HIROSHIMA MATH. J.
7 (1977), 707–730

Boundary Value Control Theory of Elastodynamic System

Kimiaki NARUKAWA

(Received May 11, 1977)

1. Introduction

Consider a linear elastic solid occupying, in its non deformed state, a bounded three dimensional domain Ω with a C^∞-boundary $\partial\Omega$. Let the medium be fixed on some part of the boundary and free on the other part. In this paper we consider the problem of controlling the deformation of the medium by applying traction forces on a small subset of the free boundary part. Let us denote by $\{u^i(x, t)\}_{i=1,2,3}$ the displacement vector at the time t of the material particle which lies at $x=\{x_i\}_{i=1,2,3}$ in the non deformed state. Then $u^i(x, t)$ $(i=1, 2, 3)$ satisfy the system of equations

$$\rho(x)\frac{\partial^2 u^i}{\partial t^2} - \frac{\partial}{\partial x_j}\left(c_{ijkl}(x)\frac{\partial u^k}{\partial x_l}\right) = 0 \qquad \text{in} \quad Q \equiv \Omega \times (0, T) \tag{1.1}$$

with initial conditions

$$u^i(x, 0) = 0 \qquad \text{in} \quad \Omega, \tag{1.2}$$

$$[\partial u^i/\partial t](x, 0) = 0 \qquad \text{in} \quad \Omega \tag{1.3}$$

and mixed boundary conditions

$$u^i(x, t) = 0 \qquad \text{on} \quad \Gamma_1 \times (0, T), \tag{1.4}$$

$$n_j c_{ijkl}(x)\frac{\partial u^k}{\partial x_l}(x, t) = g^i(x, t) \qquad \text{on} \quad \Gamma_2 \times (0, T). \tag{1.5}$$

Here $n=(n_1, n_2, n_3)$ is the outward unit normal vector on $\partial\Omega$, Γ_1 and Γ_2 are disjoint relatively open subsets of $\partial\Omega$ such that $\partial\Omega=\bar{\Gamma}_1 \cup \Gamma_2=\Gamma_1 \cup \bar{\Gamma}_2$, $L=\bar{\Gamma}_1 \cap \bar{\Gamma}_2$ is a smooth curve and T is a positive number. The coefficients $\rho(x)$ and $c_{ijkl}(x)$ are assumed to be C^∞-functions and to satisfy the following symmetry and definiteness conditions:

$$\rho_m^2 \leqq \rho(x) \leqq \rho_M^2 \qquad \text{in} \quad \Omega, \qquad 0 < \rho_m \leqq \rho_M,$$

$$|(\partial\rho/\partial x_i)(x)| \leqq \rho_0^2 \qquad \text{in} \quad \Omega, \qquad i = 1, 2, 3,$$

$$c_{ijkl}(x) = c_{klij}(x) \qquad \text{in} \quad \Omega,$$

$$c_m^2\xi_{ij}^2 \leqq c_{ijkl}(x)\xi_{ij}\xi_{kl} \leqq c_M^2\xi_{ij}^2, \qquad 0 < c_m \leqq c_M$$

for any real ξ_{ij} $(i, j=1, 2, 3)$.

Throughout this paper, all suffixes range over the values 1, 2, 3 and the usual convention of summing over repeated indices is adopted.

Let Γ_0 be a relatively open subset of Γ_2 and we denote by $\mathscr{F}$ the set of all infinitely continuously differentiable functions on $\Gamma_2 \times (0, T)$ whose supports are compact and contained in $\Gamma_0 \times (0, T)$. This space $\mathscr{F}$ is called the control space, and the set of all states $[u(T), (\partial u/\partial t)(T)]$ of the solutions of (1.1)$\sim$(1.5) when g ranges over the space $\mathscr{F}$ is called the reachable space at time T. When the reachable space at time T is dense in a certain Hilbert space, the system is said to be controllable at time T.

In case the whole boundary is free, that is, $\Gamma_1 = \emptyset$, B. M. N. Clarke [1] showed that the system (1.1)$\sim$(1.5) is not controllable at a time less than $2T_1$ and controllable at a time greater than $2T_2$ with constants T_1 and T_2 which are determined by Ω, $\rho(x)$ and $c_{ijkl}(x)$. (Cf. also D. L. Russell [8] [9].) In this paper we shall show that the same results still hold even if there is a fixed part Γ_1.

The author wishes to express his hearty thanks to Prof. A. Inoue for his kind encouragement and valuable suggestions, and to Prof. F-Y. Maeda for his comments in writing this paper.

2. The existence and uniqueness of solutions

In this section we shall show the existence and uniqueness of a solution of the initial-boundary value problem

$$\rho(x)\frac{\partial^2 u^i}{\partial t^2} - \frac{\partial}{\partial x_j}\left(c_{ijkl}(x)\frac{\partial u^k}{\partial x_l}\right) = f^i(x, t) \quad \text{in} \quad Q \equiv \Omega \times (0, T), \tag{2.1}$$

$$u^i(x, 0) = u_0^i(x) \quad \text{in} \quad \Omega, \tag{2.2}$$

$$\frac{\partial u^i}{\partial t}(x, 0) = u_1^i(x) \quad \text{in} \quad \Omega, \tag{2.3}$$

$$u^i(x, t) = 0 \quad \text{on} \quad \Gamma_1 \times (0, T), \tag{2.4}$$

$$n_j c_{ijkl}(x)\frac{\partial u^k}{\partial x_l}(x, t) = g^i(x, t) \quad \text{on} \quad \Gamma_2 \times (0, T), \tag{2.5}$$

following the lines of [2], [3] and [4].

First we introduce some function spaces on which our problem is considered. Let us denote by $H^l(\Omega)$ the Sobolev space of order l, and by $K(\Omega)$ the closure in $H^1(\Omega)$ of the space of all u each of which belongs to $C^\infty(\bar{\Omega})$ and vanishes in a neighborhood of $\overline{\Gamma_1 \cup L}$. The Gothic types $\boldsymbol{L}^2(\Omega)$, $\boldsymbol{H}^1(\Omega)$ and $\boldsymbol{K}(\Omega)$ denote the product spaces $L^2(\Omega)^3$, $H^1(\Omega)^3$ and $K(\Omega)^3$ respectively. For an element $u(x)$ in $\boldsymbol{L}^2(\Omega)$, $\boldsymbol{H}^1(\Omega)$ or $\boldsymbol{K}(\Omega)$, $u^i(x)$ $(i=1, 2, 3)$ denotes the i-th component of $u(x)$. For a Banach space X, $\mathscr{E}_t^k(X)[0, T]$ means the Banach space of k-times continuously differentiable X-valued functions in $0<t<T$. For $u, v \in L^2(\Omega)$ or $\boldsymbol{L}^2(\Omega)$, (u, v) means the inner product in either of these Hilbert spaces.

For simplicity let us put as follows:

$$v = \partial u/\partial t, \quad U = \begin{bmatrix} u \\ v \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ f \end{bmatrix},$$

$$A_{ik} = \frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial}{\partial x_l}\right), \qquad A = (A_{ik})_{i,k=1,2,3},$$

$$\mathscr{A} = \begin{bmatrix} 0 & 1 \\ \rho^{-1}A_{ik} & 0 \end{bmatrix}.$$

Then equation (2.1) is written as

$$\frac{d}{dt}U = \mathscr{A}U + F/\rho. \tag{2.6}$$

By means of $\boldsymbol{K}(\Omega)$ we define the boundary condition (2.5) in the weak sense as follows:

DEFINITION 2.1. *Let* $u(x) \in \boldsymbol{H}^1(\Omega)$, $Au(x) \in \boldsymbol{L}^2(\Omega)$ *and* $g(x) \in \boldsymbol{L}^2(\Gamma_2)$. *Then* $u(x)$ *is said to satisfy the boundary condition*

$$n_j c_{ijkl}\frac{\partial u^k}{\partial x_l} = g^i \ \textit{weakly on } \Gamma_2, \tag{2.7}$$

if it satisfies

$$-(Au, \phi) = \left(c_{ijkl}\frac{\partial u^k}{\partial x_l}, \frac{\partial \phi^i}{\partial x_j}\right) - \int_{\Gamma_2} g \cdot \phi dS$$

for any $\phi \in \boldsymbol{K}(\Omega)$.

We shall prove

THEOREM 2.2. *Let* $g(x, t) \in C_0^\infty(\Gamma_2 \times [0, T])^3$ *and* $f(x, t) \in \mathscr{E}_t^0(\boldsymbol{K}(\Omega))[0, T]$. *Then for each* $[u_0, u_1] \in \boldsymbol{K}(\Omega) \times \boldsymbol{K}(\Omega)$ *which satisfies* $Au_0 \in \boldsymbol{L}^2(\Omega)$ *and the boundary condition* $n_j c_{ijkl}[\partial u_0^k/\partial x_l] = g^i(x, 0)$ *weakly on* Γ_2, *there exists a unique solution* $u(x, t) \in \mathscr{E}_t^1(\boldsymbol{K}(\Omega))[0, T] \cap \mathscr{E}_t^2(\boldsymbol{L}^2(\Omega))[0, T]$ *of the equation* (2.1) *satisfying* (2.2), (2.3) *and*

$$n_j c_{ijkl} \frac{\partial u^k}{\partial x_l}(t) = g^i(t) \qquad \textit{weakly on } \Gamma_2$$

for each $t \in (0, T)$.

It is easy to construct a function $\tilde{u}(x, t) \in C^\infty(\bar{\Omega} \times (0, T))^3$ satisfying

$$\tilde{u}(x, t) = 0 \qquad \text{on} \quad \Gamma_1 \times [0, T]$$

and

$$n_j c_{ijkl} \frac{\partial \tilde{u}^k}{\partial x_l} = g^i \qquad \text{on} \quad \Gamma_2 \times [0, T].$$

Denoting $u = \tilde{u} + v$, we have only to solve the problem

$$(2.8) \qquad \begin{cases} \rho \dfrac{\partial^2 v^i}{\partial t^2} - \dfrac{\partial}{\partial x_j}\left(c_{ijkl} \dfrac{\partial v^k}{\partial x_l}\right) = \tilde{F}^i \qquad \text{in} \quad Q, \\ \text{where} \\ \qquad \tilde{F}^i = -\left(\rho \dfrac{\partial^2 \tilde{u}^i}{\partial t^2} - \dfrac{\partial}{\partial x_j}\left(c_{ijkl} \dfrac{\partial \tilde{u}^k}{\partial x_l}\right)\right) + f^i, \\ \qquad v(x, t) \in \mathscr{E}_t^1(\boldsymbol{K}(\Omega))\,[0, T] \cap \mathscr{E}_t^2(\boldsymbol{L}^2(\Omega))\,[0, T], \\ n_j c_{ijkl} \dfrac{\partial v^k}{\partial x_l}(x, t) = 0 \text{ weakly on } \Gamma_2 \text{ for each } t \in (0, T), \\ v(x, 0) = v_0(x) \qquad \text{in} \quad \Omega, \\ [\partial v/\partial t]\,(x, 0) = v_1(x) \qquad \text{in} \quad \Omega, \\ \text{where} \\ \qquad v_0(x) = u_0(x) - \tilde{u}(x, 0), \\ \qquad v_1(x) = u_1(x) - (\partial \tilde{u}/\partial t)(x, 0). \end{cases}$$

Now let us solve the problem (2.8) by the semi-group theory. Let $\mathscr{H}$ be the space $\boldsymbol{K}(\Omega) \times \boldsymbol{L}^2(\Omega)$ with the inner product

$$(U_1, U_2)_{\mathscr{H}} = \left(\rho^{-1} c_{ijkl} \frac{\partial u_1^k}{\partial x_l}, \frac{\partial u_2^i}{\partial x_j}\right) + (v_1, v_2) + (u_1, u_2)$$

and resulting norm $|U_1|_{\mathscr{H}} = (U_1, U_1)_{\mathscr{H}}^{1/2}$ for $U_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} \in \mathscr{H}$ $(i=1, 2)$. By the positive definiteness conditions on $c_{ijkl}(x)$ and $\rho(x)$, the norm $|\cdot|_{\mathscr{H}}$ is equivalent to the standard one in $\boldsymbol{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega)$. Let us define the domain of $\mathscr{A}$ as follows:

$$(2.9)\qquad D(\mathscr{A}) = \left\{ U = \begin{bmatrix} u \\ v \end{bmatrix} \;\middle|\; \begin{array}{l} u,\, v \in \boldsymbol{K}(\Omega),\ Au \in \boldsymbol{L}^2(\Omega),\ u \text{ satisfies the} \\ \text{boundary condition } n_j c_{ijkl}(\partial u^k/\partial x_l) = 0 \\ \text{weakly on } \Gamma_2. \end{array} \right\}.$$

Then $\mathscr{A}$ is a linear operator in $\mathscr{H}$.

LEMMA 2.3. *There exists a positive constant c_1 such that for any $U \in D(\mathscr{A})$*

$$(2.10)\qquad |(\mathscr{A}U,\, U)_{\mathscr{H}}| \leqq (c_1/2)\,|U|_{\mathscr{H}}{}^2.$$

PFOOF. If $U = \begin{bmatrix} u \\ v \end{bmatrix} \in D(\mathscr{A})$, then $\mathscr{A}U = \begin{bmatrix} v \\ \rho^{-1}Au \end{bmatrix}$.

Since u, $\rho^{-1}v \in \boldsymbol{K}(\Omega)$ and u satisfies the boundary condition $n_j c_{ijkl}(\partial u^k/\partial x_l) = 0$ weakly on Γ_2,

$$\begin{aligned}
(\mathscr{A}U,\, U)_{\mathscr{H}} &= \left(\rho^{-1} c_{ijkl} \frac{\partial v^i}{\partial x_j},\, \frac{\partial u^k}{\partial x_l}\right) + (v,\, u) + (\rho^{-1}Au,\, v) \\
&= \left(\rho^{-1} c_{ijkl} \frac{\partial v^i}{\partial x_j},\, \frac{\partial u^k}{\partial x_l}\right) + (v,\, u) \\
&\quad - \left(c_{ijkl} \frac{\partial u^i}{\partial x_j},\, \frac{\partial}{\partial x_l}(\rho^{-1} v^k)\right) \\
&= -\left(c_{ijkl} \frac{\partial u^i}{\partial x_j},\, \left(\frac{\partial}{\partial x_l}\rho^{-1}\right) v^k\right) + (v,\, u)\,.
\end{aligned}$$

Noting that $|\partial\rho^{-1}/\partial x_l| \leqq \rho_0^2/\rho_m^4$, we obtain

$$\begin{aligned}
|(\mathscr{A}U,\, U)_{\mathscr{H}}| &\leqq c' |u|_{\boldsymbol{H}^1(\Omega)} |v|_{\boldsymbol{L}^2(\Omega)} \leqq c''(|u|^2_{\boldsymbol{H}^1(\Omega)} + |v|^2_{\boldsymbol{L}^2(\Omega)}) \\
&\leqq \frac{c_1}{2} |U|^2_{\mathscr{H}}.
\end{aligned}$$

LEMMA 2.4. *For any real λ such that $|\lambda| \geqq c_1$, the estimate*

$$(2.\,11)\qquad |(\lambda I - \mathscr{A})U|_{\mathscr{H}} \geqq (|\lambda| - c_1)\,|U|_{\mathscr{H}}$$

holds for any $U \in D(\mathscr{A})$.

PFOOF By Lemma 2.3

$$\begin{aligned}
|(\lambda I - \mathscr{A})U|^2_{\mathscr{H}} &\geqq \lambda^2 |U|^2_{\mathscr{H}} - 2|\lambda|\, |(\mathscr{A}U,\, U)_{\mathscr{H}}| \\
&\geqq (\lambda^2 - |\lambda| c_1)\, |U|^2_{\mathscr{H}} \\
&\geqq (|\lambda| - c_1)^2 |U|^2_{\mathscr{H}}.
\end{aligned}$$

LEMMA 2.5. *There exists a constant c_2 such that for all real λ satisfying $|\lambda| \geqq c_2$, $\lambda I - \mathscr{A}$ is a mapping from $D(\mathscr{A})$ onto $\mathscr{H}$.*

REMARK. By Lemma 2.4, such λ belongs to the resolvent set of $\mathscr{A}$ and $|(\lambda I - \mathscr{A})^{-1}| \leqq (|\lambda| - c_2)^{-1}$ holds.

PROOF OF LEMMA 2.5. Take $F = \begin{bmatrix} f \\ g \end{bmatrix} \in \mathscr{H}$ and consider the equation

$$(\lambda I - \mathscr{A})U = F.$$

If $U = \begin{bmatrix} u \\ v \end{bmatrix}$, then this equation is equivalent to

$$\begin{cases} \lambda u - v = f \\ -\rho^{-1} A u + \lambda v = g. \end{cases} \tag{2.12}$$

By substituting the first relation in the second equation, we have

$$-Au + \rho\lambda^2 u = \rho g + \lambda \rho f. \tag{2.12'}$$

Let us put

$$B[\phi, \psi] = \left(c_{ijkl}\frac{\partial \phi^k}{\partial x_l}, \frac{\partial \psi^i}{\partial x_j}\right) + \lambda^2(\rho\phi, \psi) \tag{2.13}$$

for $\phi, \psi \in \boldsymbol{K}(\Omega)$. Then for $\lambda \neq 0$, B is a coercive bilinear form on $\boldsymbol{K}(\Omega)$, that is, there exists a constant $\delta > 0$ such that $B[\phi, \phi] \geqq \delta |\phi|^2_{\boldsymbol{H}^1(\Omega)}$ holds for any $\phi \in \boldsymbol{K}(\Omega)$. By Lax-Milgram's theorem there exists a unique function u in $\boldsymbol{K}(\Omega)$ satisfying $B[u, \phi] = (\rho g + \lambda \rho f, \phi)$ for any $\phi \in \boldsymbol{K}(\Omega)$. In particular, taking ϕ in $C_0^\infty(\Omega)^3$, we obtain the equality

$$-Au + \lambda^2 \rho u = \rho g + \lambda \rho f \quad \text{in} \quad \Omega$$

and also $Au = -\rho g - \lambda \rho f + \lambda^2 \rho u \in \boldsymbol{L}^2(\Omega)$. Thus we have, for any $\phi \in \boldsymbol{K}(\Omega)$

$$-(Au, \phi) + \lambda^2(\rho u, \phi) = B[u, \phi].$$

Hence

$$-(Au, \phi) = \left(c_{ijkl}\frac{\partial u^k}{\partial x_l}, \frac{\partial \phi^i}{\partial x_j}\right).$$

This equation means that u satisfies the boundary condition

$$n_j c_{ijkl}\frac{\partial u^k}{\partial x_l} = 0 \quad \text{weakly on} \quad \Gamma_2.$$

Put $v=\lambda u-f$. Then $v\in\boldsymbol{K}(\Omega)$. Thus $U=\begin{bmatrix} u \\ v \end{bmatrix}\in D(\mathscr{A})$ and $(\lambda I-\mathscr{A})U=F$ holds.

Now in order to apply the semi-group theory, that is, the theorem of Hille-Yosida, we have only to see that $D(\mathscr{A})$ is dense in $\mathscr{H}$. This will be proved in the following two lemmas.

LEMMA 2.6. *For any $u_0\in C^\infty(\bar{\Omega})^3$ which vanishes in a neighborhood of $\overline{\Gamma_1\cup L}$, there exists a sequence of functions $\{u_m(x)\}$ in $\boldsymbol{H}^2(\Omega)$ satisfying the following conditions.*

(i) $u_m=0$ *on a neighborhood of* $\overline{\Gamma_1\cup L}$,
(ii) $n_jc_{ijkl}(\partial u_m^k/\partial x_l)=0$ *on* Γ_2,
(iii) $u_m\longrightarrow u_0$ *in* $\boldsymbol{H}^1(\Omega)$,
(iv) $u_m-u_0\in\boldsymbol{H}_0^1(\Omega)$.

PROOF. Let Ω_0 be a domain with smooth boundary such that it is contained in Ω and contains the intersection of the support of u_0 with Ω and there exists a neighborhood of $\overline{\Gamma_1\cup L}$ disjoint from Ω_0. Now take $f\in C^\infty(\bar{\Omega}_0)^3$, and consider the boundary value problem

$$(2.14)\qquad \begin{cases} A^*Au+u=f & \text{in}\quad \Omega_0,\\[4pt] n_jc_{ijkl}\dfrac{\partial u^k}{\partial x_l}=0 & \text{on}\quad \partial\Omega_0,\\[4pt] u=u_0 & \text{on}\quad \partial\Omega_0,\end{cases}$$

where $A^*=$the formal adjoint of $A\,(=A)$. Choose $\tilde{u}\in C^\infty(\bar{\Omega}_0)^3$ such that $n_jc_{ijkl}[\partial\tilde{u}^k/\partial x_l]=0$ and $\tilde{u}=u_0$ on $\partial\Omega_0$. Then the solution of (2.14) is of the form $u=\tilde{u}+v$ with a solution v of the problem

$$(2.15)\qquad \begin{cases} A^*Av+v=f-(A^*A+1)\tilde{u} & \text{in}\quad \Omega_0,\\[4pt] n_jc_{ijkl}\dfrac{\partial v^k}{\partial x_l}=0 & \text{on}\quad \partial\Omega_0,\\[4pt] v=0 & \text{on}\quad \partial\Omega_0.\end{cases}$$

Because of the ellipticity of the boundary value problem (see e.g. [7, Chapitre 2])

$$\begin{cases} Au=f & \text{in}\quad \Omega_0\qquad (f\in\boldsymbol{L}^2(\Omega_0)),\\[4pt] n_jc_{ijkl}\dfrac{\partial u^k}{\partial x_l}=0 & \text{on}\quad \partial\Omega_0,\end{cases}$$

the inequality

$$|u|_{\boldsymbol{H}^2(\Omega_0)} \leqq \text{const.}(|Au|_{\boldsymbol{L}^2(\Omega_0)} + |u|_{\boldsymbol{L}^2(\Omega_0)})$$

holds for any $u \in \boldsymbol{H}_0^1(\Omega_0)$. Hence the bilinear form on $\boldsymbol{H}_0^2(\Omega_0)$

$$a(u, v) \equiv (Au, Av)_{\boldsymbol{L}^2(\Omega_0)} + (u, v)_{\boldsymbol{L}^2(\Omega_0)}$$

defines a norm equivalent to the standard $\boldsymbol{H}^2(\Omega_0)$-norm. Let $\boldsymbol{H}_0^2(\Omega_0)'$ be the adjoint space of $\boldsymbol{H}_0^2(\Omega_0)$. Since $f-(A^*A+1)\tilde{u} \in C^\infty(\bar{\Omega}_0)^3 \subset \boldsymbol{H}_0^2(\Omega_0)'$, there exists a unique function $v \in \boldsymbol{H}_0^2(\Omega_0)$ such that $a(v, \phi)=(f-(A^*A+1)\tilde{u}, \phi)$ holds for any $\phi \in \boldsymbol{H}_0^2(\Omega_0)$. Taking $\phi \in C_0^\infty(\Omega_0)^3$, we see that v is a solution of (2.15), and hence $u=\tilde{u}+v$ is a solution of (2.14). Moreover, since $a(\phi, \psi)= ((A^*A+1)\phi, \psi)$ for any $\phi \in C_0^\infty(\Omega_0)^3$ and $\psi \in \boldsymbol{H}_0^2(\Omega_0)$, we obtain

$$\begin{aligned} |u-(\tilde{u}+\phi)|^2_{\boldsymbol{H}^2(\Omega_0)} &\leqq c\, a(v-\phi, v-\phi) \\ &= c(f-(A^*A+1)\tilde{u}-(A^*A+1)\phi, v-\phi) \\ &\leqq c|f-(A^*A+1)\tilde{u}-(A^*A+1)\phi|_{\boldsymbol{H}_0^2(\Omega_0)'} \\ &\qquad \times |v-\phi|_{\boldsymbol{H}^2(\Omega_0)}. \end{aligned}$$

Thus

$$(2.16) \qquad |u-(\tilde{u}+\phi)|_{\boldsymbol{H}_0^2(\Omega_0)} \leqq c|f-(A^*A+1)\tilde{u}-(A^*A+1)\phi|_{\boldsymbol{H}_0^2(\Omega_0)'}.$$

Now, $\tilde{u}-u_0 \in \boldsymbol{H}_0^1(\Omega_0)$, since $\tilde{u}-u_0=0$ on $\partial\Omega_0$. Therefore, we can choose $\phi_m \in C_0^\infty(\Omega_0)^3$ $(m=1, 2, 3,\dots)$ satisfying

$$(2.17) \qquad |\phi_m+\tilde{u}-u_0|_{\boldsymbol{H}^1(\Omega_0)} \leqq \frac{1}{m}.$$

Furthermore, there exists $f_m \in C^\infty(\bar{\Omega}_0)^3$ such that

$$|f_m-(A^*A+1)\tilde{u}-(A^*A+1)\phi_m|_{\boldsymbol{H}_0^2(\Omega_0)'} \leqq \frac{1}{m}.$$

Now, let u_m be the solution of (2.14) with $f=f_m$. Then we obtain by (2.16)

$$(2.18) \qquad |u_m-(\tilde{u}+\phi_m)|_{\boldsymbol{H}^2(\Omega_0)} \leqq \frac{c}{m}.$$

Extend u_m to Ω by setting 0 outside Ω_0. Then u_m belongs to $\boldsymbol{H}^2(\Omega_0)$, since

$$n_j c_{ijkl}[\partial u_m^k/\partial x_l] = 0 \qquad \text{on} \quad \partial\Omega_0$$

and

$$u_m = 0 \qquad \text{on} \quad \Omega \cap \partial\Omega_0.$$

Thus (2.17) and (2.18) imply that

$$|u_m - u_0|_{\boldsymbol{H}^1(\Omega)} = |u_m - u_0|_{\boldsymbol{H}^1(\Omega_0)}$$
$$\leqq |u_m - (\tilde{u} + \phi_m)|_{\boldsymbol{H}^2(\Omega_0)} + |\tilde{u} + \phi_m - u_0|_{\boldsymbol{H}^1(\Omega_0)}$$
$$\leqq \frac{c}{m}.$$

Condition (iii) follows from this inequality and it is easy to see that condition (iv) holds from the fact that u_m satisfies the boundary condition $u_m = u_0$ on $\partial\Omega$.

LEMMA 2.7. *$D(\mathscr{A})$ is dense in $\mathscr{H}$.*

PROOF. For each $u_0 \in C^\infty(\bar{\Omega}_0)^3$ which vanishes on a neighborhood of $\overline{\Gamma_1 \cup L}$, let us take the functions u_m obtained in Lemma 2.6. Let $\tilde{u} \in C^\infty(\bar{\Omega})^3$ be a function satisfying $\tilde{u} = u_0$ on $\partial\Omega$ and $n_j c_{ijkl}[\partial\tilde{u}^k/\partial x_l] = 0$ on $\partial\Omega$. Then each $u_m - \tilde{u}$ belongs to $\boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^2(\Omega)$. Hence we can take $v_{mn} \in C_0^\infty(\Omega)$ $(n = 1, 2, \ldots)$ such that $v_{mn} \to u_m - \tilde{u}$ as $n \to \infty$ in $\boldsymbol{H}^1(\Omega)$. If we put $w_m = \tilde{u} + v_{mm}$, then w_m belongs to $C^\infty(\bar{\Omega})^3$ and

$$w_m = u_0 \quad \text{on} \quad \partial\Omega, \qquad n_j c_{ijkl}\frac{\partial w_m^k}{\partial x_l} = 0 \quad \text{on} \quad \Gamma_2,$$
$$w_m \to u_0 \qquad \text{in} \quad \boldsymbol{H}^1(\Omega).$$

In view of the definition of $\boldsymbol{K}(\Omega)$ and $D(\mathscr{A})$, this completes the proof.

By the preceding lemmas, we can apply the semi-group theory and complete the proof of Theorem 2.2 as follows.

In the equation (2.8), $\tilde{F}(t) = \begin{bmatrix} 0 \\ \tilde{F}(t)/\rho \end{bmatrix}$ is in $D(\mathscr{A})$, $\tilde{F}(t)$, $\mathscr{A}\tilde{F}(t)$ are in $\mathscr{E}_t^0(\mathscr{H})[0, T]$ and $V_0 = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}$ is in $D(\mathscr{A})$. By the theorem of Hille-Yosida (see e.g. [11, Chap. IX]), there is a unique solution $V(t) = \begin{bmatrix} v(t) \\ \tilde{v}(t) \end{bmatrix} \in D(\mathscr{A}) \cap \mathscr{E}_t^1(\mathscr{H})$ $[0, T]$ of the equation

$$\frac{d}{dt}V(t) = \mathscr{A}V(t) + \tilde{F}(t) \qquad \text{in} \ \ 0 < t < T, \tag{2.19}$$

with the initial condition $V(0) = V_0$. The equation (2.19) is equivalent to (2.8). Since $\tilde{v} = \partial v/\partial t$ and the weak boundary condition holds, we see that $v(x, t)$ is the unique solution of (2.8).

THEOREM 2.8 (the energy inequality). *Let $u(x, t)$ be the solution obtained in Theorem 2.2 with $g(x, t) = 0$. Then the energy inequality*

$$|u(t)|_{\boldsymbol{H}^1(\Omega)} + |[\partial u/\partial t](t)|_{\boldsymbol{L}^2(\Omega)}$$
$$\leqq C(T)\left(|u(0)|_{\boldsymbol{H}^1(\Omega)} + |[\partial u/\partial t](0)|_{\boldsymbol{L}^2(\Omega)} + \int_0^T |f(s)|_{\boldsymbol{L}^2(\Omega)} ds\right)$$

holds for any $0<t<T$, *where* $C(T)$ *is a constant not depending on* t, $u(x, t)$ *and* $f(x, t)$.

PROOF. If we put $V(t)=\begin{bmatrix} u(t) \\ [\partial u/\partial t](t) \end{bmatrix}$ and $F(t)=\begin{bmatrix} 0 \\ f(t)/\rho \end{bmatrix}$, by Lemma 2.3 it is easily seen that

$$\frac{d}{dt}|V(t)|^2_{\mathscr{H}} = 2(V(t), \frac{d}{dt}V(t))_{\mathscr{H}} = 2(V(t), \mathscr{A}V(t) + F(t))_{\mathscr{H}}$$
$$\leqq c_1|V(t)|^2_{\mathscr{H}} + 2|V(t)|_{\mathscr{H}}|F(t)|_{\mathscr{H}}$$

and from this it follows that

$$|V(t)|_{\mathscr{H}} \leqq e^{(c_1/2)t}|V(0)|_{\mathscr{H}} + \int_0^t e^{(c_1/2)(t-s)}|F(s)|_{\mathscr{H}} ds$$

Hence

$$|u(t)|_{\boldsymbol{H}^1(\Omega)} + |[\partial u/\partial t](t)|_{\boldsymbol{L}^2(\Omega)}$$
$$\leqq C(T)\left(|u(0)|_{\boldsymbol{H}^1(\Omega)} + |[\partial u/\partial t](0)|_{\boldsymbol{L}^2(\Omega)} + \int_0^T |f(s)|_{\boldsymbol{L}^2(\Omega)} ds\right).$$

3. The domain of dependence inequality

In this section we show the domain of dependence inequality. Our method of the proof is due to C. H. Wilcox [10].

THEOREM 3.1. *For* $v_0(x) \in \boldsymbol{K}(\Omega)$ *and* $v_1(x) \in \boldsymbol{L}^2(\Omega)$, *let* $v(x, t)$ *be a solution in* $\mathscr{E}^1_t(\boldsymbol{K}(\Omega))[0, T] \cap \mathscr{E}^2_t(\boldsymbol{L}^2(\Omega))[0, T]$ *of the initial-boundary value problem*

$$(3.1)\qquad \begin{cases} \rho\dfrac{\partial^2 v}{\partial t^2} - Av = 0 & \text{in } Q \equiv \Omega \times (0, T), \\ v(x, 0) = v_0(x) & \text{in } \Omega, \\ [\partial v/\partial t](x, 0) = v_1(x) & \text{in } \Omega, \\ n_j c_{ijkl}[\partial v^k/\partial x_l](x, t) = 0 & \text{weakly on } \Gamma_2 \text{ for each } t \in (0, T). \end{cases}$$

Then the following inequality holds:

$$\int_{S_r(x_0)\cap\Omega} \left\{\rho\left(\frac{\partial v^i}{\partial t}\right)^2 + c_{ijkl}\frac{\partial v^i}{\partial x_j}\frac{\partial v^k}{\partial x_l}\right\}\Big|_{t=t_1} dx$$

$$\leqq \int_{S_{r+c_1|t_1-t_0|}(x_0)\cap\Omega} \left\{\rho\left(\frac{\partial v^i}{\partial t}\right)^2 + c_{ijkl}\frac{\partial v^i}{\partial x_j}\frac{\partial v^k}{\partial x_l}\right\}\bigg|_{t=t_0} dx$$

for $0 \leqq t_0,\ t_1 \leqq T$. *Here*

$$S_r(x_0) = \{x \in \boldsymbol{R}^3 \mid |x - x_0| \leqq r\},$$

$$c_1^2 = \sup_{x\in\Omega\ \xi,\eta\in\Sigma} \frac{c_{ijkl}(x)\eta_i\eta_k\xi_j\xi_l}{\rho(x)},\ \Sigma = \{\xi \in \boldsymbol{R}^3 \mid |\xi| = 1\}.$$

Proof. We consider the case $t_1 > t_0$ because in the case $t_1 < t_0$ the inequality is proved in the same way.

Let us denote

$$\Omega_1 = S_r(x_0) \cap \Omega, \quad \Omega_0 = S_{r+c_1(t_1-t_0)}(x_0) \cap \Omega,$$

V=the subregion of the cone $\{(x, t) \mid |x - x_0| < c_1|t - t_1 - (r/c_1)|\}$ bounded by $\Omega_0 \times \{t = t_0\}$, $\Omega_1 \times \{t = t_1\}$ and $\partial\Omega \times (0, T)$.

If we put $\psi(x) = c_1^{-1}(r - |x - x_0|) + t_1$, then $V = \{(x, t) \in Q \mid \psi(x) - t > 0,\ t_0 < t < t_1\}$. Let us put $\phi(x, t) = \phi_\delta(\psi(x) - t)$ with $\phi_\delta \in C^\infty(\boldsymbol{R}^1)$ such that $\phi_\delta(\tau) = 0$ for $\tau \leqq -\delta$, $\phi_\delta(\tau) = 1$ for $\tau \geqq \delta$, $\phi_\delta'(\tau) \geqq 0$ and $0 \leqq \phi_\delta(\tau) \leqq 1$ for all $\tau \in \boldsymbol{R}^1$. If $\delta > 0$ is sufficiently small, then $\phi(x, t)$ is in $C^\infty(\Omega \times [t_0, t_1])$. Multiply the equation $\rho[\partial^2 v/\partial t^2] - Av = 0$ by $\phi[\partial v/\partial t]$, and integrate over $\Omega \times (t_0, t_1)$. Because $\phi[\partial v/\partial t] \in \mathscr{E}_t^0(\boldsymbol{K}(\Omega))[0, T]$ and $n_j c_{ijkl}[\partial v^k/\partial x_l] = 0$ weakly on $\Gamma_2 \times (0, T)$,

$$\begin{aligned}
0 &= \int_{\Omega\times(t_0,t_1)} \left(\rho\frac{\partial^2 v}{\partial t^2} - Av\right)\cdot\phi\frac{\partial v}{\partial t}\,dxdt \\
&= \int_{\Omega\times(t_0,t_1)} \frac{1}{2}\frac{\partial}{\partial t}\left[\rho\left(\frac{\partial v^i}{\partial t}\right)^2\right]\phi\,dxdt \\
&\qquad + \int_{\Omega\times(t_0,t_1)} c_{ijkl}\frac{\partial v^k}{\partial x_l}\frac{\partial}{\partial x_j}\left(\phi\frac{\partial v^i}{\partial t}\right)dxdt \\
&= \int_{\Omega\times(t_0,t_1)} \frac{1}{2}\frac{\partial}{\partial t}\left[\rho\left(\frac{\partial v^i}{\partial t}\right)^2\right]\phi\,dxdt \\
&\qquad + \int_{\Omega\times(t_0,t_1)} \frac{1}{2}\frac{\partial}{\partial t}\left[c_{ijkl}\frac{\partial v^k}{\partial x_l}\frac{\partial v^i}{\partial x_j}\right]\phi\,dxdt \\
&\qquad + \int_{\Omega\times(t_0,t_1)} c_{ijkl}\frac{\partial v^k}{\partial x_l}\frac{\partial \phi}{\partial x_j}\frac{\partial v^i}{\partial t}\,dxdt \\
&= \int_\Omega \frac{1}{2}\left[\rho\left(\frac{\partial v^i}{\partial t}\right)^2 + c_{ijkl}\frac{\partial v^k}{\partial x_l}\frac{\partial v^i}{\partial x_j}\right]\phi\,\bigg|_{t=t_0}^{t=t_1} dx \\
&\qquad - \frac{1}{2}\int_{\Omega\times(t_0,t_1)} \left[\rho\left(\frac{\partial v^i}{\partial t}\right)^2 + c_{ijkl}\frac{\partial v^k}{\partial x_l}\frac{\partial v^i}{\partial x_j}\right]\frac{\partial \phi}{\partial t}\,dxdt
\end{aligned}$$

$$+\int_{\Omega\times(t_0,t_1)} c_{ijkl}\frac{\partial v^k}{\partial x_l}\frac{\partial v^i}{\partial t}\frac{\partial\phi}{\partial x_j}dxdt.$$

By the definition of c_1,

$$\{\rho\delta_{ik}-c_{ijkl}n_jn_l\}\xi_i\xi_k\geqq 0 \qquad \text{for any} \quad \xi=(\xi_1,\xi_2,\xi_3)\in\boldsymbol{R}^3,$$

where $n_j=\partial\psi/\partial x_j=-c_1^{-1}(x_j-x_{0j})/|x-x_0|$. By substituting $\xi_j=\partial v^j/\partial t$, the inequality

$$\rho\left(\frac{\partial v^i}{\partial t}\right)^2\geqq c_{ijkl}n_jn_l\frac{\partial v^i}{\partial t}\frac{\partial v^k}{\partial t}$$

holds. Since $\partial\phi/\partial x_j=n_j\phi'_\delta$ and $\partial\phi/\partial t=-\phi'_\delta$, we have

$$\begin{aligned}
&-\left[\rho\left(\frac{\partial v^i}{\partial t}\right)^2+c_{ijkl}\frac{\partial v^k}{\partial x_l}\frac{\partial v^i}{\partial x_j}\right]\frac{\partial\phi}{\partial t}+2c_{ijkl}\frac{\partial v^k}{\partial x_l}\frac{\partial v^i}{\partial t}\frac{\partial\phi}{\partial x_j}\\
&=\left[\rho\left(\frac{\partial v^i}{\partial t}\right)^2+c_{ijkl}\frac{\partial v^k}{\partial x_l}\frac{\partial v^i}{\partial x_j}\right]\phi'_\delta+2c_{ijkl}\frac{\partial v^k}{\partial x_l}\frac{\partial v^i}{\partial t}n_j\phi'_\delta\\
&\geqq\phi'_\delta\left[c_{ijkl}n_jn_l\frac{\partial v^i}{\partial t}\frac{\partial v^k}{\partial t}+2n_jc_{ijkl}\frac{\partial v^k}{\partial x_l}\frac{\partial v^i}{\partial t}+c_{ijkl}\frac{\partial v^k}{\partial x_l}\frac{\partial v^i}{\partial x_j}\right]\\
&=\phi'_\delta c_{ijkl}\left(\frac{\partial v^k}{\partial x_l}+n_l\frac{\partial v^k}{\partial t}\right)\left(\frac{\partial v^i}{\partial x_j}+n_j\frac{\partial v^i}{\partial t}\right)\geqq 0.
\end{aligned}$$

Therefore

$$\int_\Omega\left[\rho\left(\frac{\partial v^i}{\partial t}\right)^2+c_{ijkl}\frac{\partial v^k}{\partial x_l}\frac{\partial v^i}{\partial x_j}\right]\phi\Big|_{t=t_0}^{t=t_1}dx\leqq 0.$$

As $\delta\to 0$, $\phi(x,t)=\phi_\delta(\psi(x)-t)\to\chi_V$, the characteristic function of V, boundedly. Therefore

$$\begin{aligned}
&\int_\Omega\left[\rho\left(\frac{\partial v^i}{\partial t}\right)^2+c_{ijkl}\frac{\partial v^k}{\partial x_l}\frac{\partial v^i}{\partial x_j}\right]\phi\Big|_{t=t_\nu}dx\\
&\qquad\to\int_{\Omega_\nu}\left[\rho\left(\frac{\partial v^i}{\partial t}\right)^2+c_{ijkl}\frac{\partial v^k}{\partial x_l}\frac{\partial v^i}{\partial x_j}\right]\Big|_{t=t_\nu}dx \quad (\nu=1,2).
\end{aligned}$$

This implies

$$\begin{aligned}
&\int_{\Omega_0}\left[\rho\left(\frac{\partial v^i}{\partial t}\right)^2+c_{ijkl}\frac{\partial v^k}{\partial x_l}\frac{\partial v^i}{\partial x_j}\right]\Big|_{t=t_0}dx\\
&\geqq\int_{\Omega_1}\left[\rho\left(\frac{\partial v^i}{\partial t}\right)^2+c_{ijkl}\frac{\partial v^k}{\partial x_l}\frac{\partial v^i}{\partial x_j}\right]\Big|_{t=t_1}dx.
\end{aligned}$$

4. Non-controllability at a short time

Let Γ_0 be a relatively open set in $\partial\Omega$, contained in Γ_2, with smooth boundary. To state the definition of controllability precisely, we introduce the energy space $H_E(\Omega)$. It is the space $\boldsymbol{K}(\Omega)\times\boldsymbol{L}^2(\Omega)$ with the inner product

$$([u, u'], [v, v'])_E = (\rho u', v') + \left(c_{ijkl}\frac{\partial u^i}{\partial x_j}, \frac{\partial v^k}{\partial x_l}\right)$$

for $[u, u'], [v, v'] \in \boldsymbol{K}(\Omega)\times\boldsymbol{L}^2(\Omega)$. By the definition of $\boldsymbol{K}(\Omega)$, it is easy to see that $|\cdot|_E=(\cdot, \cdot)_E^{1/2}$ defines a norm equivalent to the norm in $\boldsymbol{K}(\Omega)\times\boldsymbol{L}^2(\Omega)$. Let

$$\mathscr{F} = \{f\in C^\infty(\Gamma_2 \times (0, T))^3 |\ \mathrm{supp} f \subset \Gamma_0 \times (0, T)\}.$$

By Theorem 2.2, for given $f\in\mathscr{F}$ there is a solution $u(x, t)$ in $\mathscr{E}_t^1(\boldsymbol{K}(\Omega))[0, T] \cap \mathscr{E}_t^2(\boldsymbol{L}^2(\Omega))[0, T]$ of the initial-boundary value problem

$$(4.1)\quad \begin{cases} \rho\dfrac{\partial^2 u}{\partial t^2} - Au = 0 \quad \text{in} \quad \Omega\times(0, T), \\ n_j c_{ijkl}[\partial u^k/\partial x_l](x, t) = f^i(x, t) \quad \text{weakly on } \Gamma_2 \text{ for each } t\in(0, T), \\ u(x, 0) = 0 \quad \text{in} \quad \Omega, \\ [\partial u/\partial t](x, 0) = 0 \quad \text{in} \quad \Omega, \end{cases}$$

where $A=\left(\dfrac{\partial}{\partial x_j}\left(c_{ijkl}\dfrac{\partial}{\partial x_l}\right)\right)_{i,k=1,2,3}$. Then $[u(t), (\partial u/\partial t)(t)] \in \mathscr{E}_t^1(H_E(\Omega))[0, T]$. We define the reachable set R_T by

$$(4.2)\quad R_T = \{[u(T), (\partial u/\partial t)(T)]\ |u\colon \text{solution of (4.1)}, f\in\mathscr{F}\}.$$

DEFINITION 4.1. *When the reachable set R_T is dense in $H_E(\Omega)$, the system (4.1) is said to be controllable.*

THEOREM 4.2. *For $[g, h]\in D(\mathscr{A})$, let $v(x, t)$ be the solution in $\mathscr{E}_t^1(\boldsymbol{K}(\Omega))[0, T] \cap \mathscr{E}_t^2(\boldsymbol{L}^2(\Omega))[0, T]$ of the initial-boundary value problem*

$$(4.3)\quad \begin{cases} \rho\dfrac{\partial^2 v}{\partial t^2} - Av = 0 \quad \textit{in} \quad Q\equiv\Omega\times(0, T), \\ v(x, T) = g(x) \quad \textit{in} \quad \Omega, \\ \dfrac{\partial v}{\partial t}(x, T) = h(x) \quad \textit{in} \quad \Omega, \end{cases}$$

$$\begin{cases} v(x, t) = 0 & on \quad \Gamma_1 \times (0, T), \\ n_j c_{ijkl} \dfrac{\partial v^k}{\partial x_l}(x, t) = 0 & weakly\ on\ \Gamma_2\ for\ each\ t \in (0, T). \end{cases}$$

Then $[g, h] \in R_T^{\perp}$ (*=the orthogonal complement of* R_T *in* $H_E(\Omega)$) *if and only if*

$$\frac{\partial v}{\partial t} = 0 \quad almost\ everywhere\ on \quad \Gamma_0 \times (0, T).$$

PROOF. Let $u(x, t)$ be the solution of (4.1). Then

$$\begin{aligned} (4.4) \qquad & ([g, h], [u(T), (\partial u/\partial t)(T)])_E \\ & = \int_\Omega \left(\rho \frac{\partial v^i}{\partial t} \frac{\partial u^i}{\partial t} + c_{ijkl} \frac{\partial v^i}{\partial x_j} \frac{\partial u^k}{\partial x_l} \right) \Big|_{t=0}^{t=T} dx \end{aligned}$$

and

$$\begin{aligned} 0 &= \int_Q \left\{ \frac{\partial v}{\partial t} \cdot \left(\rho \frac{\partial^2 u}{\partial t^2} - Au \right) + \left(\rho \frac{\partial^2 v}{\partial t^2} - Av \right) \cdot \frac{\partial u}{\partial t} \right\} dxdt \\ &= \int_\Omega \rho \frac{\partial v^i}{\partial t} \frac{\partial u^i}{\partial t} \Big|_{t=0}^{t=T} dx - \int_Q \left(\frac{\partial v}{\partial t} \cdot Au + Av \cdot \frac{\partial u}{\partial t} \right) dxdt. \end{aligned}$$

Since $\partial u/\partial t, \partial v/\partial t \in \mathscr{E}_t^0(\boldsymbol{K}(\Omega))[0, T]$,

$$\begin{aligned} & - \int_Q \left(\frac{\partial v}{\partial t} \cdot Au + Av \cdot \frac{\partial u}{\partial t} \right) dxdt \\ &= \int_Q c_{ijkl} \frac{\partial u^k}{\partial x_l} \frac{\partial^2 v^i}{\partial x_j \partial t} dxdt - \int_{\Gamma_2 \times (0,T)} f \cdot \frac{\partial v}{\partial t} dSdt \\ & \qquad + \int_Q c_{ijkl} \frac{\partial v^k}{\partial x_l} \frac{\partial^2 u^i}{\partial x_j \partial t} dxdt \\ &= \int_Q c_{ijkl} \frac{\partial u^k}{\partial x_l} \frac{\partial v^i}{\partial x_j} \Big|_{t=0}^{t=T} dx - \int_{\Gamma_2 \times (0,T)} f \cdot \frac{\partial v}{\partial t} dSdt. \end{aligned}$$

Hence

$$(4.5) \quad 0 = \int_\Omega \left(\rho \frac{\partial v^i}{\partial t} \frac{\partial u^i}{\partial t} + c_{ijkl} \frac{\partial v^i}{\partial x_j} \frac{\partial u^k}{\partial x_l} \right) \Big|_{t=0}^{t=T} dx - \int_{\Gamma_2 \times (0,T)} f \cdot \frac{\partial v}{\partial t} dSdt.$$

Thus by (4.4) and (4.5)

$$(4.6) \qquad ([g, h], [u(T), (\partial u/\partial t)(T)])_E = \int_{\Gamma_2 \times (0,T)} f \cdot \frac{\partial v}{\partial t} dSdt.$$

Hence $[g, h]$ belongs to $R_T^{\perp}$ if and only if

$$\int_{\Gamma_2\times(0,T)} f\cdot\frac{\partial v}{\partial t}dSdt=0 \tag{4.7}$$

for any $f\in\mathscr{F}$. Because $\partial v/\partial t$ is in $\mathscr{E}_t^0(\boldsymbol{K}(\Omega))[0, T]$, the trace $(\partial v/\partial t)|_{\Gamma_2}$ is in $\mathscr{E}_t^0(\boldsymbol{L}^2(\Gamma_2))$. Thus $\partial v/\partial t\in\boldsymbol{L}^2(\Gamma_2\times(0, T))$, and hence (4.7) holds if and only if $\partial v/\partial t=0$ almost everywhere on $\Gamma_0\times(0, T)$.

To state non controllability, we introduce some notations. Put

$$c_1^2=\sup_{\substack{x\in\Omega\\ \xi,\eta\in\Sigma}}\frac{c_{ijkl}(x)\eta_i\eta_k\xi_j\xi_l}{\rho(x)},\quad \Sigma=\{\xi\in\boldsymbol{R}^3\,|\,|\xi|=1\}$$

as is defined in Theorem 3.1 and

$$c_2^2=\inf_{\substack{x\in\Omega\\ \xi,\eta\in\Sigma}}\frac{c_{ijkl}(x)\eta_i\eta_k\xi_j\xi_l}{\rho(x)}.$$

For $(x_0, t_0)\in\bar{Q}$, put

$$K_1^+(x_0, t_0)=\{(x, t)\in\bar{\Omega}\times[t_0, T]|c_1(t-t_0)-|x-x_0|\leqq 0\},$$

$$K_2^+(x_0, t_0)=\{(x, t)\in\bar{\Omega}\times[t_0, T]|c_2(t-t_0)-|x-x_0|\geqq 0\},$$

$$K_1^-(x_0, t_0)=\{(x, t)\in\bar{\Omega}\times[0, t_0]|c_1(t-t_0)-|x-x_0|\geqq 0\},$$

$$K_2^-(x_0, t_0)=\{(x, t)\in\bar{\Omega}\times[0, t_0]|c_2(t-t_0)-|x-x_0|\leqq 0\},$$

and for $G\subset\bar{\Omega}$, put

$$K_1^+(G; t_0)=\bigcap_{x_0\in G}K_1^+(x_0, t_0),\quad K_1^-(G; t_0)=\bigcap_{x_0\in G}K_1^-(x_0, t_0),$$

$$K_2^+(G; t_0)=\overline{\bigcup_{x_0\in G}K_2^+(x_0, t_0)},\quad K_2^-(G; t_0)=\overline{\bigcup_{x_0\in G}K_2^-(x_0, t_0)},$$

$$M(G; t_0, t_1)=K_2^+(G; t_0)\cap K_2^-(G; t_1),$$

$$N(G; t_0, t_1)=K_1^+(G; t_0)\cup K_1^-(G; t_1),$$

$$T_1=\inf\{t;\ \Omega(t)\cap K_1^+(\Gamma_0; 0)=\phi\},$$

$$T_2=\inf\{t;\ \Omega(t)\subset K_2^+(\Gamma_0; 0)\},$$

where $\Omega(t)=\Omega\times\{t\}$.

THEOREM 4.3. *At any time $T<2T_1$, the system* (4.1) *is not controllable.*

PROOF. If $T<2T_1$, then the interior in Ω of the projection of $J(T/2)\equiv$

$\Omega(T/2) \cap N(\Gamma_0; 0, T)$ to Ω is not empty. Choose $[v_0(x), v_1(x)] \in H_E(\Omega) \cap D(\mathscr{A})$ such that $|[v_0, v_1]|_E \neq 0$ and {the support of $[v_0, v_1]$} $\times \{t = T/2\}$ is contained in $J(T/2)$, and consider the problem

$$
(4.8) \quad \begin{cases} \rho \dfrac{\partial^2 v}{\partial t^2} - Av = 0 & \text{in} \quad Q, \\ v(x, T/2) = v_0(x) & \text{in} \quad \Omega, \\ (\partial v/\partial t)(x, T/2) = v_1(x) & \text{in} \quad \Omega, \\ v(x, t) = 0 & \text{on} \quad \Gamma_1 \times (0, T), \\ n_j c_{ijkl}(\partial v^k/\partial x_l)(x, t) = 0 & \text{weakly on } \Gamma_2 \text{ for each} \quad t \in (0, T). \end{cases}
$$

Since the solution $v(x, t)$ of (4.8) satisfies the equality

$$
\begin{aligned}
0 &= \left(\rho \frac{\partial^2 v}{\partial t^2} - Av, \frac{\partial v}{\partial t}\right) \\
&= \frac{\partial}{\partial t} \frac{1}{2}\left(\rho \frac{\partial v}{\partial t}, \frac{\partial v}{\partial t}\right) + \left(c_{ijkl} \frac{\partial v^k}{\partial x_l}, \frac{\partial^2 v^i}{\partial x_j \partial t}\right) \\
&= \frac{1}{2} \frac{\partial}{\partial t} |[v, \partial v/\partial t]|_E^2,
\end{aligned}
$$

the energy equality: $|[v(t), (\partial v/\partial t)(t)]|_E = |[v_0, v_1]|_E \neq 0$ holds for any $t \in [0, T]$. By Theorem 3.1, the domain of the influence of the energy does not intersect $M(\Gamma_0; 0, T)$ and thus $\partial v/\partial t = 0$ in $M(\Gamma_0; 0, T)$. Since $M(\Gamma_0; 0, T) \supset \Gamma_0 \times (0, T)$, $\partial v/\partial t = 0$ on $\Gamma_0 \times (0, T)$. If we put $g = v(T)$ and $h = (\partial v/\partial t)(T)$, then $[g, h]$ belongs to $R_T^\perp$ because of Theorem 4.2. Since $|[g, h]|_E = |[v_0, v_1]|_E \neq 0$, R_T is not dense in $H_E(\Omega)$.

5. Controllability at a sufficiently large time

To prove the controllability, first we solve the problem (4.3) for arbitrary $[g, h] \in \boldsymbol{K}(\Omega) \times \boldsymbol{L}^2(\Omega)$.

THEOREM 5.1. *For any* $[g, h] \in \boldsymbol{K}(\Omega) \times \boldsymbol{L}^2(\Omega)$, *there exists a unique solution* $v(x, t)$ *in* $\mathscr{E}_t^0(\boldsymbol{K}(\Omega))[0, T] \cap \mathscr{E}_t^1(\boldsymbol{L}^2(\Omega))[0, T] \cap \mathscr{E}_t^2(\boldsymbol{K}(\Omega)')[0, T]$ *of the initial-boundary value problem*

$$
(5.1) \quad \begin{cases} \rho \dfrac{\partial^2 v}{\partial t^2} - Av = 0 & \textit{in} \quad Q, \\ v(x, T) = g(x) & \textit{in} \quad \Omega, \\ [\partial v/\partial t](x, T) = h(x) & \textit{in} \quad \Omega, \end{cases}
$$

$$\begin{cases} -<Av, \phi> = \left(c_{ijkl}\dfrac{\partial v^k}{\partial x_l}, \dfrac{\partial \phi^i}{\partial x_j}\right) & \textit{at any} \quad t \in (0, T), \\ \textit{for any} \quad \phi \in \boldsymbol{K}(\Omega), \end{cases}$$

where $< \ , \ >$ *denotes the duality between* $\boldsymbol{K}(\Omega)'$ *and* $\boldsymbol{K}(\Omega)$.

PROOF. By virtue of Lemma 2.7, there exist $[g_n, h_n] \in D(\mathscr{A})$, $n=1, 2,\dots$, which converge to $[g, h]$ in $\boldsymbol{K}(\Omega)\times \boldsymbol{L}^2(\Omega)$. Let $v_n(t)$ be the solution in $\mathscr{E}_t^1(\boldsymbol{K}(\Omega))[0, T] \cap \mathscr{E}_t^2(\boldsymbol{L}^2(\Omega))[0, T]$ of (4.3) with $g=g_n$ and $h=h_n$. By Theorem 2.8, $v_n(t)$, $n=1, 2,\dots$, satisfy the inequality

$$\begin{aligned} (5.2) \qquad & |v_n(t) - v_m(t)|_{\boldsymbol{H}^1(\Omega)} + |(\partial v_n/\partial t)(t) - (\partial v_m/\partial t)(t)|_{\boldsymbol{L}^2(\Omega)} \\ & \leqq C(T)(|g_n - g_m|_{\boldsymbol{H}^1(\Omega)} + |h_n - h_m|_{\boldsymbol{L}^2(\Omega)}). \end{aligned}$$

Thus $\{v_n(t)\}$ is a Cauchy sequence in $\mathscr{E}_t^0(\boldsymbol{K}(\Omega))[0, T]$ and $\{(\partial v_n/\partial t)(t)\}$ is a Cauchy sequence in $\mathscr{E}_t^0(\boldsymbol{L}^2(\Omega))[0, T]$.

Since $v_n(t)$ is the solution of

$$(5.3) \qquad \frac{d}{dt}\begin{bmatrix} v_n(t) \\ \dfrac{\partial v_n}{\partial t}(t) \end{bmatrix} = \mathscr{A}\begin{bmatrix} v_n(t) \\ \dfrac{\partial v_n}{\partial t}(t) \end{bmatrix}$$

with the initial condition

$$\begin{bmatrix} v_n(T) \\ \dfrac{\partial v_n}{\partial t}(T) \end{bmatrix} = \begin{bmatrix} g_n \\ h_n \end{bmatrix} \qquad \text{in} \quad \Omega,$$

it follows that

$$(5.4) \qquad \begin{bmatrix} v_n(t) \\ \dfrac{\partial v_n}{\partial t}(t) \end{bmatrix} = \begin{bmatrix} g_n \\ h_n \end{bmatrix} + \int_T^t \mathscr{A}\begin{bmatrix} v_n(s) \\ \dfrac{\partial u_n}{\partial t}(s) \end{bmatrix} ds.$$

Let $v_n(t)$ converge to $v(t)$ in $\mathscr{E}_t^0(\boldsymbol{K}(\Omega))[0, T]$ and $(\partial v_n/\partial t)(t)$ converge to $w(t)$ in $\mathscr{E}_t^0(\boldsymbol{L}^2(\Omega))[0, T]$. Since $v_n(t)-v_m(t)$ is in $D(\mathscr{A})$, the equality

$$(5.5) \qquad -(A(v_n(t) - v_m(t)), \phi) = \left(c_{ijkl}\frac{\partial}{\partial x_l}(v_n^k(t) - v_m^k(t)), \frac{\partial \phi^i}{\partial x_j}\right)$$

holds for each $t \in (0, T)$ and $\phi \in \boldsymbol{K}(\Omega)$. Noting that the topology of $\boldsymbol{K}(\Omega)$ is induced by the $\boldsymbol{H}^1(\Omega)$-norm, by this equality we have

$$\begin{aligned} |A(v_n(t) - v_m(t))|_{\boldsymbol{K}(\Omega)'} &\leqq \sum_{i,j} |c_{ijkl}\frac{\partial}{\partial x_l}(v_n^k(t) - v_m^k(t))|_{\boldsymbol{L}^2(\Omega)} \\ &\leqq \text{const.}\, |v_n(t) - v_m(t)|_{\boldsymbol{K}(\Omega)}, \end{aligned}$$

where const. is a constant not depending on t, n and m. Thus $\{Av_n\}$ is a Cauchy sequence in $\mathscr{E}_t^0(\boldsymbol{K}(\Omega)')[0, T]$ and converges to $Av(t)$. Passage to the limit as $n\to\infty$ in (5.4) gives

$$\begin{bmatrix} v(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix} + \int_T^t \begin{bmatrix} w(s) \\ \rho^{-1}Av(s) \end{bmatrix} ds. \tag{5.6}$$

Since $\begin{bmatrix} w(t) \\ \rho^{-1}Av(t) \end{bmatrix}$ is in $\mathscr{E}_t^0(\boldsymbol{L}^2(\Omega))[0, T]\times\mathscr{E}_t^0(\boldsymbol{K}(\Omega)')[0, T]$, the equation (5.6) yields

$$\rho\frac{d}{dt}\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} \rho w \\ -Av \end{bmatrix} \quad \text{in} \quad \mathscr{E}_t^0(\boldsymbol{L}^2(\Omega))[0, T] \times \mathscr{E}_t^0(\boldsymbol{K}(\Omega)')[0, T]$$

with $\begin{bmatrix} v(T) \\ w(T) \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix}$. This means that $v(t)$ is in $\mathscr{E}_t^1(\boldsymbol{L}^2(\Omega))[0, T]\cap\mathscr{E}_t^2(\boldsymbol{K}(\Omega)')$ $[0, T]$ and satisfies the equation

$$\rho\frac{\partial^2 v}{\partial t^2}(t) = Av(t)$$

with the initial condition $v(T)=g$, $(\partial v/\partial t)(T)=h$.

Passage to the limit in (5.5) proves that $v(t)$ satisfies the weak boundary condition

$$<Av(t), \phi> = \left(c_{ijkl}\frac{\partial v^k}{\partial x_l}(t), \frac{\partial\phi^i}{\partial x_j}\right) \quad \text{at any} \quad t\in(0, T), \tag{5.7}$$

for any $\phi\in\boldsymbol{K}(\Omega)$.

LEMMA 5.2. *Suppose $u(x)\in\boldsymbol{K}(\Omega)$, $Au(x)\in\boldsymbol{L}^2(\Omega)$ and*

$$(Au, \phi) = \left(c_{ijkl}\frac{\partial u^k}{\partial x_l}, \frac{\partial\phi^i}{\partial x_j}\right) \quad \textit{for any} \quad \phi\in\boldsymbol{K}(\Omega).$$

Then $u(x)$ belongs to $\boldsymbol{H}^2_{loc}(\bar\Omega-L)$.

PROOF. Let us take any $\phi\in C^\infty(\bar\Omega)^3$ such that $(\operatorname{supp}\phi)\cap\partial\Omega\subset\Gamma_\nu$ ($\nu=1$ or 2). Then

$$A(\phi u)^i = \frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial\phi}{\partial x_l}u^k\right) + \frac{\partial\phi}{\partial x_j}c_{ijkl}\frac{\partial u^k}{\partial x_l} + \phi\frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial u^k}{\partial x_l}\right).$$

Therefore $A(\phi u)$ belongs to $\boldsymbol{L}^2(\Omega)$. Because the trace of u on $\partial\Omega$ vanishes on Γ_1, $\phi u=0$ on $\partial\Omega$ if $\nu=1$. By the ellipticity of A, ϕu belongs to $\boldsymbol{H}^2(\Omega)$ if $\nu=1$.

If $v=2$, then $u \in \boldsymbol{K}(\Omega)$ satisfies the boundary condition

$$n_j c_{ijkl} \frac{\partial(\phi u^k)}{\partial x_l} = n_j c_{ijkl} \frac{\partial \phi}{\partial x_l} u^k \qquad \text{on} \quad \partial\Omega.$$

Because $n_j c_{ijkl} \dfrac{\partial \phi}{\partial x_l} u|_{\partial\Omega} \in \boldsymbol{H}^{1/2}(\partial\Omega)$, ϕu belongs to $\boldsymbol{H}^2(\Omega)$ by the ellipticity of A (see e.g. [7, Chapitre 2]). This completes the proof.

Let S be a surface in four dimensional space-time defined by

$$S = \{(x, t) \in \boldsymbol{R}_x^3 \times \boldsymbol{R}_t^1 | \phi(x, t) = 0\},$$

where $\phi \in C^1(\boldsymbol{R}^3 \times \boldsymbol{R}^1)$.
If the matrix

$$\Lambda(x, \nabla \phi) = (\rho(\partial\phi/\partial t)^2 \delta_{ik} - c_{ijkl}(\partial\phi/\partial x_j)(\partial\phi/\partial x_l))_{i,k=1,2,3}$$

is uniformly negative definite, i.e., there exists a constant $\delta > 0$ such that

$$(\Lambda(x, \nabla \phi)\xi, \xi) \leqq -\delta|\xi|^2 \qquad \text{for any} \quad \xi \in \boldsymbol{R}^3,$$

then the surface S is called uniformly time-like. If $\det \Lambda(x, \nabla \phi) = 0$ for $(x, t) \in S$, the surface is called characteristic. Here δ_{ik} is Kronecker's delta.

From now on we suppose that $\partial\Omega$, $\rho(x)$ and $c_{ijkl}(x)$ are analytic. We state a result due to F. John [5] without proof.

LEMMA 5.3. *If $(\bar{x}, \bar{t})$ lies in the interior of $M(\Gamma_0; t_0, t_1)$ ($[t_0, t_1] \subset [0, T]$), then there is a family of uniformly time-like surfaces S_λ $(0 \leqq \lambda \leqq 1)$ with the following properties*:

(i) *S_λ is a compact subset of a relatively open analytic three dimensional surface.*

(ii) *S_λ varies analytically with respect to λ: $0 \leqq \lambda \leqq 1$.*

(iii) *$S_\lambda \subset M(\Gamma_0; t_0, t_1)$, $0 \leqq \lambda \leqq 1$, and S_0 is a subset of the interior of $\Gamma_0 \times [0, T]$.*

(iv) *If $0 \leqq \lambda \leqq 1$, then $S_0 \cup S_\lambda$ is the boundary of an open subset D_λ of $M(\Gamma_0; t_0, t_1)$ and $(\bar{x}, \bar{t}) \in D_1$.*

LEMMA 5.4. *For the solution $v(x, t) \in \mathscr{E}_t^1(\boldsymbol{K}(\Omega))[0, T] \cap \mathscr{E}_t^2(\boldsymbol{L}^2(\Omega))[0, T]$ of (4.3), $v(x, t) = 0$ almost everywhere in $\Gamma_0 \times [t_0, t_1]$ if and only if $v(x, t) = 0$ almost everywhere in $M(\Gamma_0; t_0, t_1)$.*

PROOF. The sufficiency is clear by taking the trace of $v(x, t)$ on $\Gamma_0 \times [t_0, t_1]$.

For every interior point $(\bar{x}, \bar{t})$ of $M(\Gamma_0; t_0, t_1)$, let us take the family of time-like surfaces S_λ stated in Lemma 5.3, and consider the problem

$$\rho \frac{\partial^2 z}{\partial t^2} - Az = 0 \tag{5.8}$$

with the conditions

$$z = 0, \quad (\rho \nu_t^2 \delta_{ik} - \nu_j \nu_l c_{ijkl}) \partial z^k / \partial \nu = p^i \quad \text{on} \quad S_\gamma, \qquad 0 \leqq \gamma \leqq 1, \tag{5.9}$$

where $\nu = (\nu_t, \nu_1, \nu_2, \nu_3)$ is the outward unit vector on S_γ and p^i are analytic functions. Conditions (5.9) uniquely determine all first order derivatives of z on S_γ and

$$\partial z / \partial t = \nu_t (\partial z / \partial \nu) \quad \text{on} \quad S_\gamma,$$

$$\partial z / \partial x_j = \nu_j (\partial z / \partial \nu) \quad \text{on} \quad S_\gamma$$

(see [6, p. 196]). Since $\rho(x)$, $c_{ijkl}(x)$ and S_γ are analytic and S_γ is non-characteristic by assumption, the problem (5.8), (5.9) is uniquely solvable in some neighborhood G_γ of S_γ which does not depend on p^i by Cauchy-Kowalevski's theorem. We have $G_{\gamma_0} \supset D_{\gamma_0}$ (which is stated in Lemma 5.3) for sufficiently small $\gamma_0 > 0$. For γ, $0 < \gamma < \gamma_0$, we can perform integration by parts in the following by virtue of Lemma 5.2

$$\begin{aligned} 0 &= \int_{D_\gamma} \left\{ v \cdot \left(\rho \frac{\partial^2 z}{\partial t^2} - Az \right) - z \cdot \left(\rho \frac{\partial^2 v}{\partial t^2} - Av \right) \right\} dx dt \\ &= \int_{S_0} \left\{ \rho v n_t \cdot \frac{\partial z}{\partial t} - n_j c_{ijkl} v^i \frac{\partial z^k}{\partial x_l} - \left(\rho z n_t \cdot \frac{\partial v}{\partial t} - n_j c_{ijkl} z^i \frac{\partial v^k}{\partial x_l} \right) \right\} dS \\ &\quad + \int_{S_\gamma} \left\{ \rho v \nu_t \cdot \frac{\partial z}{\partial t} - \nu_j c_{ijkl} v^i \frac{\partial z^k}{\partial x_l} - \left(\rho z \nu_t \cdot \frac{\partial v}{\partial t} - \nu_j c_{ijkl} z^i \frac{\partial v^k}{\partial x_l} \right) \right\} dS \\ &= \int_{S_0} n_j c_{ijkl} z^i \frac{\partial v^k}{\partial x_l} dS \\ &\quad + \int_{S_\gamma} \left\{ (\rho \nu_t^2 \delta_{ik} - \nu_j \nu_l c_{ijkl}) \frac{\partial z^k}{\partial \nu} v^i - \left(\rho z \nu_t \cdot \frac{\partial v}{\partial t} - \nu_j c_{ijkl} z^i \frac{\partial v^k}{\partial x_l} \right) \right\} dS, \end{aligned} \tag{5.10}$$

where $n = (n_t, n_1, n_2, n_3)$ is the outward unit normal vector on S_0, since $v = 0$ on S_0. The trace on S_0 of $n_j c_{ijkl} (\partial v^k / \partial x_l)$ vanishes because of the weak boundary condition. On S_γ, $z = 0$ and $(\rho \nu_t^2 \delta_{ik} - \nu_j \nu_l c_{ijkl}) \partial z^k / \partial \nu = p^i$. Thus we have the equality

$$\int_{S_\gamma} p \cdot v \, dS = 0 \tag{5.11}$$

for any analytic function p. Therefore $v(x, t) = 0$ almost everywhere on S_γ for any $0 < \gamma \leqq \gamma_0$. We conclude that $v(x, t) = 0$ almost everywhere in D_{γ_0}.

Now let I be the largest interval of $[0, 1]$ including 0 and having the property that $v(x, t)=0$ almost everywhere on S_λ if $\lambda \in I$. It is clear that I is a closed non-empty interval. Using the same argument as above, we can see that I is open. By the connectedness of $[0, 1]$, I is identical to $[0, 1]$. Thus $v(x, t)=0$ almost everywhere in $M(\Gamma_0; t_0, t_1)$.

THEOREM 5.5. *Let $\partial\Omega$, $\rho(x)$ and $c_{ijkl}(x)$ be analytic. Then at any time $T>2T_2$, the system* (4.1) *is controllable.*

PROOF. For any $[g, h]\in \boldsymbol{K}(\Omega)\times \boldsymbol{L}^2(\Omega)$ we can take a sequence $\{[g_n, h_n]\}$ in $D(\mathscr{A})$ which converges to $[g, h]$ in $\boldsymbol{K}(\Omega)\times \boldsymbol{L}^2(\Omega)$. Let $v_n(x, t)$ (resp. $v(x, t)$) be the solution of (4.3) (resp. (5.1)) with the initial condition $[g_n, h_n]$ (resp. $[g, h]$) and put

$$w_n(x, t) = \int_T^t v_n(x, s)ds, \tag{5.12}$$

$$w(x, t) = \int_T^t v(x, s)ds. \tag{5.13}$$

Then

$$(\partial w_n/\partial t)(x, t) = v_n(x, t), \quad (\partial w/\partial t)(x, t) = v(x, t)$$

and

$$w_n(x, t), \quad w(x, t)\in \mathscr{E}_t^1(\boldsymbol{K}(\Omega))[0, T]\cap \mathscr{E}_t^2(\boldsymbol{L}^2(\Omega))[0, T].$$

By (4.6) in the proof of Theorem 4.2, for the solution $u(t)$ of (4.1),

$$\begin{aligned}([g_n, h_n], [u(T), (\partial u/\partial t)(T)])_E &= \int_{\Gamma_2\times(0,T)} f\cdot\frac{\partial v_n}{\partial t}dSdt \\ &= \int_{\Gamma_2\times(0,T)} f\cdot\frac{\partial^2 w_n}{\partial t^2}dSdt = \int_{\Gamma_2\times(0,T)} w_n\frac{\partial^2 f}{\partial t^2}dSdt.\end{aligned} \tag{5.14}$$

Passage to the limit as $n\to\infty$ in (5.14) gives

$$([g, h], [u(T), (\partial u/\partial t)(T)])_E = \int_{\Gamma_2\times(0,T)} w\cdot\frac{\partial^2 f}{\partial t^2}dSdt. \tag{5.15}$$

Note that the convergence of the right side in (5.14) follows from the fact that $w_n(x, t)$ converges to $w(x, t)$ in $\mathscr{E}_t^1(\boldsymbol{K}(\Omega))[0, T]\cap \mathscr{E}_t^2(\boldsymbol{L}^2(\Omega))[0, T]$.

Hence, if $[g, h]\in R_T^\perp$, then

$$\int_{\Gamma_2\times(0,T)} w\cdot\frac{\partial^2 f}{\partial t^2}dSdt = 0 \qquad \text{for any } f\in\mathscr{F}.$$

This means that $(\partial^2 w/\partial t^2)(x, t)=0$ in the distribution sense on $\Gamma_0\times(0, T)$.

Because $w(x, t)|_{\Gamma_2\times(0,T)} \in \mathscr{E}_t^1(\boldsymbol{H}^{1/2}(\Gamma_2))[0, T]$, there exist functions $a_1(x)$ and $a_2(x)$ in $\boldsymbol{L}^2(\Gamma_0)$ such that

$$(5.16) \qquad w(x, t) = a_1(x)t + a_2(x) \quad \text{almost everywhere in} \quad \Gamma_0 \times (0, T).$$

For a small positive number δ, we consider the second order difference

$$(5.17) \qquad \hat{w}(x, t) \equiv w(x, t+2\delta) - 2w(x, t+\delta) + w(x, t) \quad \text{in} \quad \Omega \times [0, T-2\delta].$$

Then $\hat{w}(x, t) \in \mathscr{E}_t^1(\boldsymbol{K}(\Omega))[0, T-2\delta] \cap \mathscr{E}_t^2(\boldsymbol{L}^2(\Omega))[0, T-2\delta]$ satisfies the following:

$$(5.18) \qquad \rho\frac{\partial^2 \hat{w}}{\partial t^2} - A\hat{w} = 0 \quad \text{in} \quad \Omega \times (0, T-2\delta),$$

$$(5.19) \qquad -(A\hat{w}, \phi) = \left(c_{ijkl}\frac{\partial \hat{w}^k}{\partial x_l}, \frac{\partial \phi^i}{\partial x_j}\right) \quad \text{at any} \quad t \in (0, T-2\delta),$$

for any $\phi \in \boldsymbol{K}(\Omega)$,

$$(5.20) \qquad \hat{w}(x, t) = 0 \quad \text{on} \quad \Gamma_0 \times (0, T-2\delta).$$

To prove (5.18), it is enough to observe that

$$\begin{aligned}
\rho\frac{\partial^2 w}{\partial t^2}(t) - Aw(t) &= \rho\frac{\partial v}{\partial t}(t) - \int_T^t Av(s)ds \\
&= \int_T^t\left(\rho\frac{\partial^2 v}{\partial t^2}(s) - Av(s)\right)ds + \rho\frac{\partial v}{\partial t}(T) \\
&= \rho\frac{\partial v}{\partial t}(T).
\end{aligned}$$

Here the integrand belongs to $\mathscr{E}_t^0(\boldsymbol{K}(\Omega)')[0, T]$, so the integral is considered with respect to the topology of $\boldsymbol{K}(\Omega)'$. The equality (5.19) is shown by the following: for $\phi \in \boldsymbol{K}(\Omega)$,

$$\begin{aligned}
-(Aw(t), \phi) &= \left< A\int_T^t v(s)ds, \phi \right> = \int_T^t \left< Av(s), \phi \right> ds \\
&= \int_T^t \left(c_{ijkl}\frac{\partial v^k}{\partial x_l}(s), \frac{\partial \phi^i}{\partial x_j}\right)ds = \left(c_{ijkl}\frac{\partial w^k}{\partial x_l}(t), \frac{\partial \phi^i}{\partial x_j}\right).
\end{aligned}$$

(5.20) follows from (5.16). By applying Lemma 5.4 to this function $\hat{w}(x, t)$, we see $\hat{w}(x, t) = 0$ almost everywhere in $M(\Gamma_0; 0, T-2\delta)$.

If $T > 2T_0$, there are $\varepsilon > 0$ and $\delta > 0$ satisfying

$$\Omega \times [T/2 - \varepsilon, T/2 + \varepsilon] \subset M(\Gamma_0; 0, T - 2\delta). \tag{5.21}$$

Since $\hat{w}(x, t)=0$ in $\Omega \times [T/2-\varepsilon, T/2+\varepsilon]$, $w(x, t)$ is a polynomial in t of degree not greater than one with coefficients in $\boldsymbol{K}(\Omega)$. Differentiating $w(x, t)$ with respect to t, we see that there exists a function $\tilde{v}(x) \in \boldsymbol{K}(\Omega)$ such that

$$v(x, t) = \tilde{v}(x) \quad \text{in} \quad \Omega \times [T/2 - \varepsilon, T/2 + \varepsilon].$$

Thus $(\partial v/\partial t)(t)=0$ in $\Omega \times [T/2-\varepsilon, T/2+\varepsilon]$. By the equation

$$\rho \frac{\partial^2 v}{\partial t^2} - Av = 0 \quad \text{in} \quad \Omega \times (0, T),$$

we have the equality $Av(t)=0$ in $\Omega \times [T/2-\varepsilon, T/2+\varepsilon]$. Since $v(t)$ satisfies the weak boundary condition

$$- \langle Av(t), \phi \rangle = \left(c_{ijkl} \frac{\partial v^k}{\partial x_l}(t), \frac{\partial \phi^i}{\partial x_j}\right) \quad \text{at each} \quad t \in (0, T),$$

for any $\phi \in \boldsymbol{K}(\Omega)$,

$$\left(c_{ijkl} \frac{\partial v^k}{\partial x_l}(t), \frac{\partial \phi^i}{\partial x_j}\right) = 0 \quad \text{for each} \quad t \in [T/2 - \varepsilon, T/2 + \varepsilon].$$

Therefore for any $t \in [T/2-\varepsilon, T/2+\varepsilon]$

$$|[v(t), (\partial v/\partial t)(t)]|_E = \left(\rho \frac{\partial v}{\partial t}, \frac{\partial v}{\partial t}\right) + \left(c_{ijkl} \frac{\partial v^k}{\partial x_l}, \frac{\partial v^i}{\partial x_j}\right) = 0,$$

and hence by the energy equality, which is stated in the proof of Theorem 4.3 for $[g, h] \in D(\mathscr{A})$ and is obtained for any $[g, h] \in \boldsymbol{K}(\Omega) \times \boldsymbol{L}^2(\Omega)$ by taking a limit of convergent sequence as in the proof of Lemma 5.1,

$$|[g, h]|_E = \left|\left[v(T), \frac{\partial v}{\partial t}(T)\right]\right|_E = \left|\left[v(t), \frac{\partial v}{\partial t}(t)\right]\right|_E = 0.$$

This means that $R_T^{\perp} = \{0\}$, that is, R_T is dense in $H_E(\Omega)$.

REMARK 1. It is possible to obtain the same results for the case where the space-dimension is more than three.

REMARK 2. If the system is controllable at time T_3, then it is controllable at any time T greater than T_3. In fact, by applying no traction forces for the time $T-T_3$ we can attain the zero state at $t=T-T_3$ and after this time the system is controllable at T_3. This means that the system is controllable at $t=T$. From this it follows that there is a certain T_0 $(2T_1 \leqq T_0 \leqq 2T_2)$ such that the system is not controllable at any time less than T_0 and controllable at any time greater than T_0.

References

[1] B. M. N. Clarke, Boundary controllability of linear elastodynamic systems, Quart. J. Mech. Appl. Math., **28** (1975), 497–515.

[2] K. Hayashida, On a mixed problem for hyperbolic equations with discontinuous boundary conditions, Publ. RIMS. Kyoto Univ., **7** (1971/72), 57–67.

[3] M. Ikawa, Mixed problems for hyperbolic equations of second order, J. Math. Soc. Japan, **20** (1968), 580–608.

[4] A. Inoue, On a mixed problem for $\square$ with discontinuous boundary condition (I), J. Fac. Sci. Univ. Tokyo Sec. IA, **21** (1974), 85–92.

[5] F. John, On linear partial differential equations with analytic coefficients — Unique continuation of data, Comm. Pure Appl. Math., **2** (1947), 209–253.

[6] ———, Partial differential equations, Springer-Verlag, Berlin, 1971.

[7] J. L. Lions and E. Magenes, Probléme aux limites non homogènes et applications, vol. 1, Dunod, Paris, 1968.

[8] D. L. Russell, Boundary value control of the higher-dimensional wave equation, SIAM J. Control, **9** (1971), 29–42.

[9] ———, Boundary value control theory of the higher-dimensional wave equation, Part II, SIAM J. Control, **9** (1971), 401–419.

[10] C. H. Wilcox, Initial-boundary value problems for linear hyperbolic partial differential equations of the second order, Arch. Rat. Mech. Anal., **10** (1962), 361–400.

[11] K. Yosida, Functional analysis, Second Edition, Springer-Verlag, Berlin, 1968.

Department of Mathematics,
Faculty of Science,
Hiroshima University