_0\tau + a_1)\beta = 0,$$

we must have

$$2a_0\tau + a_1 = 0$$

and

$$6\tau^2 + 3b_1\tau + b_2 + a_0\beta = 0.$$

Hence we have again $4\tau^3+3b_1\tau^2+2b_2\tau+b_3=0$, and so our case reduces to (iv). Therefore, in our case we must have generally

$$\beta \neq 0 \quad \text{and} \quad 4\tau^3 + 3b_1\tau^2 + 2b_2\tau + b_3 + (2a_0\tau + a_1)\beta \neq 0.$$

Let the gap sequence at P_∞ is $\{1, 2, 4\}$. Then we have

$$\Delta(x) = -27x(x-1)(x-t_1)(x-t_2)\cdots(x-t_6)(x-\alpha_1)(x-\alpha_2)$$

for the discriminant. Here α_1, α_2 are suitable complex numbers. We have, first, the following relations:

(1) $$4a_3^3 - 27b_5^2 = 0,$$

(2) $$4(a_0 + a_1 + a_2 + a_3)^3 - 27(1 + b_1 + b_2 + b_3 + b_4 + b_5)^2 = 0,$$

(3) $$4(a_0 t_1^3 + \cdots + a_3)^3 - 27(t_1^5 + b_1 t_1^4 + \cdots + b_5)^2 = 0,$$

$\vdots$

(8) $$4(a_0 t_6^3 + \cdots + a_3)^3 - 27(t_6^5 + b_1 t_6^4 + \cdots + b_5)^2 = 0.$$

We consider these relations in more detail. Over the point τ on the Riemann sphere we must have a point of R whose ramification index is just one. Let the double root of y at τ be β and put $X = x - \tau$, $Y = y - \beta$. Then we have

$$\begin{aligned} Y^3\,3\beta Y^2 &- a_0 X^3 Y - (3a_0\tau + a_1)X^2 Y - (3a_0\tau^2 + 2a_1\tau + a_2)XY \\ &- \{X^5 + (5\tau + b_1)X^4 + (10\tau^2 + 4b_1\tau + b_2 + a_0\beta)X^3 \\ &\quad + (10\tau^3 + 6b_1\tau^2 + 3b_2\tau + b_3 + (3a_0\tau + a_1)\beta)X^2 \\ &\quad + (5\tau^4 + 4b_1\tau^3 + 3b_2\tau^2 + 2b_3\tau + b_4 + (3a_0\tau^2 + 2a_1\tau + a_2)\beta)X\} \\ &= 0. \end{aligned}$$

We claim that α_1 is equal to α_2. In fact, if $\alpha_1 \neq \alpha_2$ then by the Newton polygon we see that there are ramification points over α_1 and α_2. This is a contradiction. Therefore, we have

(9) $\Delta(\alpha) = 0$, i.e.,

$$4(a_0\alpha^3 + a_1\alpha^2 + a_2\alpha + a_3)^3 - 27(\alpha^5 + b_1\alpha^4 + \cdots + b_5)^2 = 0,$$

(10) $\Delta'(\alpha) = 0$, i.e.,

$$\begin{aligned} &12(a_0\alpha^3 + a_1\alpha^2 + a_2\alpha + a_3)^2(3a_0\alpha^2 + 2a_1\alpha + a_2) \\ &\quad - 54(\alpha^5 + b_1\alpha^4 + \cdots + b_5)(5\alpha^4 + 4b_1\alpha^3 + \cdots + b_4) = 0. \end{aligned}$$

We must remark here $\gamma_2(\alpha) \neq 0$ (consequently $\gamma_3(\alpha) \neq 0$).

If $\beta = 0$, then the Newton polygon shows that if

$$5\tau^4 + 4b_1\tau^3 + 3b_2\tau^2 + 2b_3\tau + b_4 \neq 0,$$

our case reduces to (iv) and if

$$5\tau^4 + 4b_1\tau^3 + 3b_2\tau^2 + 2b_3\tau + b_4 = 0,$$

our case reduces to (iv) with some restrictions. If $\beta \neq 0$, then the Newton polygon shows that if

$$5\tau^4 + 4b_1\tau^3 + 3b_2\tau^2 + 2b_3\tau + b_4 + (3a_0\tau^2 + 2a_1\tau + a_2)\beta = 0,$$

we must have

$$3a_0\tau^2 + 2a_1\tau + a_2 = 0$$

and

$$10\tau^3 + 4b_1\tau^2 + 3b_2\tau + b_3 + (3a_0\tau + a_1)\beta = 0.$$

Hence again we have $5\tau^4+4b_1\tau^3+3b_2\tau^2+2b_3\tau+b_4=0$, and so our case reduces to (iv). Therefore we must have generally

$$\beta \neq 0 \quad \text{and} \quad 5\tau^4 + 4b_1\tau^3 + \cdots + b_4 + (3a_0\tau^2 + 2a_1\tau + a_2)\beta \neq 0$$

in our case.

To express a's and b's by $t_1,\dots, t_6$ we must investigate the Jacobian of (1),..., (10). Namely, put

$$\begin{aligned} &F_i(a_0, a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_5, \alpha, \tau_i) \\ &\quad = 4(a_0\tau_i^3 + a_1\tau_i^2 + a_2\tau_i + a_3)^3 - 27(\tau_i^5 + b_1\tau_i^4 + \cdots + b_5)^2 \end{aligned}$$

for $\tau_1=0$, $\tau_2=1$, $\tau_3=t_1,\dots, \tau_8=t_6$ $(1\leq i\leq 8)$, and put

$$F_9(a_0, a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_5, \alpha) = \Delta(\alpha),$$

$$F_{10}(a_0, a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_5, \alpha) = \Delta'(\alpha).$$

Then we see that the Jacobian

$$\frac{D(F_1,\dots, F_4, F_5,\dots, F_9, F_{10})}{D(a_0,\dots, a_3, b_1,\dots, b_5, \alpha)}$$

cannot be identically zero. Thus we can express a's and b's by $t_1,\dots, t_6$ at those points which are not zeros of the Jacobian. To investigate a's and b's or $t_1,\dots, t_6$ on the Teichmüller space is interesting. However we will not discuss this problem in this paper.

§3. Weierstrass points

We take Riemann surfaces defined by

$$y^3 - \gamma_2(x)y - \gamma_3(x) = 0$$

with (#) in §2. The discriminant $\Delta(x)$ is given by

$$\Delta(x) = 4\gamma_2^3(x) - 27\gamma_3^2(x)$$

$$= -27x(x-1)(x-t_1)\cdots(x-t_6)$$

and the zeros of $\Delta(x)$ are common zeros of

$$f(x, y) = y^3 - \gamma_2(x)y - \gamma_3(x),$$

$$f_y(x, y) = 3y^2 - \gamma_2(x).$$

Then we have

$$\operatorname{div}(f_y) = P_0 + P_1 + P_{t_1} + \cdots + P_{t_6} - 8P_\infty,$$

$$\operatorname{div}(dx) = P_0 + P_1 + P_{t_1} + \cdots + P_{t_6} - 4P_\infty,$$

$$\operatorname{div}(x) = 2P_0 + P_0' - 3P_\infty,$$

$$\operatorname{div}(y) = P_{s_1} + P_{s_2} + P_{s_3} + P_{s_4} - 4P_\infty.$$

Here, P_0' is the regular point over $x=0$ and s_1, s_2, s_3 and s_4 are zeros of the polynomial $\gamma_3(x)$.

Put

$$\omega_1 = dx/f_y, \quad \omega_2 = xdx/f_y, \quad \omega_3 = ydx/f_y.$$

Then ω_1, ω_2 and ω_3 are differentials of the first kind and are linearly independent over $\mathbf{C}$, i.e., a basis of differentials.

If we take

$$y^3 - \gamma_2(x)y - \gamma_3(x) = 0$$

with (##) in §2. The discriminant $\Delta(x)$ is given by

$$\Delta(x) = 4\gamma_2^3(x) - 27\gamma_3^2(x)$$

$$= -27x(x-1)(x-t_1)(x-t_2)\cdots(x-t_6)(x-\alpha)^2.$$

For $x=\alpha$, there is a double root $y=\beta$. Then there are two points P_α, P_α' over α which correspond to these values $x=\alpha$, $y=\beta$. We denote the third point by P_α''. Then we have

$$\operatorname{div}(f_y) = P_0 + P_1 + P_{t_1} + \cdots + P_{t_6} + P_\alpha + P_\alpha' - 10P_\infty,$$

$$\operatorname{div}(dx) = P_0 + P_1 + P_{t_1} + \cdots + P_{t_6} - 4P_\infty,$$

$$\operatorname{div}(x-\alpha) = P_\alpha + P_\alpha' + P_\alpha'' - 3P_\infty,$$

$$\operatorname{div}(y-\beta) = P_\alpha + P_\alpha' + P_{\kappa_1} + P_{\kappa_2} + P_{\kappa_3} - 5P_\infty.$$

Here κ_1, κ_2 and κ_3 with a double root α are five roots of $f(x, \beta)=0$.

Put

$$\omega_1 = (x-\alpha)dx/f_y, \quad \omega_2 = (x-\alpha)^2 dx/f_y, \quad \omega_3 = (y-\beta)dx/f_y.$$

Then ω_1, ω_2 and ω_3 are differentials of the first kind and are linearly independent over $\mathbf{C}$, i.e., a basis of differentials.

In the former, i.e., in the case that the gap sequence at P_∞ is $\{1, 2, 5\}$, the Wronskian is

$$W = \begin{vmatrix} 1/f_y & x/f_y & y/f_y \\ -f'_y/f_y^2 & (f_y - xf'_y)/f_y^2 & (y'f_y - yf'_y)/f_y^2 \\ G_1/f_y^3 & G_2/f_y^3 & G_3/f_y^3 \end{vmatrix} \cdot (dx)^6 .$$

Here

$$\begin{aligned} G_1 &= -f''_y f_y + 2f'^2_y, \\ G_2 &= (-xf''_y)f_y - 2(f_y - xf'_y)f'_y, \\ G_3 &= (y''f_y - yf''_y)f_y - 2(y'f_y - yf'_y)f'_y. \end{aligned}$$

Then by a simple calculation, we have

$$W = y''f_y^3(dx/f_y)^6$$

and we have

$$\begin{aligned} y''f_y^3 = {} & (9\gamma_3\gamma''_2 + 3\gamma_2\gamma''_3 - 6\gamma'_2\gamma'_3)y^2 \\ & + (4\gamma_2^2\gamma''_2 - 2\gamma_2\gamma'_2 - 6\gamma'^2_3 + 9\gamma_3\gamma''_3)y \\ & + (\gamma_2^2\gamma''_3 - 2\gamma_2\gamma'_2\gamma'_3 + 3\gamma_2\gamma_3\gamma''_2) . \end{aligned}$$

In the latter, i.e., in the case that the gap sequence at P_∞ is $\{1, 2, 4\}$, the Wronskian is given by

$$W = \begin{vmatrix} (x-\alpha)/f_y & (x-\alpha)^2/f_y & (y-\beta)/f_y \\ K_1(x, y)/f_y^2 & K_2(x, y)/f_y^2 & K_3(x, y)/f_y^2 \\ H_1(x, y)/f_y^3 & H_2(x, y)/f_y^3 & H_3(x, y)/f_y^3 \end{vmatrix} \cdot (dx)^6.$$

Here,

$$\begin{aligned} K_1(x, y) &= f_y - (x-\alpha)f'_y, \\ K_2(x, y) &= 2(x-\alpha)f_y - (x-\alpha)^2 f'_y \end{aligned}$$

$$K_3(x, y) = y'f_y - (y - \beta)f'_y,$$

and

$$H_1(x, y) = -(x-\alpha)f_y f''_y - 2(f_y - (x-\alpha)f'_y)f'_y$$
$$H_2(x, y) = (2f_y - (x-\alpha)^2 f''_y)f_y - 2(x-\alpha)(2f_y - (x-\alpha)f'_y)f'_y,$$
$$H_3(x, y) = (y''f_y - (y-\beta)f''_y)f_y - 2(y'f_y - (y-\beta)f'_y)f'_y.$$

Then by a simple calculation, we have

$$W = \{(x-\alpha)^2 y'' - 2y'(x-\alpha) + 2(y-\beta)\}f_y^3 \cdot (dx/f_y)^6$$

and we have

$$\begin{aligned}
&\{(x-\alpha)^2 y'' - 2y'(x-\alpha) + 2(y-\beta)\}f_y^3 \\
&= (x-\alpha)^2[(9\gamma_3\gamma''_2 + 3\gamma_2\gamma''_3 - 6\gamma'_2\gamma'_3)y^2 \\
&\qquad + (4\gamma_2^2\gamma''_2 - 2\gamma_2\gamma'_2 - 6\gamma'^2_3 + 9\gamma_3\gamma''_3)y \\
&\qquad + (\gamma_2^2\gamma''_3 - 2\gamma_2\gamma'_2\gamma'_3 + 3\gamma_2\gamma_3\gamma''_2)] \\
&\quad - 2(x-\alpha)[(9\gamma_3\gamma'_2 + 3\gamma_2\gamma'_3)y^2 \\
&\qquad + (4\gamma_2^2\gamma'_2 + 9\gamma'_3\gamma_2)y + (3\gamma_2\gamma_3\gamma'_2 + \gamma_2\gamma'_3)] \\
&\quad + 2(y-\beta)f_y^3.
\end{aligned}$$

EXAMPLE 1. $y^3 - y - x^4 = 0$.

We have $\omega_1 = dx/f_y$, $\omega_2 = x\,dx/f_y$, $\omega_3 = y\,dx/f_y$, and the Wronskian is

$$\begin{aligned}
W &= y'' f_y^3 (dx/f_y)^6 \\
&= 12x^2(y^2+1)^2(dx/f_y)^6.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\operatorname{div} W &= 24P_\infty + 2(P_0 + P'_0 + P''_0 - 3P_\infty) + 2\{\operatorname{div}(y+i) + \operatorname{div}(y-i)\} \\
&= 2P_\infty + 2(P_0 + P'_0 + P''_0) \\
&\quad + 2\{({}_iP_1 + {}_iP_2 + {}_iP_3 + {}_iP_4) + ({}_{-i}P_1 + {}_{-i}P_2 + {}_{-i}P_3 + {}_{-i}P_4)\}.
\end{aligned}$$

Here ${}_iP_1, \ldots, {}_iP_4$ are points over $x_1, \ldots, x_4$ which are roots of the equation $i^3 - i - x^4 = 0$ and ${}_{-i}P_1, \ldots, {}_{-i}P_4$ are points over $x_1, \ldots, x_4$ which are roots of the equation $-i^3+i-x^4=0$. Obviously these points are distinct from each other. Therefore the number of Weierstrass points is exactly twelve.

EXAMPLE 2. $y^3-xy-x^5=0$.

We have $\omega_1=xdx/f_y$, $\omega_2=x^2dx/f_y$, $\omega_3=ydx/f_y$, and the Wronskian is

$$\begin{aligned} W &= (x^2y''-2xy'+2y)f_y^3\,(dx/f_y)^6 \\ &= 6x^3(6x^3y^2-x^7y+y+2x^4)(dx/f_y)^6 \end{aligned}$$

and we have

$$\begin{aligned} \operatorname{div}(f_y) &= P_0+Q_1+\cdots+Q_7+P_\alpha+P_0-10P_\infty, \\ \operatorname{div}(dx) &= P_0+Q_1+\cdots+Q_7-4P_\infty, \\ \operatorname{div}(x) &= 2P_0+P_\alpha-3P_\infty. \end{aligned}$$

Hence we see that

$$\operatorname{div} x^3(dx/f_y)^6 = 27P_\infty - 3P_\alpha.$$

We must evaluate $\operatorname{div}(6x^3y^2-x^7y+y+2x^4)$. From equations

$$y^3-xy-x^5=0$$

and

$$6x^3y^2-x^7y+y+2x^4=0$$

we obtain

$$x^5(x^{21}-289x^{14}-57x^7+1)=0.$$

Hence by considering the Newton polygon at $x=0$, we have from the part x^5

$$P_0+4P_\alpha-5P_\infty$$

for the divisor. From the part $(x^{21}-289x^{14}-57x^7+1)$ we have

$$P^1+\cdots+P^{21}-21P_\infty$$

for the divisor. Here P^i $(1\leq i\leq 21)$ are roots of the equation concerned.

Thus we have

$$\operatorname{div} W = P_\infty+P_0+P_\alpha+P^1+\cdots+P^{21}.$$

These points {P's} are distinct from each other and we see that the number of

Weierstrass points of this Riemann surface is exactly 24.

REMARK. The Riemann surface defined by

$$y^3 - xy - x^5 = 0$$

is conformally equivalent to the surface defined by

$$Y^7 - X(X-1)^2 = 0$$

which has the group of automorphisms of order 168 [5].

§4. Extremal Riemann surfaces

In order to determine Riemann surfaces which have exactly twelve Weierstrass points we have to prepare several propositions.

PROPOSITION 1. *Let* R *be a non-hyperelliptic Riemann surface of genus three. Assume that there exist distinct four points* P_j $(1 \leq j \leq 4)$ *on* R *which have following three conditions*:

(α) *Each* $4\mathrm{P}_j$ $(1 \leq j \leq 4)$ *is a canonical divisor, i.e., each* P_j $(1 \leq j \leq 4)$ *is a Weierstrass point whose gap sequence is* $\{1, 2, 5\}$.

(β) *The divisor* $(\mathrm{P}_1+\mathrm{P}_2+\mathrm{P}_3+\mathrm{P}_4)$ *is not a canonical divisor.*

(γ) *The divisor* $2(\mathrm{P}_1+\mathrm{P}_2+\mathrm{P}_3+\mathrm{P}_4)$ *is linearly equivalent to* 2K. *Here* K *is a canonical divisor.*

Then R *has three elliptic hyperelliptic involutions* σ_i $(1 \leq i \leq 3)$ *which have following two properties*:

(1) *Each* σ_i $(1 \leq i \leq 3)$ *is a non-trivial substitution of the set* P_j $(1 \leq j \leq 4)$.

(2) *If one of* σ*'s fixes a point* P_j, *then the other two* σ *do not fix any of* P_j $(1 \leq j \leq 4)$.

PROOF. Let ϕ: $\mathrm{R} \to J(\mathrm{R})$ be a canonical map of R into its Jacobian variety. Let $c \in \mathrm{J}(\mathrm{R})$ be the vector of Riemann constants. Put

$$f_1 = \phi(\mathrm{P}_1 + \mathrm{P}_2) + c, \quad f_2 = \phi(\mathrm{P}_3 + \mathrm{P}_4) + c.$$

Then f_1 and f_2 satisfy the assumption of the theorem of Accola [2, Part III, 6, Th. 5, p. 88] (i) $f_1 \not\equiv \pm f_2$, (ii) $2f_1 \equiv 2f_2 \not\equiv 0$, (iii) $|(2f_1), (f_1+f_2)| = 1$, (iv) the theta function vanishes at f_1 and f_2. Now, we shall prove only (iii), since the others are almost trivial. Put

$$t = 2f_1 \equiv 2f_2, \quad s = f_1 + f_2, \quad e = \phi(2\mathrm{P}_1) + c.$$

Then $e+t=\phi(2\mathrm{P}_2)+c$. Theta characteristics $[e]$ and $[e+t]$ are odd, since $l(2\mathrm{P}_1) = l(2\mathrm{P}_2) = 1$. We see that $[e+s]$ is even. Indeed, assume that it is odd. Then there exists a positive divisor A on R such that

$$e+s=\phi(\mathrm{A})+c, \quad \deg \mathrm{A}=2 \quad \text{and} \quad l(\mathrm{A})=1.$$

On the other hand we have

$$e+s=\phi(3\mathrm{P}_1+\mathrm{P}_2+\mathrm{P}_3+\mathrm{P}_4-\mathrm{K})+c.$$

Hence we have $\mathrm{P}_1+\mathrm{A}\sim\mathrm{P}_2+\mathrm{P}_3+\mathrm{P}_4$, and we have $l(\mathrm{P}_1+\mathrm{A})=2$ since R is non-hyperelliptic. Therefore there exists a point Q such that $\mathrm{Q}+\mathrm{P}_1+\mathrm{A}$ is a canonical divisor. Hence we obtain $\mathrm{A}=\mathrm{Q}+\mathrm{P}_1$ and so it follows $\mathrm{Q}=\mathrm{P}_1$. This is a contradiction. By the same way we can show that $[e+t+s]$ is also even. Therefore we obtain

$$|(t),(s)|=|[e]|\,|[e+t]|\,|[e+s]|\,|[e+t+s]|=1,$$

which shows that half periods (t) and (s) are syzygetic.

Thus we can conclude that R is elliptic-hyperelliptic.

Next, we shall construct elliptic-hyperelliptic involutions of R following Accola [1]. It is well-known that for the syzygetic subgroup of degree 2, $\Gamma=\{0,(t),(s),(t+s)\}$, there exists a unique odd characteristic $[d]$ such that $[d]$, $[d+t]$, $[d+s]$ and $[d+t+s]$ are odd characteristics. For the proof see [7, II, 2, Th. 13, p. 40]. We consider four theta characteristics $[f_1]$, $[f_2]$, $[d]$ and $[d+s]$ and we construct four complete linear systems of R_5 over R, which are half-canonical and fixed point free. Here, R_5 is a Riemann surface of genus five, which we construct as follows: It is easy to see that there exist Q_1 and Q_2 different from P_1, P_2 such that the divisor $(\mathrm{P}_1+\mathrm{P}_2+\mathrm{Q}_1+\mathrm{Q}_2)$ is canonical. There also exist Q_3 and Q_4 different from P_3, P_4 such that the divisor $(\mathrm{P}_3+\mathrm{P}_4+\mathrm{Q}_3+\mathrm{Q}_4)$ is canonical. Hence we have

$$2(\mathrm{P}_1+\mathrm{P}_2)\sim 2(\mathrm{Q}_1+\mathrm{Q}_2), \quad 2(\mathrm{P}_3+\mathrm{P}_4)\sim 2(\mathrm{Q}_3+\mathrm{Q}_4)$$

and

$$\phi(\mathrm{P}_1+\mathrm{P}_2)+t=\phi(\mathrm{Q}_1+\mathrm{Q}_2), \quad \phi(\mathrm{P}_3+\mathrm{P}_4)+t=\phi(\mathrm{Q}_3+\mathrm{Q}_4).$$

Therefore, there exists a function h whose divisor is $2(\mathrm{P}_1+\mathrm{P}_2)-2(\mathrm{Q}_1+\mathrm{Q}_2)$. Adjointing $\sqrt{h}$ to the function field of R we obtain a function field of a Riemann surface whose genus is five [4]. We denote the Riemann surface by R_5 and denote by π the canonical projection of R_5 to R. Then we see that R_5 is unramified and $\pi^{-1}(\mathrm{P}_1+\mathrm{P}_2)\sim\pi^{-1}(\mathrm{Q}_1+\mathrm{Q}_2)$ and $\pi^{-1}(\mathrm{P}_3+\mathrm{P}_4)\sim\pi^{-1}(\mathrm{Q}_3+\mathrm{Q}_4)$. Hence by a simple consideration we see that

$$l(\pi^{-1}(\mathrm{P}_1+\mathrm{P}_2))=2, \quad \deg\pi^{-1}(\mathrm{P}_1+\mathrm{P}_2)=4$$

and $\pi^{-1}(P_1+P_2)$ is a half canonical divisor on R_5 and the complete linear system $|\pi^{-1}(P_1+P_2)|$ is fixed point free. Similarly, we see that the complete linear system $|\pi^{-1}(P_3+P_4)|$ is half canonical and fixed point free. Since $[d]$ and $[d+s]$ are odd characteristics and $2d=2s=0$, there exist T_1, T_2 such that $d=\phi(T_1+T_2)+c$ and T_3, T_4 such that $d+s=\phi(T_3+T_4)+c$. Furthermore, by the same reason, there exist S_1 and S_2 different from T_1 and T_2 such that $d+t=\phi(S_1+S_2)+c$ and there also exist S_3 and S_4 different from T_3 and T_4 such that $d+s+t=\phi(S_3+S_4)+c$. We see that $\pi^{-1}(T_1+T_2)\sim\pi^{-1}(S_1+S_2)$ and $\pi^{-1}(T_3+T_4)\sim\pi^{-1}(S_3+S_4)$. Therefore, as before, we see that the complete linear systems $|\pi^{-1}(T_1+T_2)|$, $|\pi^{-1}(T_3+T_4)|$ are half canonical and fixed point free. Then, put

$$\eta_1 = \phi'(\pi^{-1}(P_1 + P_2)) + c', \quad \eta_2 = \phi'(\pi^{-1}(P_3 + P_4)) + c'$$

$$\eta_3 = \phi'(\pi^{-1}(T_1 + T_2)) + c', \quad \eta_4 = \phi'(\pi^{-1}(T_3 + T_4)) + c'.$$

Here ϕ' and c' have the same meaning in R_5 as ϕ and c in R. It is easy to see that $[\eta_1], \ldots, [\eta_4]$ are even theta characteristics such that $[\eta_1+\eta_2+\eta_3+\eta_4]=0$. Therefore applying the theorem of Accola [1, Th. 2, p. 12] to our case, we see that R_5 is elliptic-hyperelliptic, i.e., we have an elliptic-hyperelliptic involution $\bar{\sigma}$ on R_5. Since π is a fixed point free involution, again by the theorem of Accola [2, Prop. p. 86] we see that π and $\bar{\sigma}$ commute, and so we can construct an involution σ on R.

Similarly, we have the other two involutions on R, starting with

$$f'_1 = \phi(P_1 + P_3) + c, \quad f'_2 = \phi(P_2 + P_4) + c$$

and

$$f''_1 = \phi(P_1 + P_4) + c, \quad f''_2 = \phi(P_2 + P_3) + c.$$

It is easy to see that these involutions satisfy the properties (1) and (2) of Proposition 1. Here we have to notice that any group of automorphisms which fix a point is cyclic.

Proposition 2. *Under the same assumptions and notations of Prop.* 1, *let σ_1, σ_2 be involutions in Prop.* 1 *such that they do not fix any P_j. Then we have*

$$\sigma_1\sigma_2 = \sigma_2\sigma_1.$$

Proof. Assume that each of divisors

$$P_1 + P_2 + T_1 + T'_1, \; P_3 + P_4 + T_2 + T'_2; \; P_1 + P_3 + T_3 + T'_3,$$

$$P_2 + P_4 + T_4 + T'_4; \; P_1 + P_4 + T_5 + T'_5, \; P_2 + P_3 + T_6 + T'_6$$

is canonical. Since R is non-hyperelliptic, we see that at least one of T_j, T'_k $(1 \leq j,\ k \leq 6)$ is different from P_i $(1 \leq i \leq 4)$.

We see that P_i, T_j, T'_k are fixed points of $(\sigma_1\sigma_2)^4$. In fact, we may assume that

$$\sigma_1(P_1) = P_2,\ \sigma_2(P_1) = P_3,\ \sigma_1(P_3) = P_4,\ \sigma_2(P_2) = P_4.$$

Then we have

$$(\sigma_1\sigma_2)^2(P_i) = P_i \qquad (1 \leq i \leq 4).$$

Hence

$$(\sigma_1\sigma_2)^2(P_1 + P_2 + T_1 + T'_1) = P_1 + P_2 + (\sigma_1\sigma_2)^2(T_1) + (\sigma_1\sigma_2)^2(T'_1).$$

Since this is canonical, this is linearly equivalent to $(P_1+P_2+T_1+T'_1)$. Therefore, we have $(\sigma_1\sigma_2)^2(T_1)+(\sigma_1\sigma_2)^2(T'_1)=T_1+T'_1$. Thus, we have

$$(\sigma_1\sigma_2)^4(T_1) = T_1,\quad (\sigma_1\sigma_2)^4(T'_1) = T'_1.$$

Similarly we have

$$(\sigma_1\sigma_2)^4(T_i) = T_i,\quad (\sigma_1\sigma_2)^4(T'_i) = T'_i \qquad (1 \leq i \leq 6).$$

Therefore, by the theorem of Lewittes [6, Th. 6, p. 746], if one of T_j, T'_k is not a Weierstrass point then we have $(\sigma_1\sigma_2)^4=1$. However in this case we have $(\sigma_1\sigma_2)^2=1$, i.e., $\sigma_1\sigma_2=\sigma_2\sigma_1$. In fact, assume that $(\sigma_1\sigma_2)^2 \neq 1$. Since $(\sigma_1\sigma_2)^2$ is an involution which fixes four P_i $(1 \leq i \leq 4)$, the genus of $R/<(\sigma_1\sigma_2)^2>$ is zero or one. Since R is non-hyperelliptic, $R/<(\sigma_1\sigma_2)^2>$ must be elliptic. Then the divisor $(P_1+P_2+P_3+P_4)$ must be canonical. This is a contradiction. Therefore if $(\sigma_1\sigma_2)^4 \neq 1$, then all P_i, T_j, T'_k are Weierstrass points and we see easily that they are different from each other. However this conflicts with the fact which asserts that the number of fixed points of a non-trivial automorphism is at most $6\ (=2g+2)$ [3, p. 405].

Thus we can conclude that $\sigma_1\sigma_2=\sigma_2\sigma_1$.

PROPOSITION 3. *A non-hyperelliptic Riemann surface of genus three*, R, *which has two elliptic-hyperelliptic involutions* σ_1, σ_2 *such as* $\sigma_1\sigma_2=\sigma_2\sigma_1$ *is given by*

$$x^4 + y^4 + 2ax^2y^2 + 2bx^2 + 2cy^2 + 1 = 0,$$

where $a^2,\ b^2,\ c^2 \neq 1$ *and* $1+2abc-a^2-b^2-c^2 \neq 0$.

PROOF. We know that the fixed points of σ_1 are four. We denote them by $Q_i^{(1)}$ $(1 \leq i \leq 4)$. Then $\sigma_2(Q_i^{(1)})=Q_j^{(1)}$ $(i \neq j)$. If we put $\sigma_3=\sigma_1\sigma_2$, then σ_3

is an elliptic-hyperelliptic involution. In fact, put $G=\{1, \sigma_1, \sigma_2, \sigma_3\}$. Then the map: $R \to R/G$ is an elementary abelian covering of type (2, 2). R/G is the Riemann sphere and the three intermediate Riemann surfaces are all elliptic. Let fixed points of σ_2, σ_3 be $Q_i^{(2)}, Q_i^{(3)}$ $(1 \leq i \leq 4)$ respectively. We see that $\sum_{i=1}^{4} Q_i^{(1)}, \sum_{i=1}^{4} Q_i^{(2)}, \sum_{i=1}^{4} Q_i^{(3)}$ are all canonical divisors. Hereby, we take two functions x, y on R as follows:

$$\operatorname{div}(x) = \sum_{i=1}^{4} Q_i^{(1)} - \sum_{i=1}^{4} Q_i^{(3)}, \quad \operatorname{div}(y) = \sum_{i=1}^{4} Q_i^{(2)} - \sum_{i=1}^{4} Q_i^{(3)}.$$

Then we see easily that

$$\sigma_1(x) = -x, \ \sigma_1(y) = y, \ \sigma_2(x) = x, \ \sigma_2(y) = -y.$$

Consider the function space

$$L(4 \sum_{i=1}^{4} Q_i^{(3)}) = \{f \mid \operatorname{div}(f) \geq -4 \sum_{i=1}^{4} Q_i^{(3)}\}.$$

The dimension of $L(4 \sum_{i=1}^{4} Q_i^{(3)})$ is equal to 14. Therefore, there exists a linear relation in the set $\{x^i y^j \mid 0 \leq i, j \leq 4, \ i+j \leq 4\}$ and we obtain finally for an equation of R

$$f(x, y) = x^4 + y^4 + 2ax^2y^2 + 2bx^2 + 2cy^2 + 1 = 0$$

with constants a, b and c. Here we have $a^2 - 1 \neq 0$. In fact, put $u = x^2 + ay^2 + b$. Then our equation becomes to be

$$u^2 + (1 - a^2)y^4 + 2(c - ab)y^2 + (1 - b^2) = 0.$$

Therefore, if $a^2 - 1 = 0$, then the Riemann surface whose function field is $\mathbf{C}(u, y)$ is the Riemann sphere and our Riemann surface R becomes to be a covering surface of degree two over the sphere. This is a contradiction. Similarly, we have $b^2 - 1 \neq 0$ and $c^2 - 1 \neq 0$. Since $f(x, y)$ must be irreducible, we have moreover $1 + 2abc - a^2 - b^2 - c^2 \neq 0$.

REMARK. A curve defined by the equation in Prop. 3 with conditions stated there is irreducible, non-singular and of genus three. Moreover it is non-hyperelliptic. In fact, put $\operatorname{div}(x) = \sum_{i=1}^{4} Q_i^{(1)} - \sum_{i=1}^{4} Q_i^{(3)}$. Then we can put $\operatorname{div}(y) = \sum_{i=1}^{4} Q_i^{(2)} - \sum_{i=1}^{4} Q_i^{(3)}$. Let τ be an automorphism of R such that $R/\langle\tau\rangle$ is the sphere. We see that $\tau(\sum_{i=1}^{4} Q_i^{(k)}) = \sum_{i=1}^{4} Q_i^{(k)}$ $(1 \leq k \leq 3)$ [4], and so we have $\tau(Q_i^{(k)}) = Q_j^{(k)}$ $(i \neq j)$. Hence $x, y \in \mathbf{C}(R/\langle\tau\rangle)$. This is a contradiction.

PROPOSITION 4. *Let* R *be a non-hyperelliptic Riemann surface of genus* *three. Assume that* R *has just twelve Weierstrass points. We denote them by* P_i $(1 \leq i \leq 12)$. *Then, reordering if necessary, we have either*

(1) $P_1+P_2+P_3+P_4, P_5+P_6+P_7+P_8, P_9+P_{10}+P_{11}+P_{12}$ *are all canonical*

or

(2) $P_1+P_2+P_3+P_4$ *is not canonical, but* $2(P_1+P_2+P_3+P_4)\sim 2K$.

PROOF. For $i, j=1, 2,\dots, 12$, put

$$t_{ij} = \phi(2(P_i + P_j) - K) \qquad (i<j).$$

These non-zero half periods on J(R) are 66 in all. However there must be 63 $(=2^{2g}-1)$ non-zero half periods on J(R). Therefore, there must be the same points in these 66 t_{ij}. Now we may put

$$2(P_1 + P_2) \sim 2(P_3 + P_4), \text{ i.e., } 2(P_1 + P_2 + P_3 + P_4) \sim 2K.$$

Hence we have

$$t_{1,2} = t_{3,4}, \quad t_{1,3} = t_{2,4}, \quad t_{1,4} = t_{2,3}.$$

There exist the same elements in $\{t_{ij}\}$ besides these elements. Indeed, assume that there were no more the same elements in $\{t_{ij}\}$. Put

$$k = -\phi(2P_1) + t_{2,5}.$$

We see that $k \neq -\phi(2P_i)$ $(1 \leq i \leq 12)$. Put $t_i=\phi(2P_i)+k$ $(1 \leq i \leq 12)$. These are non-zero half periods and different from each other. Therefore they must coincide with a part of $\{t_{ij}\}$. Then, put $t_6=t_{ij}$. We see that i and j are not equal to $1, 2,\dots, 6$ and so we may put $t_6=t_{7,8}$. Similarly, we may put $t_9=t_{10,11}$. Finally, put $t_{12}=t_{ij}$. Then, we see that i and j are not equal to $1, 2,\dots, 12$. This is a contradiction. Thus, following only three cases can take place:

(α) $\qquad 2(P_1 + P_2) \sim 2(P_3 + P_4),\ 2(P_5 + P_6) \sim 2(P_7 + P_8)$,

(β) $\qquad 2(P_1 + P_2) \sim 2(P_3 + P_4),\ 2(P_1 + P_5) \sim 2(P_6 + P_7)$,

(γ) $\qquad 2(P_1 + P_2) \sim 2(P_3 + P_4),\ 2(P_1 + P_2) \sim 2(P_5 + P_6)$.

In case (α), we see that $2(P_9+P_{10}+P_{11}+P_{12})\sim 2K$, since we have $2\sum_{i=1}^{12} P_i\sim 6K$. Hence either (1) or (2) occurs. In case (γ), we cannot have at the same time

$$P_1 + P_2 + P_3 + P_4 \sim K \quad\text{and}\quad P_1 + P_2 + P_5 + P_6 \sim K.$$

Therefore (2) occurs. In case (β), we must carry our investigation further. Consider following 70 divisors

$$2(P_i + P_j + P_k), \qquad 1 \le i \le 7,\ 8 \le j < k \le 12.$$

Since $4(P_i + P_j + P_k) \sim 3K$, we see that there must be linearly equivalent elements in them. Therefore, (1) or (2) or the following case (*)

$$2(P_1 + P_2 + P_3 + P_4) \sim 2K,$$

$$2(P_1 + P_5 + P_6 + P_7) \sim 2K,$$

$$2(P_2 + P_5 + P_8 + P_9) \sim 2K,$$

occurs. To study case (*) put

$$t_{ij} = \phi(2(P_i + P_j) - K), \qquad 5 \le i, j \le 12,$$

$$s_{ijk} = \phi(2(P_5 + P_i + P_j + P_k) - 2K), \qquad 6 \le i < j < k \le 12.$$

If some of them coincide or some of them are equal to zero, then case (*) becomes (1) or (2). Therefore it remains to consider the case that these 28 t_{ij} and 35 s_{ijk} are all the non-zero half periods on J(R). Then, since t_{ij}, for example, is a half period, $t_{3,6}$ must coincide with one of these 28 t_{ij} and 35 s_{ijk} and so we are led to (1) or (2) or, for example if it is equal to $t_{8,10}$, $2(P_3 + P_6 + P_8 + P_{10}) \sim 2K$. Then we have $2(P_4 + P_7 + P_9 + P_{10}) \sim 2K$ by adding $2(P_3 + P_6 + P_8 + P_{10}) \sim 2K$ to (*). Further we see that

$$P_1 + P_2 + P_3 + P_4,\ P_1 + P_5 + P_6 + P_7,\ P_2 + P_5 + P_8 + P_9,$$

$$P_3 + P_6 + P_8 + P_{10},\ P_4 + P_7 + P_9 + P_{10},$$

are not all canonical. Thus, we are led to (2).

Now we are in position to prove the following theorem:

THEOREM 2. *Let* R *be a non-hyperelliptic compact Riemann surface of genus three. Assume that* R *has exactly twelve Weierstrass points on it. Then* R *is given by*

(1) $$y^3 - y - x^4 = 0$$

or

(2) $$x^4 + y^4 + \frac{6}{5}x^2y^2 + \frac{6}{\sqrt{5}}(x^2 + y^2) + 1 = 0.$$

REMARK. (1) is birationally equivalent to

(1′) $$x^4 + y^4 = 1$$

and (2) is birationally equivalent to

(2′) $$x^4 + y^4 + 3(x^2y^2 + x^2 + y^2) + 1 = 0.$$

It is easy to see that the former has exactly 96 automorphisms and the latter has exactly 24 automorphisms.

PROOF OF THEOREM 2. By Proposition 4, our proof is divided into two cases:

Case (1): We take P_1 instead of P_∞ in § 2. There exists a function x such that

$$\operatorname{div}(x) = P_2 + P_3 + P_4 - 3P_1.$$

Then as we see in § 2 the equation of R is given by

$$f(x, y) = y^3 - \gamma_2(x)y - \gamma_3(x),$$

$$\gamma_2(x) = a_0x^2 + a_1x + a_2,$$

$$\gamma_3(x) = x^4 + b_1x^3 + b_2x^2 + b_3x + b_4.$$

Consider the Wronskian of R. Then we have

$$\operatorname{div}(y''f_y^3) + 6\operatorname{div}(dx/f_y) = 2\sum_{i=1}^{12} P_i.$$

Here $\operatorname{div}(dx/f_y) = 4P_1$. Hence we obtain

$$\operatorname{div}(y''f_y^3) = 2(\sum_{i=2}^{12} P_i - 11P_1).$$

On the other hand, there exist functions

$$A_1x + B_1y + C_1, \quad A_2x + B_2y + C_2$$

such that

$$\operatorname{div}(A_1x + B_1y + C_1) = P_5 + P_6 + P_7 + P_8 - 4P_1,$$

$$\operatorname{div}(A_2x + B_2y + C_2) = P_9 + P_{10} + P_{11} + P_{12} - 4P_1,$$

with suitable constants A's, B's and C's. Here we must notice $B_1B_2 \neq 0$. Therefore we obtain

$$y''f_y^3 = x^2(A_1x + B_1y + C_1)^2(A_2x + B_2y + C_2)^2.$$

We shall express both sides of this equation by the linear combinations of a basis of the function space $L(22P_1)$;

$$\{x^iy^j;\ 0 \leq i \leq 7,\ 0 \leq j \leq 2,\ 3i + 4j \leq 22\}.$$

On the left, we have

$$y''f_y^3 = (9\gamma_2''\gamma_3 + 3\gamma_2\gamma_3'' - 6\gamma_2'\gamma_3')y^2$$
$$+ (4\gamma_2^2\gamma_2'' - 2\gamma_2\gamma_2'^2 + 9\gamma_3\gamma_3'' - 6\gamma_3'^2)y$$
$$+ (3\gamma_2\gamma_2''\gamma_3 - 2\gamma_2\gamma_2'\gamma_3' + \gamma_2^2\gamma_3'') .$$

On the right, we have

$$\begin{aligned}
&x^2(A_1x + B_1y + C_1)^2(A_2x + B_2y + C_2)^2\\
&\quad = x^2\{B_1^2B_2^2\gamma_2 + B_2^2(A_1x + C_1)^2 + B_1^2(A_2x + C_2)^2\\
&\qquad + 4B_1B_2(A_1x + C_1)\cdot(A_2x + C_2)\}y^2\\
&\quad + x^2\{B_1^2B_2^2\gamma_3 + 2B_1B_2(B_2(A_1x + C_1) + B_1(A_2x + C_2))\gamma_2\\
&\qquad + 2(A_1x + C_1)(A_2x + C_2)(B_2(A_1x + C_1) + B_1(A_2x + C_2))\}y\\
&\quad + x^2\{2B_1B_2(B_2(A_1x + C_1) + B_1(A_2x + C_2))\gamma_3\\
&\qquad + (A_1x + C_1)^2(A_2x + C_2)^2\} .
\end{aligned}$$

Comparing the coefficients of both sides we obtain

$$2B_1B_2(B_2A_1 + B_1A_2) = 0 \qquad (\text{in } x^7),$$
$$B_1^2B_2^2 = 12,\ 18b_1 = b_1B_1^2B_2^2 \qquad (\text{in } x^6y,\ x^5y) .$$

Hence we have

$$B_2A_1 + B_1A_2 = 0 \quad \text{and} \quad b_1 = 0.$$

From the coefficients of x^3y^2 we have $A_1C_2+A_2C_1=0$. Hence we have

$$a_0a_1 = 0 \qquad (\text{in } x^5) .$$

Now, if $a_0=0$, then we have $A_1A_2=0$ (in x^4y^2). Hence we obtain

$$b_2 = 0,\ B_2C_1 + B_1C_2 = 0 \qquad (\text{in } x^4y,\ x^6) .$$

Therefore,

$$a_1 = 0 \quad (\text{in } x^4), \qquad b_3 = b_4 = 0 \quad (\text{in } x^3y,\ x^2y) .$$

Thus for the equation of R we have

$$y^3 - a_2y - x^4 = 0 \qquad (a_2 \neq 0) .$$

It is easy to see that this is birationally equivalent to the Riemann surface defined by

$$y^3 - y - x^4 = 0.$$

If $a_1=0$, then we have $b_3=0$ (in x^3y) and the equation becomes

$$f(x, y) = y^3 - a_0x^2y - a_2y - (x^4 + b_2x^2 + b_4) = 0.$$

We may put the coordinates of P_i $(0, \beta_i)$ $(2 \leq i \leq 4)$. Then

$$f_x(0, \beta_i) = 0, \quad \operatorname{div}(y - \beta_i) = 4P_i - 4P_1.$$

Therefore, for $i=2$, 3 and 4 the equation

$$f(x, \beta_i) = -x^4 - (a_0\beta_i + b_2)x^2 = 0$$

has $x=0$ as a quadruple root. Thus we have

$$a_0\beta_i + b_2 = 0 \qquad (2 \leq i \leq 4).$$

Hence we have $a_0=b_2=0$ and our case is reduced to the former.

Case (2): By Propositions 1, 2 and 3 we have

$$f(x, y) = x^4 + y^4 + 2ax^2y^2 + 2bx^2 + 2cy^2 + 1 = 0$$

for the equation of R.

First, we have to remark following two properties:

(#) A necessary and sufficient condition for $Q_k^{(3)}$ $(1 \leq k \leq 4)$ to be Weierstrass points is $b=c=0$.

In fact, if we put $u=1/x$, $v=y/x$, then the equation comes to be

$$g(u, v) = u^4 + v^4 + 2cu^2v^2 + 2bu^2 + 2av^2 + 1 = 0.$$

We may assume that the coordinates of $Q_k^{(3)}$ is $(u, v)=(0, \alpha_k)$ $(1 \leq k \leq 4)$. Since $g_u(0, \alpha_k)=0$, $u=0$ must be a quadruple root of $g(u, \alpha_k)=0$. Hence we obtain $b=c=0$, and vice versa.

By the same way, the necessary and sufficient condition for $Q_k^{(1)}$ $(1 \leq k \leq 4)$ to be Weierstrass points is $a=b=0$ and for $Q_k^{(2)}$ $(1 \leq k \leq 4)$ to be Weierstrass points is $c=a=0$.

(##) A necessary and sufficient condition for one of $Q_k^{(1)}$ $(1 \leq k \leq 4)$ to be a Weierstrass point is $a^2+b^2-2abc=0$.

In fact, let the coordinates of $Q_k^{(1)}$ be $(0, \beta)$. Then since $x=0$ must be a quadruple root of $f(x, \beta)=0$, we have

$$\beta^4 + 2c\beta^2 + 1 = 0, \quad a\beta^2 + b = 0.$$

Hence we have $a^2+b^2-2abc=0$, and vice versa.

By the same way, a necessary and sufficient condition for one of $Q_k^{(2)}$ $(1\leq k\leq 4)$ to be a Weierstrass point is $a^2+c^2-2abc=0$ and for one of $Q_k^{(3)}$ $(1\leq k\leq 4)$ to be a Weierstrass point is $b^2+c^2-2abc=0$.

Now we shall prove Theorem 2.

First, assume that for some $i\,(1\leq i\leq 3)$, $Q_1^{(i)},\ldots, Q_4^{(i)}$ are Weierstrass points. For example, if $i=3$, we see that $b=c=0$ by (♯) and the equation comes to be

$$Y^4 = X^4 + 2aX^2 + 1$$

by a suitable birational transformation and by the standard method used in § 3, we see that for the Wronskian of R

$$W = (aX^4 + 3X^3 - a^2X^2 + a)(dX)^6/Y^{16}.$$

To have just twelve Weierstrass points it must be $a=\pm 3$ and we have

$$f(x, y) = x^4 + y^4 \pm 6x^2y^3 + 1 = 0.$$

It is easy to see that this is birationally equivalent to

$$x^4 + y^4 = 1, \quad \text{i.e.,} \quad y^3 - y - x^4 = 0.$$

Secondly, we shall show that the other cases are reduced to the case (♯) or to the case where we have $a=\pm 3/5$ and $b^2=c^2=9/5$. Now let P be a Weierstrass point whose gap sequence is $\{1, 2, 5\}$ and is not a fixed point of any of σ_i $(1\leq i\leq 3)$. Since $4P\sim\sum_{k=1}^{4} Q_k^{(3)}$, there exists a function $\lambda x+\mu y+\nu$ such that $4P-\sum_{k=1}^{4} Q_k^{(3)}=\operatorname{div}(\lambda x+\mu y+\nu)$. By the assumption of P we may assume that $\lambda\mu\nu\neq 0$. Hence, we rewrite

$$4P - \sum_{k=1}^{4} Q_k^{(3)} = \operatorname{div}(\lambda x + \mu y + 1).$$

Put

$$F(x) = f(x, -(\lambda x + 1)/\mu)\mu^4$$
$$= A_0x^4 + 4A_1x^3 + 2A_2x^2 + 4A_3x + A_4.$$

Here

$$A_0 = \lambda^4 + \mu^4 + 2a\lambda^2\mu^2, \quad A_1 = \lambda(\lambda^2 + a\mu^2),$$
$$A_2 = 3\lambda^2 + a\mu^2 + b\mu^4 + c\lambda^2\mu^2,$$

$$A_3 = \lambda(1 + c\mu^2), \quad A_4 = 1 + \mu^4 + 2c\mu^2.$$

Since the zeros of $\lambda x+\mu y+1$ are only P, $F(x)$ must have a quadruple root in x. Hence we have

$$3A_1^2 - A_0A_2 = 0\ (1), \quad A_1A_2 - 3A_0A_3 = 0\ (2),$$
$$A_1A_3 - A_0A_4 = 0\ (3).$$

Put $l=\lambda^2$, $m=\mu^2$. From (2)

$$g_1(l, m) = -2cl^2 + (ab - 3c)m^2 + (b - 5ac)lm - 2al + (a^2 - 3)m$$
$$= 0 \qquad (4).$$

From $(1)\times A_3-(3)\times 3A_1$,

$$(3a - bc)m^2 + (3 - c^2)lm + 2cl + (5ac - b)m + 2a = 0 \qquad (5).$$

Use y instead of x. Then we have

$$g_2(l, m) = (ac - 3b)l^2 - 2bm^2 + (c - 5ab)lm + (a^2 - 3)l - 2am = 0 \qquad (4'),$$
$$(3a - bc)l^2 + (3 - b^2)lm + (5ab - c)l + 2bm + 2a = 0 \qquad (5').$$

If $(4)=(4')$, then by a simple calculation we have $b=c=0$. However this conflicts with our assumption by (#). Therefore we may assume that $(4)\neq(4')$. If $(5)=(5')$, by a simple calculation we have $b=c=0$ or

$$a = 3/5,\ b^2 = 3a = 9/5 \qquad \text{for} \quad b = c \neq 0,$$
$$a = -3/5,\ b^2 = -3a = 9/5 \qquad \text{for} \quad b = -c \neq 0.$$

Hence we are led to the case (#) or to the case $a=\pm 3/5$ and $b^2=c^2=9/5$. Therefore we may assume that $(5)\neq(5')$. Then we have further

$$g_3(l, m) = (3a - bc)(l^2 - m^2) + (c^2 - b^2)lm + (5ab - 3c)l$$
$$- (5ac - 3b)m = 0 \qquad (6).$$

Assume that any of twelve Weierstrass points $\{P_i\}$ is not fixed by any of σ_1, σ_2 and σ_3. Then we have three distinct (l, m) for the twelve Weierstrass points $\{P_i\}$. Therefore, equations (4), (4′) and (6) have four common points. Of course (0, 0) is one of these common points. Since (4) is a conic different from (4′), there exist k and k' such that

$$kg_1(l, m) + g_2(l, m) = k'g_3(l, m).$$

This means that we have

$$\frac{-2ck+ac-3b}{3a-bc}=\frac{(ab-3c)k-2b}{bc-3a}=\frac{(b-5ac)k+c-5ab}{c^2-b^2}$$
$$=\frac{-2ak+a^2-3}{5ab-3c}=\frac{(a^2-3)k-2a}{3b-5ac}.$$

By an elementary calculation we see that

$$a=b=0 \quad \text{or} \quad b=c=0 \quad \text{or} \quad b^2=c^2 \neq 0.$$

The first two cases conflict with our assumption by (♯). For the last case R has two more elliptic-hyperelliptic involutions τ_1 and τ_2 defined by

$$\tau_1\colon (x, y) \longrightarrow (y, x),\ \tau_2\colon (x, y) \longrightarrow (-y, -x), \qquad \text{if} \quad b=c,$$

and

$$\tau_1\colon (x, y) \longrightarrow (-iy, ix),\ \tau_2\colon (x, y) \longrightarrow (iy, -ix), \qquad \text{if} \quad b=-c.$$

Then we have $\tau_3=\tau_1\tau_2=\tau_2\tau_1$ $(=\sigma_3)$. We remark that if we put x to ix then the case $b=-c$ is reduced to the case $b=c$.

Now, assume that there exists a Weierstrass point which is not fixed by any of σ_i, τ_i $(1\leq i\leq 3)$. Then eight points

$$\text{P},\ \tau_1(\text{P}),\ \sigma_1(\text{P}),\dots,\ \sigma_3(\text{P}),\ \sigma_1\tau_1(\text{P}),\dots,\ \sigma_3\tau_1(\text{P})$$

are Weierstrass points which are not fixed by any of σ_i, τ_i $(1\leq i\leq 3)$. Therefore we see that there are fixed points of some of σ_1, σ_2, σ_3 and τ_1, τ_2, τ_3 $(=\sigma_3)$ in the twelve Weierstrass points. We remark here that if P is a Weierstrass point which is not fixed by any of σ_i $(1\leq i\leq 3)$, then four points P, $\sigma_1(\text{P}),\dots,\sigma_3(\text{P})$ are point of the same kind. Use τ, if necessary, instead of σ. Then by the same argument as above, we are led to the case (♯) or to the case (♯♯) satisfied for two i's or to the case $a=\pm 3/5$ and $b^2=c^2=9/5$. Further, for the case (♯♯) satisfied for two i's it is reduced to the case (♯) or the case where we have $a=\pm 3/5$ and $b^2=c^2=9/5$.

Finally we have two cases: The first is

$$x^4+y^4=1$$

and the second is

$$x^4+y^4+\frac{6}{5}x^2y^2+\frac{6}{\sqrt{5}}(x^2+y^2)+1=0.$$

The former has its Weierstrass points by fours on each three straight lines but the latter has its Weierstrass points by fours with common points on each six straight lines. Therefore they are not conformally equivalent.

REMARK. The second one is birationally equivalent to

$$x^4 + y^4 + 3(x^2y^2 + x^2 + y^2) + 1 = 0$$

and we can represent this equation in homogeneous coordinates as

$$x^4 + y^4 + z^4 + 3(x^2y^2 + y^2z^2 + z^2x^2) = 0.$$

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