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The Steenrod Operations in the Eilenberg-Moore Spectral Sequence

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Introduction

R. Vázquez García [19] and S. Araki [1] introduced two kinds of the Steenrod operations into the mod p Serre spectral sequence $\{E_r\}$, that is, the squaring operations

(a) $Sq^i: E_r^{s,t} \longrightarrow E_r^{s,t+i} \qquad (i < t)$,

(b) $Sq^i: E_r^{s,t} \longrightarrow E_r^{s+i-t,2t} \qquad (i \geq t)$,

for $p=2$, and the reduced power operations

(a) $\beta^\varepsilon P^i: E_r^{s,t} \longrightarrow E_r^{s,t+2i(p-1)+\varepsilon} \qquad (2i < t;\ \varepsilon = 0, 1)$,

(b) $\beta^\varepsilon P^i: E_r^{s,t} \longrightarrow E_r^{s+(2i-t)(p-1)+\varepsilon,pt} \qquad (2i \geq t;\ \varepsilon = 0, 1)$,

for p an odd prime; and they discussed the properties of these operations. Also L. Kristensen [6], [7] obtained the results by using the simplical method.

On the other hand, along with the establishment of the Eilenberg-Moore spectral sequence, J. P. May conjectured at the Conference on Algebraic Topology at Chicago Circle in 1968 that one might introduce the Steenrod operations into the mod p Eilenberg-Moore spectral sequence; and then D. Rector [10] and L. Smith [15], [16] showed that the mod p Eilenberg-Moore spectral sequence is a spectral sequence of modules over the mod p Steenrod algebra with respect to the operations of type (a).

Further, in his work [9], J. P. May developed a general theory to introduce the Steenrod operations into a spectral sequence, and W. M. Singer [14] introduced the squaring operations of both types (a) and (b) into a class of spectral sequences such as the change of ring spectral sequence, the Eilenberg-Moore spectral sequence and the Serre spectral sequence. It remains to introduce the Steenrod reduced powers into such spectral sequences.

The purpose of this paper is to introduce and study the Steenrod operations of both types (a) and (b) for any prime p in such a class of spectral sequences of Eilenberg-Moore type. The main results are Theorems 1.2, 1.3, 1.4, 1.5 and 1.6. Our results extend those obtained by W. M. Singer [14] who worked when $p=2$. The method is slightly different from [14]. The key lemma is Lemma

2.3, which follows from A. Dold [3; Satz 1.12], and this enables us to work for any prime p.

The paper is motivated by introducing the Steenrod operations into the Eilenberg-Moore spectral sequence to calculate the cohomology of the classifying spaces of Lie groups. To have the Steenrod operations in the spectral sequence is helpful in at least two ways: first in proving the collapsing of the spectral sequence and second in reproducing the data lost in passing to quotient. The applications are found in the works of M. Mimura and M. Mori*), A. Kono and M. Mimura**), M. Mimura and Y. Sambe***), and M. Mori****), in which they calculate the cohomology of the classifying spaces of some Lie groups whose integral homology groups have torsion groups.

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§1. Results

Let p be a prime, and $\mathcal{O}$ be the category of finite ordered sets and non-decreasing maps. A *simplicial Z_p-module* R is a contravariant functor from $\mathcal{O}$ to the category of Z_p-modules, that is, R is a collection of Z_p-modules R_n $(n\geq 0)$ together with morphisms d_i: $R_n\to R_{n-1}$, s_i: $R_n\to R_{n+1}$ $(0\leq i\leq n)$, called the face operators and the degeneracy operators, which satisfy the simplicial identities (see J. P. May [8; Definitions 1.1 and 2.1]). Then we write CR for the Z_p-complex such that $C_nR=R_n$, $d=\sum(-1)^i d_i$, and CR forms a differential Z_p-coalgebra with coproduct ξD: $CR\to C(R\times R)\to CR\otimes CR$, where D is the diagonal map and ξ is the Alexander-Whitney map. A *simplicial Z_p-coalgebra* is a simplicial Z_p-module equipped with the coproduct ξD.

A *bisimplicial Z_p-module* is a contravariant functor from $\mathcal{O}\times\mathcal{O}$ to the category of Z_p-modules. We write d_i^h, s_i^h for the horizontal face and degeneracy operators and d_i^v, s_i^v for the vertical face and degeneracy operators. Let K be a bisimplicial Z_p-module. We write CK for the double Z_p-complex such that $C_{m,n}K=K_{m,n}$, $d^h=\sum(-1)^i d_i^h$, $d^v=\sum(-1)^i d_i^v$, and TK for the total Z_p-complex such that $T_nK=\sum_{s+t=n}C_{s,t}K$, $d=d^h+(-1)^s d^v$ on $C_{s,t}K$. Then we can give CK

*) *The squaring operations in the Eilenberg-Moore spectral sequence and the classifying space of an associative H-space, I,* Publ. Res. Inst. Math. Sci., Kyoto Univ. **13** (1977), 755–776.

**) *On the cohomology* mod 2 *of the classifying space of* AdE_7, J. Math. Kyoto Univ., 18 (1978), 535–542.

***) *On the mod p cohomology of the classifying spaces of the exceptional groups, I, II, III, IV,* J. Math. Kyoto Univ., to appear.

****) *The* mod 2 *cohomology of the classifying space of the semi-spinor group Ss*(12), mimeographed note.

the bigraded Z_p-coalgebra structure in the above way, and a bisimplicial Z_p-module with this structure is called a *bisimplicial Z_p-coalgebra*. Apparently the coalgebra structure on CK induces the ones on TK and on K_{0*}.

An *augmentation* ε: $K \to R$ for a bisimplicial Z_p-coalgebra K is a morphism such that $\varepsilon = 0$ on K_{s*} for $s > 0$ and that ε: $K_{0*} \to R_*$ is a morphism of simplicial Z_p-coalgebras satisfying $\varepsilon d_1^h = \varepsilon d_0^h$: $K_{1*} \to R_*$, where R is a simplicial Z_p-coalgebra.

Dualizing them, we can define a *cosimplicial Z_p-module*, *-algebra*, a *bicosimplicial Z_p-module*, *-algebra*, and a *coaugmentation*, etc. Obvious notation and terminology are similarly used (see, for example, [2], [11]).

We now state our results. Let R be a simplicial Z_p-coalgebra and K a bisimplicial Z_p-coalgebra. Then $\mathrm{Hom}(R, Z_p)$ and $\mathrm{Hom}(K, Z_p)$ form a cosimplicial Z_p-algebra and a bicosimplicial Z_p-algebra, respectively, and hence $H^*(CR)$ and $H^*(TK)$ have the products. We shall define the Steenrod operations on $H^*(TK)$ as well as on $H^*(CR)$, and prove the following proposition in § 2:

PROPOSITION 1.1. *Let ε: $K \to R$ be an augmentation. Then ε^*: $H^*(CR) \to H^*(TK)$ preserves the products and the Steenrod operations.*

We define an increasing filtration on TK by

$$F_r T_n K = \sum_{\substack{s+t=n \\ s \leq r}} K_{s,t}.$$

This gives rise to a spectral sequence passing to $H_*(TK)$. Dually, putting $T^n K = \mathrm{Hom}(T_n K, Z_p)$, we define a decreasing filtration on T^*K by

$$F^r T^n K = \{ f \in T^n K \mid f(F_{r-1} T_n K) = 0 \},$$

which gives rise to a spectral sequence $\{E_r\}$ passing to $H^*(TK)$.

This spectral sequence $\{E_r\}$ is a spectral sequence of Z_p-algebras. Further we shall define the 'Steenrod operations' on E_r, $r \geq 2$, (see § 3):

(a) Sq^i: $E_r^{s,t} \longrightarrow E_r^{s,t+i} \quad (i < t)$,

$\beta^\varepsilon P^i$: $E_r^{s,t} \longrightarrow E_r^{s,t+2i(p-1)+\varepsilon} \quad (2i < t;\ \varepsilon = 0, 1)$,

(b) Sq^i: $E_r^{s,t} \longrightarrow E_r^{s+i-t,2t} \quad (i \geq t)$,

$\beta^\varepsilon P^i$: $E_r^{s,t} \longrightarrow E_r^{s+(2i-t)(p-1)+\varepsilon,pt} \quad (2i \geq t;\ \varepsilon = 0, 1)$.

Here we always assume that the underlying coefficient ring is Z_2 for the squaring operations and Z_p, p an odd prime, for the reduced power operations.

THEOREM 1.2. *The Steenrod operations on E_2 determine those on E_r for all $r \geq 2$.*

THEOREM 1.3. *Let* $u \in E_r^{s,t}$.

(i) *If* $i<t-r+1$, *then* $d_r Sq^i u = Sq^i d_r u$. *If* $2i<t-r+1$, *then*

$$d_r \beta^\varepsilon P^i u = (-1)^\varepsilon \beta^\varepsilon P^i d_r u.$$

(ii) *If* $t-r+1 \leq i < t$, *then* $Sq^i u$ *survives to* $E_q^{s,t+i}$ *where* $q=2r+i-t-1$, $Sq^i d_r u$ *survives to* $E_q^{s+q,2t-2r+2}$, *and*

$$d_q Sq^i u = Sq^i d_r u.$$

If $t-r+1 \leq 2i < t$, *then* $\beta^\varepsilon P^i u$ *survives to* $E_q^{s,t+2i(p-1)+\varepsilon}$ *where* $q=r+(2i-t+r-1)(p-1)+\varepsilon$, $\beta^\varepsilon P^i d_r u$ *survives to* $E_q^{s+q,t+2i(p-1)+\varepsilon+q-1}$, *and*

$$d_q \beta^\varepsilon P^i u = (-1)^\varepsilon \beta^\varepsilon P^i d_r u.$$

(iii) *If* $t \leq i$, *then* $Sq^i u$ *survives to* $E_q^{s+i-t,2t}$ *where* $q=2r-1$, $Sq^i d_r u$ *survives to* $E_q^{s+q+i-t,2t-2r+2}$, *and*

$$d_q Sq^i u = Sq^i d_r u.$$

If $t \leq 2i$, *then* $\beta^\varepsilon P^i u$ *survives to* $E_q^{s+(2i-t)(p-1)+\varepsilon,pt}$ *where* $q=rp-p+1+\varepsilon$, $\beta^\varepsilon P^i d_r u$ *survives to* $E_q^{s+(2i-t)(p-1)+\varepsilon+q,pt+q-1}$, *and*

$$d_q \beta^\varepsilon P^i u = (-1)^\varepsilon \beta^\varepsilon P^i d_r u.$$

THEOREM 1.4. *Let* $\rho\colon F^{s,t}=F^s H^{s+t}(TK) \to E_\infty^{s,t}$ *be the natural projection and* $u \in F^{s,t}$.

(i) *If* $i < t$, *then* $Sq^i u \in F^{s,t}$ *and* $\rho Sq^i u = Sq^i \rho u$.
If $2i < t$, *then* $\beta^\varepsilon P^i u \in F^{s,t}$ *and* $\rho\beta^\varepsilon P^i u = \beta^\varepsilon P^i \rho u$.

(ii) *If* $t \leq i$, *then* $Sq^i u \in F^{s+i-t,2t}$ *and* $\rho Sq^i u = Sq^i \rho u$.
If $t \leq 2i$, *then* $\beta^\varepsilon P^i u \in F^{s+(2i-t)(p-1)+\varepsilon,t-(2i-t)(p-1)-\varepsilon}$ *and* $\rho\beta^\varepsilon P^i u = \beta^\varepsilon P^i \rho u$.

Proofs of Theorems 1.2, 1.3 and 1.4 will be given in § 3.

The Eilenberg-Moore spectral sequence is a typical example of this spectral sequence ([5], [10], [12], [13]). Let G be a connected associative H-space. Let X be a right G-space and Y a left G-space. Then we have the Eilenberg-Moore spectral sequence

$$E_2 \cong \mathrm{Cotor}_{H^*(G;Z_p)}(H^*(X;Z_p), H^*(Y;Z_p)) \Longrightarrow H^*(X \times_G Y; Z_p),$$

to which our results are applicable (see § 4).

It is known in [9], [18] that two kinds of the Steenrod operations are defined on $\mathrm{Cotor}_{H^*(G;Z_p)}(H^*(X;Z_p), H^*(Y;Z_p))$ (=Cotor), that is, the vertical squaring operations

$$Sq_V^i\colon \mathrm{Cotor}^{s,t} \longrightarrow \mathrm{Cotor}^{s,t+i},$$

the diagonal squaring operations

$$Sq_D^i\colon \mathrm{Cotor}^{s,t} \longrightarrow \mathrm{Cotor}^{s+i-t,2t},$$

for $p=2$, and the vertical reduced power operations

$$\beta^\varepsilon P_V^i\colon \mathrm{Cotor}^{s,t} \longrightarrow \mathrm{Cotor}^{s,t+2i(p-1)+\varepsilon},$$

the diagonal reduced power operations

$$\beta^\varepsilon P_D^i\colon \mathrm{Cotor}^{s,t} \longrightarrow \mathrm{Cotor}^{s+(2i-t)(p-1)+\varepsilon,pt},$$

for p an odd prime. The vertical operations are induced by the topological Steenrod operations and the diagonal operations are algebraically defined on Cotor. These operations satisfy the usual properties such as the Cartan formula and the Adem relations (see § 4).

We shall always assume that the coefficient ring in Cotor is Z_2 when we consider these squaring operations, and Z_p, p an odd prime, when we consider these reduced power operations.

THEOREM 1.5. *Through the isomorphism*

$$E_2 \cong \mathrm{Cotor}_{H^*(G;Z_p)}(H^*(X;Z_p), H^*(Y;Z_p))$$

in the Eilenberg-Moore spectral sequence, (i) *the squaring operation* Sq^i *of type* (a) *coincides with the vertical squaring operation* Sq_V^i *if* $i<t$, *and the reduced power operation* $\beta^\varepsilon P^i$ *of type* (a) *coincides with the vertical reduced power operation* $\beta^\varepsilon P_V^i$ *if* $2i<t$, *and* (ii) *the squaring operation* Sq^i *of type* (b) *coincides with the diagonal squaring operation* Sq_D^i *if* $i\geq t$, *and the reduced power operation* $\beta^\varepsilon P^i$ *of type* (b) *coincides with the diagonal reduced power operation* $\beta^\varepsilon P_D^i$ *if* $2i\geq t$.

Since the usual properties of the Steenrod operations such as the Cartan formula and the Adem relations hold on Cotor, these properties inherit on the E_r-term for $r\geq 2$ in the Eilenberg-Moore spectral sequence by Theorems 1.2 and 1.5.

NOTATION.

$$\overline{Sq}_D^i = Sq_D^{i+t}\colon \mathrm{Cotor}^{s,t} \longrightarrow \mathrm{Cotor}^{s+i,2t},$$

$$\beta^\varepsilon \overline{P}_D^i = \beta^\varepsilon P_D^{i+t}\colon \mathrm{Cotor}^{s,2t} \longrightarrow \mathrm{Cotor}^{s+2i(p-1)+\varepsilon,2pt}.$$

THEOREM 1.6.

(i) $Sq_V^{2a}\overline{Sq}_D^b u = \overline{Sq}_D^b Sq_V^a u,\quad Sq_V^{2a+1}\overline{Sq}_D^b u = 0,\qquad$ *for* $u\in \mathrm{Cotor}^{s,t}$.

(ii) $P_V^{pa}\overline{P}_D^b u = \overline{P}_D^b P_V^a u,\quad P_V^{pa+i}\overline{P}_D^b u = 0,\qquad$ *for* $u\in \mathrm{Cotor}^{s,2t}$, *where* $0<i<p$.

Proofs of Theorems 1.5 and 1.6 will be given in § 4.

§ 2. The Steenrod operations

After J. P. May [9], we introduce some categories on which the Steenrod operations will be defined.

Let p be a prime. Let π be a cyclic group of order p with generator α and Σ_p the symmetric group on p-letters. Then π is regarded as a subgroup of Σ_p by $\alpha(1,\dots, p)=(p, 1,\dots, p-1)$.

Let W be the standard $Z_p\pi$-free resolution of Z_p, which has one generator e_i in each dimension $i\geq 0$ (see [9; p. 157]). Let V be a $Z_p\Sigma_p$-free resolution of Z_p and $j\colon W\to V$ be a morphism of $Z_p\pi$-complexes over Z_p. We regard W as a cochain complex by setting $\deg e_i=-i$ so that the differential is of degree $+1$, and also V as a cochain complex in a similar way.

Define a category $\mathscr{C}(p)$ as follows. The objects of $\mathscr{C}(p)$ are pairs (K, θ), where K is a homotopy associative differential Z_p-algebra with differential of degree $+1$ and $\theta\colon W\otimes K^p\to K$ is a morphism of $Z_p\pi$-complexes, where π acts on $K^p=K\otimes\cdots\otimes K$ (p-times) as a permutation, on $W\otimes K^p$ diagonally, and on K trivially, such that (i) the restriction of θ to $e_0\otimes K^p$ is π-homotopic to the iterated product $K^p\to K$ associated in some fixed order, and (ii) θ is π-homotopic to a composition $\xi(j\otimes 1)\colon W\otimes K^p\to V\otimes K^p\to K$, where ξ is a morphism of $Z_p\Sigma_p$-complexes. A morphism $f\colon (K, \theta)\to(K', \theta')$ in $\mathscr{C}(p)$ is a morphism $f\colon K\to K'$ of $Z_p\pi$-complexes such that $\theta'(1\otimes f^p)$ is π-homotopic to $f\theta$.

The category $\mathscr{C}(p)$ is essentially the same as $\mathscr{C}(\pi, \infty, Z_p)$ defined in [9; p. 160]. The only difference between them is the sign convention of degree of differentials.

A morphism $f\colon (K, \theta)\to(K', \theta')$ is said to be *perfect* if $\theta'(1\otimes f^p)=f\theta$, and $\mathscr{P}(p)$ denote the subcategory of $\mathscr{C}(p)$ having the same objects (K, θ) and all perfect morphisms between them. A *unital object, a reduced* mod p *object, a Cartan object* and an *Adem object* in $\mathscr{C}(p)$ are defined in the same way as [9; p. 161, pp. 173–4].

For a simplicial Z_p-module R, let $C(R)$ denote the normalized chain complex.

LEMMA 2.1. *Let π be a cyclic group of order p and W the standard $Z_p\pi$-free resolution of Z_p. Then there is a natural morphism of Z_p-complexes*

$$\Phi\colon W\otimes C(R^p)\longrightarrow W\otimes C(R)^p,$$

where $R^p=R\times\cdots\times R$ (p-times) and $C(R)^p=C(R)\otimes\cdots\otimes C(R)$ (p-times), which satisfies the following properties:

(i) Φ *is π-equivariant,*

(ii) Φ *is the identity homomorphism on* $W\otimes C_0(R^p)$,

(iii) $\Phi(e_0\otimes k_1\times\cdots\times k_p)=e_0\otimes\xi(k_1\times\cdots\times k_p)$ *if* $k_i\in R$, *where* $\xi\colon C(R^p)\to C(R)^p$ *is the Alexander-Whitney map, and*

(iv) $\Phi(W\otimes C_j(R^p))\subset\sum_{k\leq pj}W\otimes[C(R)^p]_k$.

PROOF. See A. Dold [3; Satz 1.12], and J. P. May [9; Lemma 7.1].

q. e. d.

We write ϕ for the composite

$$\phi=(\varepsilon\otimes 1)\Phi\colon W\otimes C(R^p)\xrightarrow{\Phi}W\otimes C(R)^p\xrightarrow{\varepsilon\otimes 1}C(R)^p,$$

where $\varepsilon\colon W\to Z_p$ is an augmentation.

Let $C^*(R)=\mathrm{Hom}\,(C(R),Z_p)$, $(C(R)^p)^*=\mathrm{Hom}\,(C(R)^p,Z_p)$, and $U\colon C^*(R)^p\to(C(R)^p)^*$ be the natural shuffle map. We define a $Z_p\pi$-morphism

$$\theta\colon W\otimes C^*(R)^p\longrightarrow C^*(R)$$

by

$$\theta(w\otimes x)(t)=(-1)^{\deg w\deg x}U(x)\phi(w\otimes t^p),$$

where $w\in W$, $x\in C^*(R)^p$, $t\in C(R)$.

LEMMA 2.2. *$(C^*(R),\theta)$ is a reduced* mod *p object of the category* $\mathscr{C}(p)$.

PROOF. This is immediate from Lemma 2.1 (see [9; pp. 194–5]).

q. e. d.

Let K be a bisimplicial Z_p-module. Let $C(K)$ denote the normalized double Z_p-complex and $T(K)$ the normalized total complex, and set $C^*(K)=\mathrm{Hom}\,(C(K),Z_p)$ and $T^*(K)=\mathrm{Hom}\,(T(K),Z_p)$.

LEMMA 2.3. *There exists a natural morphism of Z_p-complexes*

$$\phi\colon W\otimes T(K)\longrightarrow T(K)^p=T(K)\otimes\cdots\otimes T(K)\ (p\text{-times}),$$

which satisfies the following properties:

(i) ϕ *is π-equivariant,*

(ii) $\phi(w\otimes t)=t^p$, *where t is a 0-simplex and* $w\in W$,

(iii) $\phi(e_0\otimes t)=e_0\otimes\xi(t^p)$, *where* $t\in T(K)$ *and ξ is the Alexander-Whitney map, and*

(iv) $\phi(W\otimes T_j(K))\subset\sum_{k\leq pj}[T(K)^p]_k$.

PROOF. The map ϕ is defined componentwise as follows:

$$W_k \otimes C_{s,t}(K) \xrightarrow{D\otimes D} \sum_{i+j=k} W_i \otimes W_j \otimes C_{s,t}(K^p)$$

$$\xrightarrow{1\otimes\phi^v} \sum_{i+j=k} W_i \otimes \sum_{t_1+\cdots+t_p=t+j} C_{s,t_1}(K) \otimes\cdots\otimes C_{s,t_p}(K)$$

$$\xrightarrow{\phi^h} \sum_{i+j=k} \sum_{\substack{t_1+\cdots+t_p=t+j \\ s_1+\cdots+s_p=s+i}} C_{s_1,t_1}(K) \otimes\cdots\otimes C_{s_p,t_p}(K).$$

Here D is the diagonal map, and ϕ^v and ϕ^h are constructed with respect to the vertical degree and the horizontal degree, respectively, by using Lemma 2.1. Now the lemma is proved by using Lemma 2.1 again. q.e.d.

Let $(T(K)^p)^* = \mathrm{Hom}\,(T(K)^p, Z_p)$, and $U: T^*(K)^p \longrightarrow (T(K)^p)^*$ be the natural shuffle map. We define a $Z_p\pi$-morphism

$$\theta: W\otimes T^*(K)^p \longrightarrow T^*(K)$$

by

$$\theta(w \otimes x)(t) = (-1)^{\deg w \deg x} U(x)\phi(w \otimes t),$$

where $w \in W$, $x \in T^*(K)^p$, $t \in T(K)$.

Lemma 2.4. *$(T^*(K), \theta)$ is a reduced* mod p *object of the category $\mathscr{C}(p)$.*

Proof. By Lemma 2.3, this is proved in the same way as Lemma 2.2. q.e.d.

Now we shall introduce the Steenrod operations, following J. P. May [9]. Let (K, θ) be an object of $\mathscr{C}(p)$. θ induces a morphism $\theta: W\otimes_\pi K^p \to K$ of Z_p-complexes, and we define

$$D^i: H^p(K) \longrightarrow H^{pq-i}(K)$$

by

$$D^i(x) = \theta^*(e_i \otimes x^p) \qquad \text{for} \quad x \in H^q(K).$$

Notation. When p is an odd prime, we set

$$m = (p-1)/2,$$

$$\nu(-q) = (-1)^j (m!)^\varepsilon, \quad \text{where} \quad q = 2j - \varepsilon,\ \varepsilon = 0 \quad \text{or} \quad 1.$$

If $p=2$, then we define $Sq^i: H^q(K) \to H^{q+i}(K)$ by

$$Sq^i(x) = \begin{cases} D^{q-i}(x) & (i \le q) \\ 0 & (i > q). \end{cases}$$

If $p>2$, then we define $P^i\colon H^q(K)\to H^{q+2i(p-1)}(K)$ and $\beta P^i\colon H^q(K)\to H^{q+2i(p-1)+1}(K)$ by

$$P^i(x)=\begin{cases}(-1)^i\nu(-q)D^{(q-2i)(p-1)}(x) & (2i\le q)\\ 0 & (2i>q),\end{cases}$$

$$\beta P^i(x)=\begin{cases}(-1)^i\nu(-q)D^{(q-2i)(p-1)-1}(x) & (2i\le q)\\ 0 & (2i>q).\end{cases}$$

By virtue of Lemmas 2.2 and 2.4, we can define, in the above way, the Steenrod operations in $H^*(TK)$ as well as in $H^*(CR)$. Further, by [9; p. 162], the operation βP^i on $H^*(TK)$ and on $H^*(CR)$ is the composite of P^i and the Bockstein β.

PROOF OF PROPOSITION 1.1. Since $\varepsilon^*\colon C^*(R)\to T^*(K)$ is a morphism of differential Z_p-algebras, the first half follows immediately. By the definitions of θ's, we have the following commutative diagram

$$\begin{tikzcd} W\otimes C^*(R)^p \arrow[r, "\theta"] \arrow[d, "1\otimes(\varepsilon^*)^p"] & C^*(R) \arrow[d, "\varepsilon^*"] \\ W\otimes T^*(K)^p \arrow[r, "\theta"] & T^*(K). \end{tikzcd}$$

Thus the second half follows from the above definition of the Steenrod operations.

q. e. d.

§3. The Steenrod operations in the spectral sequence

Let K be a bisimplicial Z_p-coalgebra. As is described in §1, the decreasing filtration $\{F^rT^*(K)\}$ on the total complex $T^*(K)=\mathrm{Hom}(T(K), Z_p)$ gives rise to a spectral sequence $\{E_r\}$ passing to $H^*(TK)$. In this section we shall introduce the Steenrod operations into the spectral sequence $\{E_r\}$ and prove Theorems 1.2, 1.3 and 1.4.

We first define functions $Sq^i\colon T^q(K)\to T^{q+i}(K)$ and $\beta^\varepsilon P^i\colon T^q(K)\to T^{q+2i(p-1)+\varepsilon}(K)$, $\varepsilon=0, 1$, after S. Araki [1] and J. P. May [9].

Let $a\in T^q(K)$ and $da=b\in T^{q+1}(K)$. Assume that $p>2$. Define $t_l\in T^*(K)^p$ $(1\le l\le p)$ by

$$t_{2k}=\sum_I(-1)^{kq}(k-1)!b^{i_1}a^2b^{i_2}a^2\cdots b^{i_{k+1}}a^2, \qquad 1\le k\le m,$$

where $I=(i_1,\dots,i_k)$ with $\sum i_j=p-2k$, and

$$t_{2k+1}=\sum_I(-1)^{kq}k!b^{i_1}a^2b^{i_2}a^2\cdots b^{i_{k+1}}a, \qquad 0\le k\le m,$$

where $I=(i_1,\dots,i_{k+1})$ with $\sum i_j=p-2k-1$. Then

$$\deg t_{2k} = p(q+1) - 2k, \quad \deg t_{2k+1} = p(q+1) - 2k - 1.$$

Put $j=(q-2i+1)(p-1)$. Define

$$c = \textstyle\sum_{k=0}^{m}(-1)^k e_{j-2k} \otimes t_{2k+1} - \sum_{k=1}^{m}(-1)^k e_{j-2k+1} \otimes (\alpha^{-1}-1)^{p-2} t_{2k},$$

$$c' = \textstyle\sum_{k=0}^{m}(-1)^k e_{j-2k-1} \otimes t_{2k+1} + \sum_{k=1}^{m}(-1)^k e_{j-2k} \otimes t_{2k}.$$

Then

$$\deg c = q + 2i(p-1), \quad \deg c' = q + 2i(p-1) + 1.$$

An easy calculation shows that

$$dc = e_j \otimes b^p, \quad dc' = -e_{j-1} \otimes b^p.$$

Now define functions P^i and βP^i by

$$P^i a = (-1)^i \nu(-q+1)\theta(c),$$

$$\beta P^i a = (-1)^i \nu(-q+1)\theta(c').$$

If $p=2$, we define Sq^i by

$$Sq^i a = \theta(c), \quad \text{where} \quad c = e_{q-i-1} \otimes b \otimes a + e_{q-i} \otimes a \otimes a.$$

Then, we see immediately the following (see J. P. May [9])

LEMMA 3.1. *These functions $Sq^i\colon T^q(K) \to T^{q+i}(K)$ and $\beta^\varepsilon P^i\colon T^q(K) \to T^{q+2i(p-1)+\varepsilon}(K)$ satisfy the following properties*:

(i) $dSq^i = Sq^i d$, $d\beta^\varepsilon P^i = (-1)^\varepsilon \beta^\varepsilon P^i d$.

(ii) *If a is a cocycle which represents $x \in H^*(TK)$, then $Sq^i a$ and $\beta^\varepsilon P^i a$ are cocycles which represent $Sq^i x$ and $\beta^\varepsilon P^i x$, respectively.*

(iii) *If $f\colon (T^*(K), \theta) \to (T^*(K'), \theta')$ is a morphism in $\mathscr{P}(p)$, then $fSq^i = Sq^i f$ and $f\beta^\varepsilon P^i = (-1)^\varepsilon \beta^\varepsilon P^i f$.*

We now estimate the filtration degree. We define a filtration on $T^*(K)^p$ by

$$F^r T^*(K)^p = \sum_{r_1+\cdots+r_p \leq r} F^{r_1} T^*(K) \otimes \cdots \otimes F^{r_p} T^*(K).$$

Then the following lemmas and corollary follow immediately from definitions.

LEMMA 3.2. *If $a \in F^s T^*(K)$ and $da = b \in F^{s+r} T^*(K)$, then*

$$t_{2k} \in F^{sp+(p-2k)r} T^*(K)^p,$$

$$t_{2k+1} \in F^{sp+(p-2k-1)r} T^*(K)^p.$$

LEMMA 3.3.

$$\theta(W_k \otimes F^sT^*(K)^p) \subset F^{s-k}T^*(K),$$

$$\theta(W_k \otimes F^sT^*(K)^p) \subset F^{\mathrm{lig}(s/p)}T^*(K),$$

where $\mathrm{lig}(x)$ *is the least integer greater than or equal to* x.

COROLLARY 3.4. *Let* $a \in F^{s,t} = F^{s,t}T^*(K)$. *Then*

$$\begin{aligned} &Sq^i a \in F^{s,t+i} && \text{if } i < t, \\ &Sq^i a \in F^{s+i-t,2t} && \text{if } i \geq t, \\ &\beta^\varepsilon P^i a \in F^{s,t+2i(p-1)+\varepsilon} && \text{if } 2i < t, \\ &\beta^\varepsilon P^i a \in F^{s+(2i-t)(p-1)+\varepsilon,pt} && \text{if } 2i \geq t. \end{aligned}$$

Therefore in the E_0-term of the spectral sequence passing to $H^*(TK)$, the functions Sq^i and $\beta^\varepsilon P^i$ are defined as follows:

$$Sq^i a = \theta(e_{q-i} \otimes a^2),$$

$$\beta P^i a = (-1)^i \nu(-q+1)\theta(c_0),$$

$$\beta P^i a = (-1)^i \nu(-q+1)\theta(c_0'),$$

for $a \in E_0^{s,t}$, where $q = s+t$ and

$$c_0 = (-1)^{m+mq} m! e_{(q-2i)(p-1)} \otimes a^p,$$

$$c_0' = (-1)^{m+mq} m! e_{(q-2i)(p-1)-1} \otimes a^p.$$

Thus the functions Sq^i and $\beta^\varepsilon P^i$ are homomorphisms on the E_0-term. Generally, recalling the usual formula

$$E_r^{s,t} = Z_r^{s,t} / (dZ_{r-1}^{s-r+1,t+r-2} + Z_r^{s+1,t-1}),$$

$$Z_r^{s,t} = \{x \in F^sT^*(K) | dx \in F^{s+r}T^*(K)\}, \qquad r \geq 1,$$

we obtain homomorphisms

$$\begin{aligned} &Sq^i \colon E_r^{s,t} \longrightarrow E_r^{s,t+i} && (i<t), \\ &Sq^i \colon E_r^{s,t} \longrightarrow E_r^{s+i-t,2t} && (i \geq t), \\ &\beta^\varepsilon P^i \colon E_r^{s,t} \longrightarrow E_r^{s,t+2i(p-1)+\varepsilon} && (2i<t), \\ &\beta^\varepsilon P^i \colon E_r^{s,t} \longrightarrow E_r^{s+(2i-t)(p-1)+\varepsilon,pt} && (2i \geq t), \end{aligned}$$

for all $r \geq 0$.

LEMMA 3.5. *The functions Sq^i and $\beta^\varepsilon P^i$ are homomorphisms on the E_r-terms for all $r \geq 0$.*

We now have

LEMMA 3.6. *Let $a \in Z_r^{s,t}$. Then*

$Sq^i a \in Z_r^{s,t+i}$ *if* $i < t - r + 1$,

$Sq^i a \in Z_q^{s,t+i}$ *where* $q = i - t + 2r - 1$ *if* $t - r + 1 \leq i < t$,

$Sq^i a \in Z_q^{s+i-t,2t}$ *where* $q = 2r - 1$ *if* $i \geq t$,

$\beta^\varepsilon P^i a \in Z_r^{s,t+2i(p-1)+\varepsilon}$ *if* $2i < t - r + 1$,

$\beta^\varepsilon P^i a \in Z_q^{s,t+2i(p-1)+\varepsilon}$ *where* $q = r + (2i - t + r - 1)(p - 1) + \varepsilon$ *if* $t - r + 1 \leq 2i < t$,

$\beta^\varepsilon P^i a \in Z_r^{s+(2i-t)(p-1)+\varepsilon,pt}$ *where* $q = rp - p + 1 + \varepsilon$ *if* $2i \geq t$.

PROOF. Calculate $dSq^i a$, $d\beta^\varepsilon P^i a$ and estimate the filtration degree. Then the lemma follows from Corollary 3.4 and the definitions. q.e.d.

PROOFS OF THEOREMS 1.2, 1.3 AND 1.4. Theorem 1.2 follows immediately from Lemmas 3.1 and 3.5; Theorem 1.3 from Lemmas 3.1 and 3.6, and Theorem 1.4 from Lemma 3.1 and Proposition 1.1. q.e.d.

§4. The Eilenberg-Moore spectral sequence

Let G be a connected associative H-space. Let X be a right G-space and Y a left G-space. The geometric bar construction on X and Y over G, to be denoted by $\boldsymbol{G}=\boldsymbol{G}(X, G, Y)$, is defined as follows. Put

$$\boldsymbol{G}_n = \boldsymbol{G}_n(X, G, Y) = X \times G \times \cdots \times G \times Y, \qquad n \geq 0,$$

where the factor G occurs n-times. Define face operators $\delta_i\colon \boldsymbol{G}_n \to \boldsymbol{G}_{n-1}$ by

$$\delta_i(x, g_1, \dots, g_n, y) = \begin{cases} (xg_1, g_2, \dots, g_n, y) & (i = 0) \\ (x, g_1, \dots, g_i g_{i+1}, \dots, g_n, y) & (1 \leq i \leq n-1) \\ (x, g_1, \dots, g_{n-1}, g_n y) & (i = n) \end{cases}$$

and degeneracy operators $\sigma_i\colon \boldsymbol{G}_n \to \boldsymbol{G}_{n+1}$ by

$$\sigma_i(x, g_1, \dots, g_n, y) = (x, g_1, \dots, g_i, e, g_{i+1}, \dots, g_n, y) \qquad (0 \leq i \leq n)$$

where $e \in G$ is the identity. It is easy to check the simplicial identities in $\boldsymbol{G}(X, G, Y)$.

Let $S_*(T)$ denote the singular chain complex of a space T in coefficient Z_p with all vertices at the base point and $C_*(T)$ denote the normalization of $S_*(T)$. Let $S^*(T) = \mathrm{Hom}(S_*(T), Z_p)$. The complex $S_*(T)$ is regarded as a simplicial Z_p-coalgebra and $S^*(T)$ as a simplicial Z_p-algebra through the Eilenberg-Zilber map.

We now obtain a bisimplicial Z_p-coalgebra K by setting $K_{n,*} = S_*(\boldsymbol{G}_n)$. Here the horizontal face and degeneracy operators are $d_i^h = (\delta_i)_*$ and $s_i^h = (\sigma_i)_*$, respectively, and the vertical operators are the usual ones in $S_*(\boldsymbol{G}_n)$. Dualizing this, we obtain a bicosimplicial Z_p-algebra $K^{**} = \mathrm{Hom}(K_{**}, Z_p)$.

Let $p: \boldsymbol{G}_0 = X \times Y \to X \times_G Y$ be the projection. Then the map

$$p^*: S^*(X \times_G Y) \longrightarrow S^*(X) \otimes S^*(Y),$$

is regarded as a map

$$p^*: S^*(X \times_G Y) \longrightarrow S^*(X) \square_{S^*(G)} S^*(Y),$$

and induces a coaugmentation

$$\eta: S^*(X \times_G Y) \longrightarrow K^{**}.$$

The cohomology of the bicosimplicial Z_p-algebra K^{**} is, by definition, $\mathrm{Cotor}_{C^*(G)}(C^*(X), C^*(Y))$. Now J. C. Moore [10] states that the map η induces an isomorphism

$$H^*(X \times_G Y; Z_p) \cong \mathrm{Cotor}_{C^*(G)}(C^*(X), C^*(Y)).$$

Filter the total complex $T^*(K)$ as in § 1. Then we have the Eilenberg-Moore spectral sequence $\{E_r\}$ such that

$$E_2 \cong \mathrm{Cotor}_{H^*(G; Z_p)}(H^*(X; Z_p), H^*(Y; Z_p)) \Longrightarrow H^*(X \times_G Y; Z_p),$$

into which the Steenrod operations are introduced as is discussed in §§ 2, 3.

We shall recall two kinds of the Steenrod operations in $\mathrm{Cotor}_{H^*(G;Z_p)}(H^*(X; Z_p), H^*(Y; Z_p))$.

Define $H_*(\boldsymbol{G}) = H_*(X) \otimes TH_*(G) \otimes H_*(Y)$, where $TH_*(G)$ is the tensor algebra of $H_*(G)$ and the coefficient ring is Z_p. Then $H_*(\boldsymbol{G})$ forms a simplicial Z_p-coalgebra and the normalization $\boldsymbol{B} = CH_*(\boldsymbol{G})$ coincides, up to sign, with the bar construction. The usual notation $x[g_1|\cdots|g_n]y$ is used for an element in $\boldsymbol{B}$. The differential in $\boldsymbol{B}$ is given by

$$\begin{aligned} d(x[g_1|\cdots|g_n]y) = {} & xg_1[g_2|\cdots|g_n]y \\ & + \sum(-1)^i x[g_1|\cdots|g_i g_{i+1}|\cdots|g_n]y \\ & + (-1)^n x[g_1|\cdots|g_{n-1}]g_n y. \end{aligned}$$

(Remark that the sign convention differs from the usual one.)

LEMMA 4.1. *Let π be a cyclic group of order p and let W be the standard $Z_p\pi$-free resolution of Z_p such that $W_0=Z_p\pi$ with generator e_0. Form $W\otimes\boldsymbol{B}$ and bigrade it by*

$$[W\otimes\boldsymbol{B}]_{s,t}=\sum_{i+j=s}W_i\otimes\boldsymbol{B}_{j,t}.$$

Then there exists a morphism of bigraded $Z_p\pi$-complexes

$$\phi\colon W\otimes\boldsymbol{B}\longrightarrow\boldsymbol{B}^p=\boldsymbol{B}\otimes\cdots\otimes\boldsymbol{B}$$

which is natural in the $\boldsymbol{B}$ and satisfies the following properties:

(i) $\phi(w\otimes b)=0$ *if* $b\in\boldsymbol{B}_0$ *and* $w\in W_i$, $i>0$,

(ii) $\phi(e_0\otimes b)=D(b)$ *if* $b\in\boldsymbol{B}$, *where D is the iterated coproduct,*

(iii) *if $X=G$, then ϕ is a morphism of left $H_*(G)$-modules, where $H_*(G)$ operates on $W\otimes\boldsymbol{B}$ by*

$$a(w\otimes b)=(-1)^{\deg w\deg a}w\otimes ab,$$

(iv) $\phi(W_i\otimes\boldsymbol{B}_{s,t})=0$ *if* $i>(p-1)s$.

PROOF. See, for example, J. P. May [9; Lemma 11.3]. q.e.d.

Define $H^*(\boldsymbol{G})=H^*(X)\otimes TH^*(G)\otimes H^*(Y)$. Then $H^*(\boldsymbol{G})$ forms a cosimplicial Z_p-algebra and let $\boldsymbol{C}=CH^*(\boldsymbol{G})$ denote the normalization of $H^*(\boldsymbol{G})$. Apparently $\boldsymbol{C}$ is the dual to $\boldsymbol{B}$ and is a differential module over the mod p Steenrod algebra.

DEFINITION. Let $U\colon\boldsymbol{C}^p\to(\boldsymbol{B}^p)^*$ be the natural shuffle map and define a $Z_p\pi$-morphism

$$\theta\colon W\otimes\boldsymbol{C}^p\longrightarrow\boldsymbol{C}$$

by

$$\theta(w\otimes x)(k)=(-1)^{\deg w\deg x}U(x)\phi(w\otimes k),$$

for $w\in W$, $x\in\boldsymbol{C}^p$, $k\in\boldsymbol{B}$.

Using the terminology of [9], we have apparently

LEMMA 4.2. *$(\boldsymbol{C},\theta)$ is a reduced* mod *p object, a unital object, a Cartan object and an Adem object of $\mathscr{C}(p)$.*

Consequently we have

THEOREM 4.3. *There exist natural homomorphisms Sq_D^i and $\beta^\varepsilon P_D^i$ for*

$i \geq 0$, $\varepsilon = 0, 1$, *called the diagonal Steenrod operations, defined on* Cotor$=$ $\mathrm{Cotor}_{H^*(G;Z_p)}(H^*(X;Z_p), H^*(Y;Z_p))$, *that is,*

$$Sq_D^i\colon \mathrm{Cotor}^{s,t} \longrightarrow \mathrm{Cotor}^{s+i-t,2t},$$

$$\beta^\varepsilon P_D^i\colon \mathrm{Cotor}^{s,t} \longrightarrow \mathrm{Cotor}^{s+(2i-t)(p-1)+\varepsilon,pt}.$$

These operations satisfy the following properties:

(i) $Sq_D^i = 0$ *if* $i < t$ *or* $i > s+t$,
$P_D^i = 0$ *if* $2i < t$ *or* $2i > s+t$,
$\beta P_D^i = 0$ *if* $2i < t$ *or* $2i \geq s+t$,

(ii) $Sq_D^i x = x^2$ *if* $i = s+t$,
$P_D^{2i} x = x^p$ *if* $i = s+t$, *for* $x \in \mathrm{Cotor}^{s,t}$,

(iii) *the Cartan formula and the Adem relations hold.*

Note that $Sq_D^0 \neq 1$, $P_D^0 \neq 1$.

NOTATION.

$$\overline{Sq}_D^i = Sq_D^{i+t}\colon \mathrm{Cotor}^{s,t} \longrightarrow \mathrm{Cotor}^{s+i,2t},$$

$$\beta^\varepsilon \bar{P}_D^i = \beta^\varepsilon P_D^{i+t}\colon \mathrm{Cotor}^{s,2t} \longrightarrow \mathrm{Cotor}^{s+2i(p-1)+\varepsilon,2pt}.$$

On the other hand, since $\boldsymbol{C}$ is a differential module over the mod p Steenrod algebra, the following Steenrod operations are induced on Cotor:

$$Sq_V^i\colon \mathrm{Cotor}^{s,t} \longrightarrow \mathrm{Cotor}^{s,t+i},$$

$$\beta^\varepsilon P_V^i\colon \mathrm{Cotor}^{s,t} \longrightarrow \mathrm{Cotor}^{s,t+2i(p-1)+\varepsilon},$$

for $i \geq 0$, $\varepsilon = 0, 1$. These operations are called the vertical Steenrod operations and satisfy, a priori, the usual properties such as the Cartan formula and the Adem relations.

LEMMA 4.4. *Let π be a cyclic group of order p. Then the $Z_p\pi$-morphism*

$$\theta\colon W \otimes \boldsymbol{C}^p \longrightarrow \boldsymbol{C},$$

defined after Lemma 4.1, *is a morphism of modules over the* mod p *Steenrod algebra $\mathscr{A}_p$, where $\mathscr{A}_p$ acts on $W \otimes \boldsymbol{C}^p$ by*

$$a(w \otimes c) = (-1)^{\deg w \deg a} w \otimes ac,$$

for $a \in \mathscr{A}_p$, $w \in W$, $c \in \boldsymbol{C}^p$.

PROOF OF THEOREM 1.5. Let $u \in E_2^{s,t}$ be represented by $a \in T^q K$ such that *$a \in F^s TK$ and $da \in F^{s+2} TK$.* Let $p > 2$. Then $\beta^\varepsilon P^i u$ is represented by

$$\beta^\varepsilon P^i a = (-1)^i \nu(-q)\theta(e_{(q-2i)(p-1)-\varepsilon} \otimes a^p)$$

(see § 3). Recall from Lemma 3.6 that

$$\beta^\varepsilon P^i a \in Z_2^{s,t+2i(p-1)+\varepsilon}, \qquad \text{when} \quad 2i < t,$$

$$\beta^\varepsilon P^i a \in Z_2^{s+(2i-t)(p-1)+\varepsilon,pt}, \qquad \text{when} \quad 2i \geq t.$$

Now we have, for $k \in T_*(K)$,

$$(\beta^\varepsilon P^i a)(k) = (-1)^i \nu(-q)\theta(e_{(q-2i)(p-1)-\varepsilon} \otimes a^p)(k)$$
$$= (-1)^{i+\varepsilon pq} \nu(-q) U(a^p)\phi(e_{(q-2i)(p-1)-\varepsilon} \otimes k).$$

(i) Assume that $2i<t$. Then estimating a filtration degree by Lemma 3.2, we need only pick out from k the component which lies in $C_{s,t+2i(p-1)+\varepsilon}$ and consider the composite

$$W_{(q-2i)(p-1)-\varepsilon} \otimes C_{s,t+2i(p-1)+\varepsilon}(K)$$
$$\xrightarrow{D\otimes D} W_{s(p-1)} \otimes W_{(t-2i)(p-1)-\varepsilon} \otimes C_{s,t+2i(p-1)+\varepsilon}(K^p)$$
$$\xrightarrow{1\otimes\phi^v} W_{s(p-1)} \otimes C_{s,t}(K)^p$$
$$\xrightarrow{\phi^h} C_{s,t}(K)^p.$$

Recall from [7; Lemma 8.2] that

$$\phi^h(e_{s(p-1)} \otimes k_1 \otimes \cdots \otimes k_p) = (-1)^{ms} \nu(-s)^{-1} k_1 \otimes \cdots \otimes k_p,$$

and an easy calculation shows that $\beta^\varepsilon P^i a$ represents $\beta^\varepsilon P^i_V u$ on Cotor.

(ii) Assume that $2i \geq t$. Then, estimating a filtration degree, we need only pick out from k the component which lies in $C_{s+(2i-t)(p-1)+\varepsilon,pt}$ and consider the composite

$$W_{(q-2i)(p-1)-\varepsilon} \otimes C_{s+(2i-t)(p-1)+\varepsilon,pt}(K)$$
$$\xrightarrow{D\otimes D} W_{(q-2i)(p-1)-\varepsilon} \otimes W_0 \otimes C_{s+(2i-t)(p-1)+\varepsilon,pt}(K^p)$$
$$\xrightarrow{1\otimes\phi^v} W_{(q-2i)(p-1)-\varepsilon} \otimes C_{s+(2i-t)(p-1)+\varepsilon,t}(K)^p$$
$$\xrightarrow{\phi^h} C_{s,t}(K)^p.$$

Since $\phi^v(e_0 \otimes k_1 \times \cdots \times k_p) = \xi(k_1 \times \cdots \times k_p)$ by Lemma 2.1, $\phi^v D$ is the diagonal map. Remark that ϕ^h commutes with the internal differential. Then an easy calculation shows that $\beta^\varepsilon P^i a$ represents $\beta^\varepsilon P^i_D u$ on Cotor.

If $p=2$, then the proof is similar. q.e.d.

PROOF OF THEOREM 1.6. Let $p>2$. Let $u \in \mathrm{Cotor}^{s,2t} \cong E_2^{s,2t}$ be represented by $x \in T^*(K)$. Then by Lemma 4.4, $P_V^{pa}\bar{P}_D^b u$ is represented by

$$\begin{aligned}(*) &= (-1)^i v(-q) P^{pa}\theta(e_{(s-2b)(p-1)} \otimes x^p) \\ &= (-1)^{i'} v(-q')\theta(e_{(s-2b)(p-1)} \otimes (P^a x)^p) \\ &\quad + \sum \theta(e_{(s-2b)(p-1)} \otimes P^{i_1}x \otimes \cdots \otimes P^{i_p}x),\end{aligned}$$

where $i=b+t$, $q=s+2t$, $i'=t+a(p-1)$, $q'=s+2t+2a(p-1)$. Since the second term is contained in the image of the boundary, $(*)$ represents $\bar{P}_D^b P_V^a u$.

If $p=2$, then the proof is similar. q.e.d.

§5. The Serre spectral sequence

Let $f: E \to B$ be the Serre fibration, where B is simply connected. According to A. Dress [4], there is a bisimplicial Z_p-coalgebra K and an augmentation $\varepsilon: K \to S_*(E)$ such that $\varepsilon^*: H^*(E; Z_p) \to H^*(TK)$ is an isomorphism. Thus the filtration on TK as in §1 gives rise to the Serre spectral sequence

$$E_2^{s,t} \cong H^s(B; H^t(F_b; Z_p)) \Longrightarrow H^{s+t}(E; Z_p), \qquad b \in B,$$

where $F_b = f^{-1}(b)$, and Theorems 1.2, 1.3 and 1.4 recover those in [1], [6], [7] and [19].

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