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On the Existence of Non-tangential Limits of Polyharmonic Functions

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1. Introduction and statement of results

Let R^n $(n \geqq 2)$ be the n-dimensional Euclidean space. A point x of R^n will be written also as $(x', x_n) \in R^{n-1} \times R^1$. We denote by R^n_+ the set of all points $x=(x', x_n) \in R^n$ such that $x_n > 0$, and by R^n_0 its boundary ∂R^n_+. For a function $u \in C^\infty(R^n_+)$, we define the gradient of order k by

$$\nabla^k u(x) = (D^\gamma u(x))_{|\gamma|=k}, \qquad x \in R^n_+,$$

where $\gamma=(\gamma_1,\dots, \gamma_n)$ is a multi-index with length $|\gamma| = \sum_{i=1}^n \gamma_i$ and $D^\gamma = (\partial/\partial x_1)^{\gamma_1} \cdots (\partial/\partial x_n)^{\gamma_n}$. A function $u \in C^\infty(R^n_+)$ is said to be polyharmonic of order m in R^n_+ if $\Delta^m u=0$ on R^n_+, and to have a non-tangential limit at $\xi \in R^n_0$ if

$$\lim_{\substack{x \to \xi \\ x \in \Gamma(\xi; a)}} u(x)$$

exists and is finite for all $a>0$, where Δ^m is the Laplace operator iterated m times and

$$\Gamma(\xi; a) = \{x = (x', x_n) \in R^n_+;\ |(x', 0) - \xi| < a x_n,\ |x - \xi| \leqq 1\}.$$

Our first aim is to show the following theorem:

THEOREM 1. *Let k and m be positive integers such that $k \geqq m$, $1<p<\infty$ and $-\infty<\alpha<kp$. If u is a function polyharmonic of order m in R^n_+ which satisfies*

$$\iint_G |\nabla^k u(x', x_n)|^p x_n^\alpha dx' dx_n < \infty \qquad \textit{for any bounded open set } G \subset R^n_+,$$

then there exists a Borel set $E \subset R^n_0$ such that $B_{k-\alpha/p,p}(E)=0$ and u has a non-tangential limit at each point of $R^n_0 - E$.

Here $B_{\beta,p}$ $(\beta>0)$ is the Bessel capacity of index (β, p) (cf. [2]). Theorem 1 is a generalization of a result of the first author [3; Theorem 1] $(k=m=1)$. In case $-1<\alpha<kp-1$, Theorem 1 is the best possible as to the size of the exceptional set in the following sense:

THEOREM 2. *Let* $1<p<\infty$, k *be a positive integer and* $-1<\alpha<kp-1$. *Let* E *be a subset of* R_0^n *with* $B_{k-\alpha/p,p}(E)=0$. *Then there is a harmonic function* u *in* R_+^n *such that* $\int_{R_+^n}|\nabla^k u(x)|^p x_n^\alpha dx<\infty$ *and* $\lim_{x\to\xi, x\in R_+^n} u(x)=\infty$ *for any* $\xi\in E$.

2. Proof of Theorem 1

To prove Theorem 1, we need the following lemmas.

LEMMA 1. *Let* $\beta>0$ *and* f *be a non-negative function in* $L^p(R^n)$, $1<p<\infty$, *with compact support. Then*

$$\int|x-y|^{\beta-n}f(y)dy=\infty \qquad \text{if and only if} \qquad \int g_\beta(x-y)f(y)dy=\infty$$

for $x\in R^n$, *where* g_β *denotes the Bessel kernel of order* β (cf. [2]).

PROOF. If $0<\beta<n$, then for any compact set K in R^n, there exists a constant $c_1>0$ such that

$$c_1^{-1}|x|^{\beta-n}\leqq g_\beta(x)\leqq c_1|x|^{\beta-n} \qquad \text{whenever} \quad x\in K,$$

so that the lemma easily follows in this case. If $\beta\geqq n$, then $g_\beta\in L^{p'}(R^n)$ for any $p'>1$, and hence

$$\int|x-y|^{\beta-n}f(y)dy<\infty \quad \text{and} \quad \int g_\beta(x-y)f(y)dy<\infty$$

for all $x\in R^n$. For the properties of Bessel kernels, see e.g. [2].

In what follows, $c_2, c_3,\dots$, are positive constants.

LEMMA 2. *Let* $b>0$, i *be a positive integer and* $u\in C^\infty(R_+^n)$.

If $\int_{\Gamma(\xi;b)}|\xi-y|^{i+1-n}|\nabla^{i+1}u(y)|dy<\infty$, *then* $\int_{\Gamma(\xi;b)}|\xi-y|^{i-n}|\nabla^i u(y)|dy<\infty$.

PROOF. Let γ be a multi-index with $|\gamma|=i$. Then

$$D^\gamma u(y)=-\int_r^1(\partial/\partial s)\,[D^\gamma u(\xi+s\sigma)]ds+D^\gamma u(\xi+\sigma),$$

where $r=|\xi-y|$ and $\sigma=(y-\xi)/r$. Hence it follows that

$$|\nabla^i u(y)|\leqq c_2\left\{\int_r^1|\nabla^{i+1}u(\xi+s\sigma)|ds+|\nabla^i u(\xi+\sigma)|\right\}.$$

Therefore,

$$\int_{\Gamma(\xi;b)}|\xi-y|^{i-n}|\nabla^i u(y)|dy$$

$$\leqq c_2\int_{S(b)}\left\{\int_0^1 r^{i-n}r^{n-1}dr\right\}|\nabla^i u(\xi+\sigma)|dS(\sigma)$$

$$+c_2\int_{S(b)}\left[\int_0^1\left\{\int_r^1|\nabla^{i+1}u(\xi+s\sigma)|ds\right\}r^{i-n}r^{n-1}dr\right]dS(\sigma)$$

$$\leqq c_3\left[A_u+\int_{S(b)}\left\{\int_0^1 s^i|\nabla^{i+1}u(\xi+s\sigma)|ds\right\}dS(\sigma)\right]$$

$$=c_3\left[A_u+\int_{\Gamma(\xi;b)}|\xi-y|^{i+1-n}|\nabla^{i+1}u(y)|dy\right],$$

where $S(b)=\{x\in\Gamma(O;b);\ |x|=1\}$ and $A_u=\int_{S(b)}|\nabla^i u(\xi+\sigma)|dS(\sigma)<\infty$. The proof of our lemma is thus complete.

PROOF OF THEOREM 1. Let k, m, p, α, u be as in Theorem 1. Given $N>1$, let us consider the existence of non-tangential limits of u at points of $B_N=\{\xi\in R_0^n;\ |\xi|<N\}$. Set

$$f(x)=\begin{cases}|\nabla^k u(x)|x_n^{\alpha/p}, & \text{if } x=(x',x_n)\in R_+^n \text{ and } |x|<2N,\\ 0, & \text{otherwise.}\end{cases}$$

Then $f\in L^p(R^n)$ by our assumption. If we set

$$E=\left\{x\in R^n;\ \int|x-y|^{k-\alpha/p-n}f(y)dy=\infty\right\},$$

then $B_{k-\alpha/p,p}(E)=0$ on account of Lemma 1. Let $\xi\in B_N-E$ and $a>0$ be fixed. Since there is a constant $c_4>0$ such that $x_n\leqq|\xi-x|<c_4x_n$ for $x=(x',x_n)\in\Gamma(\xi;b)$, $b>0$,

$$\int_{\Gamma(\xi;b)}|\xi-y|^{k-n}|\nabla^k u(y)|dy\leqq c_4^{|\alpha|/p}\int_{\Gamma(\xi;b)}|\xi-y|^{k-\alpha/p-n}f(y)dy<\infty.$$

Hence Lemma 2 gives

$$\sum_{i=1}^k\int_{\Gamma(\xi;b)}|\xi-y|^{i-n}|\nabla^i u(y)|dy<\infty \tag{1}$$

for any $b>0$. By (1), we have

$$\int_{S(a)}\left\{\int_0^1|\nabla u(\xi+r\sigma)|dr\right\}dS(\sigma)=\int_{\Gamma(\xi;a)}|\xi-y|^{1-n}|\nabla u(y)|dy<\infty,$$

so that there is $\sigma^* \in S(a)$ with $A_{\sigma^*} = \int_0^1 |\nabla u(\xi + r\sigma^*)| dr < \infty$. Since $\int_0^1 |(\partial/\partial r) u(\xi + r\sigma^*)| dr \leqq A_{\sigma^*}$, $\lim_{r \downarrow 0} u(\xi + r\sigma^*)$ exists and is finite.

We shall show that $x_n |\nabla u(x)| \to 0$ as $x \to \xi$, $x \in \Gamma(\xi; a)$. In view of [1; (15)],

$$(2) \qquad v(x) = \sum_{i=1}^{m-1} \frac{(-1)^i}{i!} \rho^{2i} \frac{1}{\omega_n} \int_S \left(\frac{\partial}{\partial \rho^2} \right)^i v(x + \rho\sigma) dS(\sigma)$$

for any v polyharmonic of order m in R^n_+, where $B(x, \rho) = \{y \in R^n;\ |x - y| \leqq \rho\} \subset R^n_+$, $S = \partial B(0, 1)$ and ω_n is the area of S. Since $\rho^{2i}(\partial/\partial\rho^2)^i$ is of the form $\sum_{j=0}^{i} a_j \rho^j (\partial/\partial\rho)^j$, a_j being constants depending only on i and j, (2) can be written as

$$v(x) = \sum_{i=0}^{m-1} a'_i \rho^i \int_S \left(\frac{\partial}{\partial \rho} \right)^i v(x + \rho\sigma) dS(\sigma)$$

with constants a'_i depending only on m and n. Multiplying both sides by ρ^{n-1} and integrating them with respect to ρ over the interval $(0, x_n/2)$ then yield

$$(3) \qquad v(x) = \sum_{i=0}^{m-1} a''_i x_n^{-n} \int_{|x-y|<x_n/2} \left(\frac{\partial}{\partial \rho} \right)^i v(y) |x - y|^i dy,$$

where $x = (x', x_n) \in R^n_+$ and a''_i are constants depending only on m and n. Applying (3) with $v = \partial u/\partial x_j$, $j = 1, \ldots, n$, we obtain

$$\begin{aligned} x_n |\nabla u(x)| &\leqq c_5 \sum_{i=1}^{m} x_n^{i-n} \int_{|x-y|<x_n/2} |\nabla^i u(y)| dy \\ &\leqq c_6 \sum_{i=1}^{m} \int_{|x-y|<x_n/2} |\xi - y|^{i-n} |\nabla^i u(y)| dy \\ &\longrightarrow 0 \quad \text{as } x \longrightarrow \xi,\ x = (x', x_n) \in \Gamma(\xi; a) \end{aligned}$$

on account of (1), because we can find $b > 0$ such that $B(x, x_n/2) \subset \Gamma(\xi; b)$ whenever $x \in \Gamma(\xi; a)$.

For $x = (x', x_n) \in \Gamma(\xi; a)$, denote by x_{σ^*} the point on the half line $\{\xi + r\sigma^*;\ r > 0\}$ whose n-th coordinate is equal to x_n, and by $\ell(x)$ the line segment between x and x_{σ^*}. It follows that

$$|u(x) - u(x_{\sigma^*})| \leqq |x - x_{\sigma^*}| \sup_{\ell(x)} |\nabla u| \leqq 2a x_n \sup_{\ell(x)} |\nabla u|$$

tends to zero as $x \to \xi$, $x = (x', x_n) \in \Gamma(\xi; a)$. Therefore $\lim_{x \to \xi, x \in \Gamma(\xi; a)} u(x)$ exists and is finite. This implies that u has a non-tangential limit at $\xi \in B_N - E$. Since N is arbitrary, our theorem is proved.

3. Proof of Theorem 2

By our assumption that $B_{k-\alpha/p,p}(E)=0$, there is a non-negative function $f \in L^p(R^n)$ such that $\int g_{k-\alpha/p}(\xi-y)f(y)dy=\infty$ for every $\xi \in E$. We denote by F the restriction of $\int g_{k-\alpha/p}(x-y)f(y)dy$ to R^{n-1}, i.e.,

$$F(x') = \int g_{k-\alpha/p}((x', 0) - y)f(y)dy, \qquad x' \in R^{n-1}.$$

We note that the function F belongs to the Lipschitz space $\Lambda_\beta^{p,p}(R^{n-1})$ with $\beta=k-(\alpha+1)/p>0$ (cf. [5; Chap. VI, §4.3]). Let u be the Poisson integral of F with respect to R_+^n. By the fact in [5; p. 152] we have

$$\int_0^\infty \left[x_n^{k_0-\beta} \left\{ \int_{R^{n-1}} \left| \left(\frac{\partial}{\partial x_n} \right)^{k_0} u(x', x_n) \right|^p dx' \right\}^{1/p} \right]^p x_n^{-1} dx_n < \infty,$$

k_0 being the smallest integer greater than β. This implies

$$\int_0^\infty \left[x_n^{k'-\beta} \left\{ \int_{R^{n-1}} \left| \left(\frac{\partial}{\partial x_n} \right)^{k'} u(x', x_n) \right|^p dx' \right\}^{1/p} \right]^p x_n^{-1} dx_n < \infty$$

for any positive integer k' greater than β, which is equivalent to

$$\int_{R_+^n} |\nabla^{k'} u(x)|^p x_n^{p(k'-\beta)-1} dx < \infty,$$

by the observation given after Lemma 4′ in [5; Chap. V]. In particular, taking $k'=k$ $(>\beta)$, we obtain

$$\int_{R_+^n} |\nabla^k u(x)|^p x_n^\alpha dx < \infty.$$

Moreover we see from the property of the Poisson integral and the lower semi-continuity of F that $\lim_{x\to\xi, x\in R_+^n} u(x)=\infty$ for every $\xi \in E$. Thus u satisfies all the conditions in the theorem.

4. Remark

In this section, we shall write a point $x \in R^n$ as

$$x = (x', x'') \in R^{n'} \times R^{n''},$$

where n' and n'' are positive integers such that $n'' \geqq 2$ and $n=n'+n''$. We shall consider functions u polyharmonic of order m in $R^n - R^{n'} \times \{O\}$ satisfying the condition of the form:

$$\iint_G |\nabla^k u|^p |x''|^\alpha dx' dx'' < \infty \tag{4}$$

for any bounded open set $G \subset R^n - R^{n'} \times \{O\}$.

The following theorem can be proved in the same way as Theorem 1:

THEOREM 1'. *Let k, m, p and α be as in Theorem 1. Let u be a function polyharmonic of order m in $R^n - R^{n'} \times \{O\}$ which satisfies (4). Then we can find a Borel set $E \subset R^{n'} \times \{O\}$ such that $B_{k-\alpha/p,p}(E) = 0$ and if $\xi \in R^{n'} \times \{O\} - E$, then*

$$\lim_{\substack{x'' \to O \\ (x', x'') \in \Gamma^*(\xi; a)}} u(x', x'')$$

exists and is finite for any $a > 0$, where

$$\Gamma^*(\xi; a) = \{x = (x', x'') \in R^n;\ |(x', O) - \xi| < a|x''|,\ |x - \xi| \leqq 1\}.$$

In case $kp - \alpha \leqq n''$, our theorem does not give any new information since $B_{k-\alpha/p,p}(R^{n'} \times \{O\}) = 0$; however, under the additional assumption that $\alpha = 0$ and $(2m - k)p/(p - 1) \leqq n''$, any function polyharmonic of order m in $R^n - R^{n'} \times \{O\}$ and satisfying (4) can be extended to a function polyharmonic of order m in R^n (cf. [4]).

In case $kp - \alpha > n''$, we do not know whether Theorem 1' is the best possible as to the size of the exceptional set or not.

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