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# *Fine Differentiability of Riesz Potentials*

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## 1. Introduction

In the $n$-dimensional Euclidean space $R^n$, we are concerned with the differentiability properties of Riesz potential $U_\alpha^\mu$ of order $\alpha$, $0<\alpha<n$, of a non-negative measure $\mu$. The potential $U_\alpha^\mu$ may fail to be differentiable at any point of $R^n$, since $U_\alpha^\mu$ may take the value $\infty$ on a countable dense subset of $R^n$. We are therefore motivated to relax the requirement in the definition of differentiability; in fact, if we restrict the set of approach to $x^0$, then we may be able to conclude

$$\lim_{x\to x^0, x\notin E} \frac{|U_\alpha^\mu(x) - U_\alpha^\mu(x^0) - L(x-x^0)|}{|x-x^0|} = 0,$$

where $L=L_{x^0}$ is a linear function. The following problems are proposed here:

(i) Characterize the excluded set $E$ in an appropriate manner.

(ii) Evaluate the size of the set of all $x^0$ at which $U_\alpha^\mu$ is not differentiable in such a sense.

Before finding answers to these problems, we fix some notation which will be used in this note. For a point $x=(x_1,\dots, x_n)\in R^n$ and a multi-index $\gamma=(\gamma_1,\dots,\gamma_n)$, we define

$$x^\gamma = x_1^{\gamma_1}\cdots x_n^{\gamma_n}, \quad (\partial/\partial x)^\gamma = (\partial/\partial x_1)^{\gamma_1}\cdots(\partial/\partial x_n)^{\gamma_n},$$

$$\gamma! = \gamma_1!\cdots\gamma_n!, \quad |\gamma| = \gamma_1 + \cdots + \gamma_n.$$

We denote by $R_\alpha$ the Riesz kernel of order $\alpha$. Fix a point $x^0\in R^n$ and set

$$K_m(x, y) = R_\alpha(x-y) - \sum_{|\gamma|\leqq m} \frac{1}{\gamma!}(x-x^0)^\gamma \frac{\partial^\gamma R_\alpha}{\partial x^\gamma}(x^0-y)$$

for a positive integer $m$.

A set $E$ is said to be $\alpha$-thin at $x^0$ either if $x^0\notin \overline{E\setminus\{x^0\}}$ (the closure of $E\setminus\{x^0\}$) or if $x^0\in \overline{E\setminus\{x^0\}}$ and there is a non-negative measure $\mu$ satisfying

$$\liminf_{x\to x^0, x\in E\setminus\{x^0\}} U_\alpha^\mu(x) > U_\alpha^\mu(x^0).$$

Our first aim is to prove the following theorem.

THEOREM 1. *Let $\mu$ be a non-negative measure such that*

$$(1) \qquad \int |x^0 - y|^{\alpha-m-n} d\mu(y) < \infty.$$

*Then there exists a Borel set $E \subset R^n$ which is $\alpha$-thin at $x^0$ and satisfies*

$$\lim_{x \to x^0, x \notin E} |x - x^0|^{-m} \int K_m(x, y) d\mu(y) = 0.$$

Next we consider the case when $\mu$ has a density in $L^p(R^n)$, $1<p<\infty$. Letting $G$ be an open set in $R^n$, we define the capacity

$$C_{\alpha,p}(E; G) = \inf \|g\|_p^p, \qquad E \subset R^n,$$

where the infimum is taken over all non-negative measurable functions $g$ such that $g$ vanishes outside $G$ and $U_\alpha^g(x) \geqq 1$ for all $x \in E$. A set $E \subset R^n$ is said to be $(\alpha, p)$-thin at $x^0$ if

$$\sum_{k=1}^{\infty} [2^{k(n-\alpha p)} C_{\alpha,p}(E_k; B(x^0, 2))]^{1/(p-1)} < \infty, \qquad \text{in case} \quad \alpha p \leqq n,$$

$$x^0 \notin \overline{E \setminus \{x^0\}}, \qquad \text{in case} \quad \alpha p > n,$$

where $E_k = E \cap B(x^0, 2^{-k+1}) \setminus B(x^0, 2^{-k})$, $B(x^0, r)$ being the open ball with center at $x^0$ and radius $r$.

REMARK. In case $\alpha p < n$, $E$ is $(\alpha, p)$-thin at $x^0$ if and only if

$$\int_0^1 [r^{\alpha p - n} C_{\alpha,p}(E \cap B(x^0, r); B(x^0, 2))]^{1/(p-1)} \frac{dr}{r} < \infty.$$

Thus our thinness in this case coincides with the thinness defined by Meyers [4]. For a proof, see the Appendix. We also refer to Adams and Meyers [1].

THEOREM 2. *Let $f$ be a non-negative function in $L^p(R^n)$ such that $\int |x^0 - y|^{\alpha-m-n} f(y) dy < \infty$ and*

$$\int_0^1 \left[ r^{(\alpha-m)p-n} \int_{B(x^0,r)} f(y)^p dy \right]^{1/(p-1)} \frac{dr}{r} < \infty.$$

*Then there exists a set $E \subset R^n$ which is $(\alpha, p)$-thin at $x^0$ and satisfies*

$$\lim_{x \to x^0, x \notin E} |x - x^0|^{-m} \int K_m(x, y) f(y) dy = 0.$$

Our theorem corresponding to the case $m=0$ has been proved by Meyers [4; Theorem 3.1].

## 2. Proof of Theorem 1

To prove Theorem 1, we use the following elementary lemmas.

LEMMA 1 ([2; Theorem 5.2]). *A set $E \subset R^n$ is $\alpha$-thin at $x^0$ if and only if*

$$\sum_{k=1}^{\infty} 2^{k(n-\alpha)} C_\alpha(E_k) < \infty,$$

*where $E_k = E \cap B(x^0, 2^{-k+1}) \backslash B(x^0, 2^{-k})$ and $C_\alpha$ denotes the Riesz capacity of order $\alpha$. This is equivalent to*

$$\int_0^1 r^{\alpha-n} C_\alpha(E \cap B(x^0, r)) \frac{dr}{r} < \infty.$$

LEMMA 2. *Let $\lambda$ be a non-negative measure and set $A = \{x \in R^n;\ U_\alpha^\lambda(x) \geqq 1\}$. Then $C_\alpha(A) \leqq 2^{n-\alpha}\lambda(R^n)$.*

Lemma 2 follows readily from a maximum principle (cf. [2; Theorem 1.10]).

LEMMA 3. *There exists a constant $C > 0$ such that*

$$|K_m(x, y)| \leqq C|x - x^0|^m |y - x^0|^{\alpha-m-n}$$

*whenever $|x-y| \geqq |x-x^0|/2$.*

PROOF. In the case where $|x-y| \geqq |x-x^0|/2$ and $|x^0-y| \leqq 2|x-x^0|$, it suffices to evaluate each term of $K_m$ separately. In the case where $|x^0-y| > 2|x-x^0|$, we apply the mean value theorem for the function

$$f(t) = |x^0 - y + t(x - x^0)|, \qquad t > 0,$$

and obtain the desired result.

PROOF OF THEOREM 1. By Lemma 3, we can apply Lebesgue's dominated convergence theorem to obtain

$$\lim_{x \to x^0} |x - x^0|^{-m} \int_{|x-y| \geqq |x-x^0|/2} K_m(x, y) d\mu(y) = 0.$$

For each integer $k$, we set

$$a_k = \int_{2^{-k-1} \leqq |y-x^0| < 2^{-k+2}} |y - x^0|^{\alpha-m-n} d\mu(y).$$

Then $\sum_{k=1}^{\infty} a_k < \infty$ by our assumption, and hence we can find a sequence $\{b_k\}$ of positive numbers such that $\lim_{k\to\infty} b_k = \infty$ and $\sum_{k=1}^{\infty} a_k b_k < \infty$. Consider the set

$$E^{(k)} = \Big\{x \in R^n;\ 2^{-k} \leqq |x - x^0| < 2^{-k+1},$$

$$\int_{|x-y|<|x-x^0|/2} |x - y|^{\alpha-n} d\mu(y) \geqq 2^{-km} b_k^{-1}\Big\}.$$

Then from Lemma 2 it follows that

$$C_\alpha(E^{(k)}) \leqq 2^{n-\alpha} 2^{km} b_k \int_{2^{-k-1} \leqq |y-x^0| < 2^{-k+2}} d\mu(y)$$

$$\leqq 2^{2m+3(n-\alpha)} 2^{-k(n-\alpha)} a_k b_k,$$

so that $\sum_{k=1}^{\infty} 2^{k(n-\alpha)} C_\alpha(E^{(k)}) < \infty$. If we set $E = \bigcup_{k=1}^{\infty} E^{(k)}$, then we see from Lemma 1 that $E$ is $\alpha$-thin at $x^0$. Moreover,

$$\lim_{x \to x^0, x \notin E} |x - x^0|^{-m} \int_{|x-y|<|x-x^0|/2} |x - y|^{\alpha-n} d\mu(y) = 0,$$

which yields

$$\lim_{x \to x^0, x \notin E} |x - x^0|^{-m} \int_{|x-y|<|x-x^0|/2} K_m(x, y) d\mu(y) = 0.$$

Thus we obtain $\lim_{x \to x^0, x \notin E} |x - x^0|^{-m} \int K_m(x, y) d\mu(y) = 0$.

COROLLARY. *Let $1 < \alpha < n$ and let $\mu$ be a non-negative measure such that $U_\alpha^\mu(x^0) < \infty$ and $U_{\alpha-1}^\mu(x^0) < \infty$. Then there is a Borel set $E \subset \partial B(O, 1)$ such that $C_\alpha(E) = 0$ and*

$$\lim_{r \downarrow 0} \frac{U_\alpha^\mu(x^0 + r\xi) - U_\alpha^\mu(x^0)}{r} = (n-\alpha) \int (y - x^0, \xi) |y - x^0|^{\alpha-2-n} d\mu(y)$$

*for every $\xi \in \partial B(O, 1) \backslash E$, where $(\cdot, \cdot)$ denotes the usual inner product in $R^n$.*

For this, we have only to note the following lemma which can be proved by Lemma 1.

LEMMA 4. *For a set $A \subset R^n$, denote by $A^\sim$ the set of all points $z \in \partial B(x^0, 1)$ such that $x^0 + r(z - x^0) \in E$ for some $r > 0$. If $E$ is $\alpha$-thin at $x^0$, then $C_\alpha(\bigcap_{k=1}^{\infty} (E \cap B(x^0, k^{-1}))^\sim) = 0$.*

REMARK 1. We say that $U_\alpha^\mu$ is $\alpha$-finely $m$ times differentiable at $x^0$ if the conclusion of Theorem 1 holds. If $0 < m < \alpha$ and $U_\alpha^\mu \not\equiv \infty$, then $U_\alpha^\mu$ is $\alpha$-finely $m$ times differentiable on $R^n$ except possibly for a set $A$ with $C_{\alpha-m}(A) = 0$; in fact, $A = \{x \in R^n;\ U_{\alpha-m}^\mu(x) = \infty\}$.

REMARK 2. In Theorem 1, Condition (1) is needed. For example, we can find a non-negative measure $\mu$ such that $U_\alpha^\mu(O)<\infty$ and $\lim_{x\to O, x\in E}|x|^{-1}[U_\alpha^\mu(x)-U_\alpha^\mu(O)]=-\infty$ for some $E$ which is not $\alpha$-thin at $O$. It is easy to see that this $U_\alpha^\mu$ is not $\alpha$-finely differentiable at $O$.

To construct such $\mu$, let

$$A=\{y=(y', y_n)\in R^{n-1}\times R^1;\ |y'|<y_n/2,\ |y|<1\},$$

$$B=\{y=(y', y_n)\in R^{n-1}\times R^1;\ (y', -y_n)\in A\},$$

and consider the potential

$$u(x)=\int_B |x-y|^{\alpha-n}|y|^{-\alpha+1/2}dy.$$

Then $u(x)<\infty$ for all $x\in R^n$. We shall prove $\lim_{x\to O, x\in A}|x|^{-1}[u(O)-u(x)]=\infty$. There is a constant $C>0$ such that

$$|y|^{\alpha-n}-|x-y|^{\alpha-n}\geqq C|x|\cdot|y|^{\alpha-1-n}$$

whenever $x\in A$, $y\in B$ and $|y|>2|x|$. Noting that $|x-y|>|y|$ whenever $x\in A$ and $y\in B$, we obtain

$$|x|^{-1}[u(O)-u(x)]\geqq C\int_{\{y\in B;\,|y|>2|x|\}}|y|^{\alpha-1-n}|y|^{-\alpha+1/2}dy$$

$$\longrightarrow\infty \quad \text{as} \quad x\longrightarrow O,\ x\in A.$$

With the aid of Lemma 1, one sees easily that $A$ is not $\alpha$-thin at $O$.

REMARK 3. Let $E$ be a set which is $\alpha$-thin at $x^0$. Then there exists a non-negative measure $\mu$ such that $\int|x^0-y|^{\alpha-m-n}d\mu(y)<\infty$ and $\lim_{x\to x^0, x\in E}|x-x^0|^{-m}\int K_m(x, y)d\mu(y)=\infty$.

To prove this fact, take sequences $\{a_k\}$, $\{b_k\}$ of positive numbers such that $\lim_{k\to\infty}a_k=\infty$ and

$$\sum_{k=1}^{\infty}2^{k(n-\alpha)}a_k[C_\alpha(E_k)+b_k]<\infty,\quad E_k=E\cap B(x^0, 2^{-k+1})\backslash B(x^0, 2^{-k}).$$

For each $k$ we can find a non-negative measure $\mu_k$ with support in $B(x^0, 2^{-k+2})\backslash B(x^0, 2^{-k-1})$ such that $\mu_k(R^n)\leqq C_\alpha(E_k)+b_k$ and $U_\alpha^{\mu_k}(x)\geqq 1$ for all $x\in E_k$. Set $\mu=\sum_{k=1}^\infty 2^{-km}a_k\mu_k$. Then $\int|x^0-y|^{\alpha-m-n}d\mu(y)<\infty$. This gives:

$$\lim_{x\to x^0}|x-x^0|^{i-m}\int_{|x-y|<|x-x^0|/2}|x^0-y|^{\alpha-i-n}d\mu(y)=0 \tag{2}$$

for $i=0, 1,\dots, m$;

$$\lim_{x\to x^0}|x-x^0|^{-m}\int_{|x-x^0|/2\leqq|x-y|<5|x-x^0|}|x-y|^{\alpha-n}d\mu(y)=0; \tag{3}$$

$$\lim_{x\to x^0}|x-x^0|^{-m}\int_{|x-y|\geqq|x-x^0|/2}K_m(x, y)d\mu(y)=0. \tag{4}$$

On the other hand we have

$$\liminf_{x\to x^0, x\in E}|x-x^0|^{-m}\int_{|x-y|<5|x-x^0|}|x-y|^{\alpha-n}d\mu(y)\geqq\lim_{k\to\infty}\inf_{x\in E_k}2^{-m}a_k U_\alpha^{\mu_k}(x)=\infty,$$

which together with (3) implies

$$\lim_{x\to x^0, x\in E}|x-x^0|^{-m}\int_{|x-y|<|x-x^0|/2}|x-y|^{\alpha-n}d\mu(y)=\infty. \tag{5}$$

By (2), (4) and (5), we obtain

$$\lim_{x\to x^0, x\in E}|x-x^0|^{-m}\int K_m(x, y)d\mu(y)=\infty.$$

REMARK 4. Let $m$ and $\mu$ be as in Theorem 1. If, in addition, there are constants $C>0$ and $r_0>0$ such that $\mu(B(x, r))\leqq Cr^{n+m-\alpha}$ for all $x\in B(x^0, r_0)$ and all $r>0$ with $r<r_0$, then

$$\lim_{x\to x^0}|x-x^0|^{-m}\int K_m(x, y)d\mu(y)=0,$$

i.e., $U_\alpha^\mu$ is $m$ times differentiable at $x^0$.

For this it suffices to prove

$$\lim_{x\to x^0}|x-x^0|^{-m}\int_{|x-y|<|x-x^0|/2}|x-y|^{\alpha-n}d\mu(y)=0. \tag{6}$$

Set $a(r)=\int_{|y-x^0|<r}|y-x^0|^{\alpha-m-n}d\mu(y)$ for $r>0$. Then $a(r)\to 0$ as $r\downarrow 0$ and

$$\mu(B(x^0, r))\leqq a(r)r^{n+m-\alpha}\qquad\text{for}\quad r>0.$$

Hence for $b>0$ and $r=|x-x^0|$,

$$\begin{aligned}\int_{br\leqq|x-y|<r/2}|x-y|^{\alpha-n}d\mu(y)&\leqq(br)^{\alpha-n}\mu(B(x^0, 2r))\\&\leqq 2^{n+m-\alpha}a(2r)b^{\alpha-n}r^m.\end{aligned}$$

On the other hand,

$$\int_{|x-y|<br} |x-y|^{\alpha-n} d\mu(y) = \int_0^{br} \mu(B(x, t)) d(-t^{\alpha-n})$$
$$\leqq (n-\alpha) C m^{-1} b^m r^m$$

if $|x-x^0|<r_0$ and $br<r_0$. Taking $b=a(2r)^{1/2(n-\alpha)}$, we obtain (6).

## 3. Proof of Theorem 2

First we shall be concerned with the case $\alpha p \leqq n$. In the proof of Theorem 1, we have shown that

$$\lim_{x \to x^0} |x-x^0|^{-m} \int_{|x-y| \geqq |x-x^0|/2} K_m(x, y) f(y) dy = 0.$$

Let $a(r)$ be a non-increasing positive function of $r>0$ such that $\lim_{r \downarrow 0} a(r) = \infty$, $a(r) \leqq 2a(2r)$ and

$$\int_0^1 \left[ r^{(\alpha-m)p-n} a(r) \int_{B(x^0, r)} f(y)^p dy \right]^{1/(p-1)} \frac{dr}{r} < \infty.$$

Consider the set

$$E^{(k)} = \Big\{ x \in R^n;\ 2^{-k} \leqq |x-x^0| < 2^{-k+1},$$
$$\int_{|x-y|<|x-x^0|/2} |x-y|^{\alpha-n} f(y) dy \geqq 2^{-km} a(2^{-k})^{-1/p} \Big\}$$

for each positive integer $k$. By definition,

$$C_{\alpha,p}(E^{(k)}; B(x^0, 2)) \leqq 2^{kmp} a(2^{-k}) \int_{2^{-k-1}<|y-x^0|<2^{-k+2}} f(y)^p dy.$$

Setting $E = \cup_{k=1}^{\infty} E^{(k)}$, we have

$$\sum_{k=1}^{\infty} [2^{k(n-\alpha p)} C_{\alpha,p}(E \cap B(x^0, 2^{-k+1}) \backslash B(x^0, 2^{-k}); B(x^0, 2))]^{1/(p-1)}$$
$$\leqq \sum_{k=1}^{\infty} \left[ 2^{k(n-\alpha p+mp)} a(2^{-k}) \int_{|y-x^0|<2^{-k+2}} f(y)^p dy \right]^{1/(p-1)} < \infty,$$

which implies that $E$ is $(\alpha, p)$-thin at $x^0$. We also derive

$$\lim_{x \to x^0, x \notin E} |x-x^0|^{-m} \int_{|x-y|<|x-x^0|/2} K_m(x, y) f(y) dy = 0$$

and thus obtain our theorem for $\alpha p \leqq n$.

Next we treat the case $\alpha p > n$. For this purpose it suffices to prove

$$\lim_{x \to x^0} |x - x^0|^{-m} \int_{|x-y| < |x-x^0|/2} |x - y|^{\alpha-n} f(y) dy = 0.$$

Let $b(r) = \int_0^{2r} \left[ s^{(\alpha-m)p-n} \int_{B(x^0,s)} f(y)^p dy \right]^{1/(p-1)} s^{-1} ds$, $r > 0$. Then $\int_{B(x^0,r)} f(y)^p dy \leqq$ const. $b(r)^{p-1} r^{n-(\alpha-m)p}$. Consequently, using Hölder's inequality, we obtain for $p' = p/(p-1)$

$$\begin{aligned} & |x - x^0|^{-m} \int_{|x-y| < |x-x^0|/2} |x - y|^{\alpha-n} f(y) dy \\ & \leqq |x - x^0|^{-m} \left\{ \int_{|x-y| < |x-x^0|/2} |x - y|^{p'(\alpha-n)} dy \right\}^{1/p'} \left\{ \int_{|x-y| < |x-x^0|/2} f(y)^p dy \right\}^{1/p} \\ & \leqq \text{const.}\, b(2|x - x^0|)^{1/p'} \longrightarrow 0 \quad \text{as} \quad x \longrightarrow x^0. \end{aligned}$$

The proof is now complete.

REMARK 1. Let $f$ be a non-negative function in $L^p(R^n)$ and set

$$E_1 = \left\{ x \in R^n;\ \int |x - y|^{\alpha-m-n} f(y) dy = \infty \right\},$$

$$E_2 = \left\{ x \in R^n;\ \int_0^1 \left[ r^{(\alpha-m)p-n} \int_{B(x,r)} f(y)^p dy \right]^{1/(p-1)} \frac{dr}{r} = \infty \right\}.$$

If $0 < m < \alpha$ and $U^f_{\alpha-m} \not\equiv \infty$, then $C_{\alpha-m,p}(E_1 \cap B(O, a); B(O, 2a)) = 0$ for every $a > 0$, which is equivalent to $B_{\alpha-m,p}(E_1) = 0$. Here $B_{\alpha-m,p}$ denotes the Bessel capacity of index $(\alpha - m, p)$ (cf. [3]). We also have $B_{\alpha-m,p}(E_2) = 0$ on account of [4; Theorem 2.1]. By these facts and Theorem 2, we may state that the potential $U^f_\alpha$ is $(\alpha, p)$-finely $m$ times differentiable $B_{\alpha-m,p}$-q.e. on $R^n$ if $U^f_\alpha \not\equiv \infty$.

REMARK 2. Let $\alpha p \leqq n$ and let $E$ be a set satisfying

$$\sum_{k=1}^{\infty} [2^{k(n-\alpha p)} C_{\alpha,p}(E_k;\ G_k)]^{1/p} < \infty,$$

where $E_k = E \cap B(x^0, 2^{-k+1}) \setminus B(x^0, 2^{-k})$ and $G_k = B(x^0, 2^{-k+2}) \setminus \overline{B(x^0, 2^{-k-1})}$. (This is stronger than the condition that $E$ is $(\alpha, p)$-thin at $x^0$.) Then there is a non-negative function $f \in L^p(R^n)$ with the following properties:

$$\int |x^0 - y|^{\alpha-m-n} f(y) dy < \infty; \tag{7}$$

$$\int_0^\infty \left[ r^{(\alpha-m)p-n} \int_{B(x^0,r)} f(y)^p dy \right]^{1/p} \frac{dr}{r} < \infty; \tag{8}$$

$$\lim_{x \to x^0, x \in E} |x - x^0|^{-m} \int K_m(x, y) f(y) dy = \infty. \tag{9}$$

To construct such $f$, take sequences $\{a_k\}$, $\{b_k\}$ of positive numbers such that $\lim_{k\to\infty} a_k = \infty$ and

$$\sum_{k=1}^{\infty} [2^{k(n-\alpha p)} a_k^p \{C_{\alpha,p}(E_k;\ G_k) + b_k\}]^{1/p} < \infty.$$

Then for each $k$, there exists a non-negative function $f_k$ such that $f_k = 0$ outside $G_k$, $\|f_k\|_p^p < C_{\alpha,p}(E_k;\ G_k) + b_k$ and $U_\alpha^{f_k} \geqq 1$ on $E_k$. Consider the function $f = \sum_{k=1}^{\infty} 2^{-mk} a_k f_k$. Then (7) is fulfilled and

$$\sum_{k=1}^{\infty} \left[ 2^{k(n-\alpha p+mp)} \int_{B(x^0, 2^{-k})} f(y)^p dy \right]^{1/p} < \infty,$$

which yields (8). As in the proof of Remark 3 in § 2, $f$ is seen to satisfy (9).

## 4. Appendix

We shall show below that in case $\alpha p < n$, a set $E \subset R^n$ is $(\alpha, p)$-thin at $x^0$ if and only if

$$\int_0^1 [r^{\alpha p - n} C_{\alpha,p}(E \cap B(x^0, r);\ B(x^0, 2))]^{1/(p-1)} \frac{dr}{r} < \infty,$$

which is equivalent to

$$\int_0^1 [r^{\alpha p - n} B_{\alpha,p}(E \cap B(x^0, r))]^{1/(p-1)} \frac{dr}{r} < \infty.$$

If $p > 2$, then $1/(p-1) < 1$ and

$$\begin{aligned} &\sum_{j=1}^{\infty} [2^{j(n-\alpha p)} C_{\alpha,p}(E \cap B(2^{-j+1});\ B(x^0, 2))]^{1/(p-1)} \\ &\leqq \sum_{j=1}^{\infty} 2^{j(n-\alpha p)/(p-1)} \sum_{k=j}^{\infty} [C_{\alpha,p}(E_k;\ B(x^0, 2))]^{1/(p-1)} \\ &\leqq \text{const.} \sum_{k=1}^{\infty} [2^{k(n-\alpha p)} C_{\alpha,p}(E_k;\ B(x^0, 2))]^{1/(p-1)}, \end{aligned}$$

where $E_k = E \cap B(x^0, 2^{-k+1}) \backslash B(x^0, 2^{-k})$. If $p \leqq 2$, then we have

$$C_{\alpha,p}(E \cap B(x^0, r);\ B(x^0, 2)) \leqq \int_0^r C_{\alpha,p}(E \cap B(x^0, 2s) \backslash B(x^0, s/2);\ B(x^0, 2)) \frac{ds}{s},$$

so that the inequality of Hardy (cf. [5; Appendices, A.4]) gives

$$\int_0^1 [r^{\alpha p-n} C_{\alpha,p}(E \cap B(x^0, r); B(x^0, 2))]^{1/(p-1)} \frac{dr}{r}$$

$$\leqq \text{const.} \int_0^1 [s^{\alpha p-n} C_{\alpha,p}(E \cap B(x^0, 2s) \backslash B(x^0, s/2); B(x^0, 2))]^{1/(p-1)} \frac{ds}{s}.$$

These arguments readily yield the required assertion.

## References


[1] D. R. Adams and N. G. Meyers, Thinness and Wiener criteria for non-linear potentials, Indiana Univ. Math. J. **22** (1972), 169–197.

[2] N. S. Landkof, Foundations of modern potential theory, Springer-Verlag, Berlin, 1972.

[3] N. G. Meyers, A theory of capacities for potentials of functions in Lebesgue classes, Math. Scand. **26** (1970), 255–292.

[4] N. G. Meyers, Continuity properties of potentials, Duke Math. J. **42** (1975), 157–166.

[5] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, 1970.



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