Structure of Rings Satisfying (Hm) and (Ham)

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All rings considered in this paper are commutative but may not have a unity. An ideal $A$ of a ring $R$ is said to be a multiplication ideal if for every ideal $B$ of $R$, $B \subseteq A$, there is an ideal $C$ of $R$ such that $B = AC$. An ideal $A$ is said to be an $M$-ideal if for every ideal $B$ containing $A$, there is an ideal $C$ of $R$ such that $A = BC$. $R$ is said to be a multiplication ring if every ideal of $R$ is a multiplication ideal (equivalently every ideal is an $M$-ideal). A ring $R$ is said to be an $(AM)$-ring if for any two ideals $A$ and $B$ of $R$, $A < B$, there is an ideal $C$ of $R$ such that $A = BC$. An ideal $A$ is said to be simple if there is no ideal $A'$ with $A' < A$. A ring $R$ is said to be primary if $R$ has at most one proper prime ideal. $R$ is said to be a special primary ring if $R$ has a prime ideal $P$ such that every ideal of $R$ is a power of $P$. If $S$ is a multiplicatively closed subset of $R$ and $A$ is any ideal then $A^e$ denotes the extension of $A$ to the quotient ring $R_S$ and $A^{ec}$ denotes the contraction of $A^e$ to $R$. $R$ is said to satisfy ($\ast$)-condition if every ideal with prime radical is primary. $R$ is said to satisfy (Hm) or (Ham) according as every proper homomorphic image of $R$ is a multiplication ring or an $(AM)$-ring.

The purpose of this note is to determine the structure of rings satisfying (Hm) and (Ham) and the desired structure is given by Theorems 1.7 and 2.5.

1. Let $R$ be a ring and $N$ be its set of nilpotent elements. For any subset $S$ of $R$, define $S_1 = (N : S) =$ set of all $x$ in $R$ such that $xS \subseteq N$[7, p. 434]. The following lemma is due to Griffin [7, Lemma 7].

**Lemma 1.1.** If for any element $x$ of a ring $R$ there exists an ideal $D$ such that $(x) = D(N + (x) + x^1)$ then there is an idempotent $e \in (x^1)^1$ and a positive integer $n$ such that $x^n = ex^n$.

**Lemma 1.2.** If $R$ is a ring satisfying (Hm) and $x \in R$ such that $x^2 \neq 0$ then $(x)$ is an $M$-ideal.

**Proof.** Suppose $A$ is any ideal of $R$ such that $x \in A$. Now $(x)/(x^2) \subseteq A/(x^2)$ in $R/(x^2)$ which is a multiplication ring. There is an ideal $I$ containing $x^2$ such that $(x)/(x^2) = (A/(x^2))(I/(x^2))$. Thus $(x) = AI + (x^2) = A(I + (x)) + (x^2) = A(I + (x))$, since $x^2 \in A(I + (x))$. Therefore $(x)$ is an $M$-ideal.

**Corollary 1.3.** If $R$ is a ring satisfying (Hm) such that $\text{rad}(0) = (0)$ then $R$ is a multiplication ring.
COROLLARY 1.4. If $R$ is a ring satisfying (Hm) and $x \in R$ with $x^2 \neq 0$ then there are an idempotent $e \in (x^1)^1$ and an integer $n$ such that $x^n = ex^n$.

PROOF. It follows from Lemmas 1.1 and 1.2.

LEMMA 1.5. If $R$ is a ring satisfying (Hm) such that $x^2 \neq 0$ for some $x \in R$ then $R$ is idempotent.

PROOF. Since $R/(x^2)$ is a multiplication ring, $(R/(x^2))^2 = R/(x^2)$. Thus $R = R^2 + (x^2) = R^2$.

THEOREM 1.6. If $R$ is a ring satisfying (Hm) and $x \in R$ such that $x^2 \neq 0$ then there exists an idempotent $e$ such that $x = ex$.

PROOF. Since $x^2 \neq 0$, $(x)$ is an $M$-ideal. There is an ideal $J$ of $R$ such that $(x) = IR = IR^2 = (IR)R = xR$. Let $x = xy$, $y \in R$. Now $0 \neq x^2 = x^2y^2$ implies that $y^2 \neq 0$ and by Corollary 1.4 we get an idempotent $e$ and an integer $n$ such that $y^n = ey^n$. Then $x = xy = xy^2 = \cdots = xy^n = x(ey^n) = e(xy^n) = ex$.

NOTATION. Let $R$ be a ring and $x$ a non-zero element of $R$. If there exists a prime integer $p$ such that $px = 0 = x^2$ then we denote $I^*_p = \{x, 2x, \ldots, px = 0\}$ which is isomorphic to $\mathbb{Z}/(p)$ as a $\mathbb{Z}$-module.

THEOREM 1.7.* A ring $R$ satisfies (Hm) if and only if $R$ satisfies one of the following:

I. $R$ is a multiplication ring.
II. $x^2 = 0$ for each $x \in R$ and $R = l^p$ type.
III. $R$ has a unity and a unique maximal ideal $M$ such that
   (i) $M^2 = (0)$.
   (ii) If $x, y \in M$ such that $(x) \neq (y)$ and $(y) \neq (x)$ then $M = (x) + (y)$.
   (iii) There is an ideal $A$ such that $(0) < A < M$ and every such $A$ is principal.
   (iv) $R$ does not contain a chain of five ideals.
   (v) $R$ is noetherian.

PROOF. Assume $R$ satisfies (Hm). Suppose II does not hold. Let $x \in R$ such that $x^2 \neq 0$. By Theorem 1.6 there exists an idempotent $e$ such that $x = ex$. Let $A = eR$ and $B = \{r - er : r \in R\}$. Then $A$ and $B$ are ideals of $R$ and it is easy to see that $R = A \oplus B$. Clearly $A \neq (0)$. If $A < R$ then $B \neq (0)$ and hence $A(\cong R/B)$ and $B(\cong R/A)$ are multiplication rings and consequently $R$ is a multiplication ring. If $A = R$ then $e$ is the unity of $R$ and (i) to (v) of III follow from [14, Theorem 2.5 and Theorem 3.12]. Now suppose $x^2 = 0$ for each $x \in R$. If $(0) < (x) < R$,

*) I am indebted to the referee, whose comments enabled me to put Theorem 1.7 in the present form.
then $R/(x)$ is a multiplication ring. Let $\bar{e}=e+(x)$ be any non-zero idempotent in $R/(x)$. It can be easily seen that $\bar{e}^2 \neq 0$ which is impossible. Thus $R=(x)$ for every $x \neq 0$ in $R$. It is now plain that $R=I_p^*$ type for some prime integer $p$.

The converse is trivial, for if $R$ satisfies I or II then $R$ evidently satisfies $(Hm)$ and if $R$ satisfies III then $R$ satisfies $(Hm)$ by [14, Theorem 3.12].

**Corollary 1.8.** A ring satisfying $(Hm)$ satisfies $(\ast)$-condition.

**Proof.** This follows from Theorem 1.7 and [6, Theorem 7].

2. In this section we establish the structure of rings satisfying $(Ham)$. The structure of $(AM)$-rings was established by Mori [10] and Griffin [7].

**Lemma 2.1.** If $R$ is an $(AM)$-ring then $R$ satisfies one of the following:
I. $R=R^2$ and hence $R$ is a multiplication ring.
II. $R \neq R^2$ and every non-zero ideal of $R$ is principal and a power of $R$.

**Proof.** This is [7, Proposition 4].

**Lemma 2.2.** Let $R$ be a ring satisfying $(Ham)$. If $A<B$ are ideals of $R$ such that $AB \neq (0)$ then there is an ideal $C$ of $R$ such that $A=CB$.

**Proof.** Let $a \in A$ and $b \in B$ such that $ab \neq 0$. Since $A/(a)<B/(a)$, there is an ideal $I$ containing $(a)$ such that $A/(a)=(I/(a))(B/(a))$. Thus $A=IB+(a)$. Again $(a)/(ab)<B/(ab)$ implies that there is an ideal $J$ containing $(ab)$ such that $(a)=JB+(ab)$. Thus $A=IB+JB+(ab)=(I+J)B+(ab)=(I+J)B$.

**Corollary 2.3.** If $R$ is a ring satisfying $(Ham)$ without nilpotent elements then $R$ is an $(AM)$-ring.

**Lemma 2.4.** If $A$ is any ideal of a ring $R$ such that there is no ideal of $R$ properly between $A$ and $A^2$ then for every positive integer $n$, the only ideals between $A$ and $A^n$ are $A, A^2, A^3, ..., A^n$.

**Proof.** This is [3, Lemma 3].

**Theorem 2.5.** A ring $R$ satisfies $(Ham)$ if and only if $R$ satisfies one of the following:
I. $R=R^2$ and $R$ satisfies $(Hm)$.
II. $R \neq R^2$ but $R^2=(0)$ such that every non-zero proper ideal of $R$ is of the type $I_p^*$ and every two proper distinct ideals $I_p^*$ and $I_q^*$ intersect at $(0)$ and $R=I_p^* \oplus I_q^*$.
III. Either $R$ is an $(AM)$-ring or there is a non-zero proper prime ideal $P$ of $R$ satisfying the following:
   (i) $P^2=(0)$ and $P=I_p^*$ type.
(ii) \( P < R^2 \) or \( R = R^2 \oplus P \).

(iii) The only ideals of \( R \) are \((0), P, R, R^2, \ldots\). Each ideal of \( R \) is generated by at most two elements.

**Proof.** Suppose \( R \) satisfies \((Hm)\).

Case I. \( R = R^2 \). We shall prove that \( R \) satisfies I. Let \( A \neq (0) \) be any ideal of \( R \). Since \( R/A \) is an \((AM)\)-ring and \( (R/A)^2 = R/A \), we deduce from Lemma 2.1 that \( R/A \) is a multiplication ring. Thus \( R \) satisfies \((Hm)\).

Case II.* \( (0) \neq R^2 < R \). In this case the ideals of \( R \) are the \( Z \)-submodules of the additive group \( R \). By Lemma 2.1, every homomorphic image of \( R \) is simple and isomorphic to \( Z/(p) \) for some prime \( p \). It follows that \( R \) is a finitely generated abelian group. By Lemma 2.1, \( R \) satisfies the condition II.

Case III. \( (0) < R^2 < R \). Let \( 0 \neq y \in R^2 \). Suppose there is an ideal \( I \) such that \( R^2 < I < R \). Then \( R/(y) \) is an \((AM)\)-ring and \( (R/(y))^2 = R^2/(y) \leqslant R/(y) \). Lemma 2.1 implies that every non-zero ideal of \( R/(y) \) is a power of \( R/(y) \) which is impossible since \( (R/(y))^2 < I/(y) < R/(y) \). Thus there is no ideal of \( R \) properly between \( R \) and \( R^2 \). Using Lemma 2.4 we deduce that the only ideals of \( R \) between \( R \) and \( R^n \) are \( R, R^2, \ldots, R^n \) for every integer \( n \). Hence every ideal of \( R \) properly containing \( (y) \) is a power of \( R \). Let \( A \) be any ideal of \( R \). If \( A^2 \neq (0) \) then every ideal of \( R \) properly containing \( A^2 \) is a power of \( R \). In particular if \( A^2 < A \) then \( A \) is a power of \( R \). Hence for every ideal \( A \) of \( R \), either \( A^2 = (0) \) or \( (0) \neq A = A^2 \) or \( A \) is a power of \( R \). Suppose \( A^2 \neq (0) \) and \( A \) is not a power of \( R \). Then \( A = A^2 \). Let \( 0 \neq x \in A^2 \). Then every ideal of \( R \) properly containing \( (x) \) is a power of \( R \). As \( (x) \subseteq A \) and \( A \) is not a power of \( R \), we get \( (x) = A \). Since \( A = A^2 \), \( (x^2) = (x^3) = \cdots \). Let \( x = rx^2, r \in R \). Then \( (rx)^2 = rx \). Denote \( e = rx \). Then \( e \) is a non-zero idempotent and \( A = (x) = (e) \). Let \( B = \{ r - er : r \in R \} \). Then \( R = A \oplus B \). \( A \cong R/B \) and \( A^2 = A \) implies that \( A \) is a multiplication ring. Since \( R \) is not a multiplication ring, \( B \) is not a multiplication ring. But \( B \cong R/A \) is an \((AM)\)-ring. Therefore \( B^2 \neq B \). Hence \( B^2 = (0) \) or \( B = R^k \) for some integer \( k > 1 \). If \( B^2 = (0) \), then \( R^2 = A^2 \oplus B^2 = A^2 = A \subseteq R \). We get that \( A = R^2 \) which is impossible. Now suppose that \( B = R^k, k > 1 \). Then \( R = A \oplus R^k = A^2 \oplus R^k \subseteq R^2 \) which is again impossible. Thus for every ideal \( A \) of \( R \), either \( A \) is a power of \( R \) or \( A^2 = (0) \). If \( A \) is any proper ideal of \( R \) such that \( A \not\subseteq R^2 \), then \( R = R^2 + A \). If there is a non-zero \( y \in R^2 \cap A \), then \( A \) is a power of \( R \) or \( A = (y) \subseteq R^2 \), a contradiction. Hence \( R = R^2 \oplus A \). Let \( 0 \neq a \in A \). Then as above \( R = R^2 \oplus (a) \) and therefore \( A = (a) \). Thus every non-zero ideal \( A \) of \( R \) satisfies one of the following:

*) I am thankful to the referee for suggesting me the proof of Case II which has considerably simplified my original proof.
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(i) \(A\) is a power of \(R\).

(ii) \(A^2=(0)\), \(A\) is a principal ideal generated by every non-zero element of \(A\) such that either \(R=R^2\oplus A\) or \(A<R^2\).

Also \(R^2\neq(0)\). Let \(a, b \in R\) such that \(ab\neq 0\). If \((ab)<(a)\) then \((a)\) is a power of \(R\) and if \((ab)<(b)\) then \((b)\) is a power of \(R\). If \((ab)\equiv(a)=(b)\) then we get \((a)\equiv(a^2)\) and such a case is impossible, as we have already proved. Thus for some \(k\), \(R^k=(x)\) is a principal ideal. If \(k=1\) then every ideal of \(R\) is principal. Suppose \(k>1\). Let \(R^t\) be any power of \(R\). We can find a least integer \(m\) such that \(t<2mk\). If \(R^t=R^{2mk}\) then \(R^t\) is a principal ideal. If \(R^t>R^{2mk}\) and \(R^{2mk}\neq(0)\) then \(R^t/R^{2mk}\) is a non-zero ideal of \(R/R^{2mk}\) which is an \((AM)\)-ring whose every ideal is principal. Since \(R^{2mk}\) and \(R^t/R^{2mk}\) are principal ideals, \(R^t\) is generated by at most two elements. If \(R^{2mk}=(0)\) then by Lemma 2.4, the only ideals of \(R\) are powers of \(R\) and hence \(R\) is an \((AM)\)-ring.

Consider now \(\text{rad}(0)\). If \(\text{rad}(0)=R\) then every element of \(R\) is nilpotent. Thus \(R^4=(x)\) is nilpotent, showing that \(R\) is an \((AM)\)-ring. If \(\text{rad}(0)\neq R\) then there is a prime ideal \(P\), \((0)<P<R\). Clearly \(P\) is not a power of \(R\). Thus \(P^2=(0)\) and \(P\) is the principal ideal generated by every non-zero element of \(P\) such that either \(R=R^2\oplus P\) or \(P<R^2\). Suppose \(A\neq(0)\) be any ideal of \(R\) which is not a power of \(R\). Then \(A^2=(0)\) and it implies that \(A\subseteq P\). Since \(P\) is generated by every non-zero element of \(P\), \(A=P\). Thus \(P\) is the only non-zero ideal of \(R\) which is not a power of \(R\). Hence either \(R\) is an \((AM)\)-ring or there is a prime ideal \(P\) of \(R\) such that \(P=I_2^t\) type, \(P<R^2\) or \(R=R^2\oplus P\).

Now assume that \(R\) satisfies any one of I, II, III. If \(R\) satisfies I then clearly \(R\) satisfies \((Ham)\). Suppose \(R\) satisfies II. If \(A\) is any non-zero proper ideal of \(R\) then \(R=A\oplus I_2^t\) type by II. Since \(I_2^t\) is an \((AM)\)-ring, \(R/A(\cong I_2^t)\) is an \((AM)\)-ring and hence \(R\) satisfies \((Ham)\). Lastly assume that \(R\) satisfies III. If \(R^k\neq(0)\) for any \(k\) then \(R/R^k\) is clearly an \((AM)\)-ring. It remains only to verify that \(R/P\) is an \((AM)\)-ring. Now any non-zero ideal of \(R/P\) is \((R^k+P)/P\) for any \(k\) an integer such that \(R^k\notin P\). Now \((R^k+P)/P=(R/P)^k\) and hence \(R/P\) is an \((AM)\)-ring.

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References


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