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Existence of Various Boundary Limits of Beppo Levi Functions of Higher Order

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1. Introduction

Let U be the unit open ball with center at the origin in the n-dimensional Euclidean space R^n. It is well known that a non-negative function u harmonic in U has a non-tangential limit at almost every boundary point of U. Diederich [4] proved that such a function u has an mc (mean continuous) limit at almost every point of ∂U; we say that u has an mc limit ℓ at $\xi \in \partial U$ if

$$\lim_{r \downarrow 0} \frac{1}{r^n} \int_{B(\xi,r) \cap U} |u(x) - \ell| \, dx = 0,$$

$B(\xi, r)$ being the open ball with center at ξ and radius r. By the mean value property of harmonic functions, we can show easily that u has a non-tangential limit at every point of ∂U at which u has an mc limit.

Now let f be a function defined on U whose (partial) derivatives of the first order exist a.e. in U and satisfy

$$\text{(a)} \qquad \int_U |\operatorname{grad} f|^2 (1 - |x|)^\alpha dx < \infty, \qquad 0 \leqq \alpha < 1.$$

This condition only does not necessarily ensure the existence of non-tangential limits of f (see Proposition 2 in Sec. 4). In case $n=2$, assuming that f is continuous in U, Carleson [2; Theorem 3 in Sec. V] proved the existence of radial limits of f. Wallin [19; Theorem 1] generalized Carleson's theorem to higher dimensional case with f defined on the upper half space R^n_+ and satisfying the condition analogous to (a):

$$\text{(b)} \qquad \int \cdots \int_G |\operatorname{grad} f|^2 x_n^\alpha dx_1 \cdots dx_n < \infty, \qquad 0 \leqq \alpha < 1,$$

for any bounded open set $G \subset R^n_+$. He also proved that if in addition f is harmonic in R^n_+, then the non-tangential limit of f exists at $\xi \in \partial R^n_+$ except for a set whose Riesz capacity of order $2-\alpha$ is zero ([19; Theorem 3]).

In this paper we are concerned with Beppo Levi functions f of order m defined on R^n_+ which satisfy

(c) $$\sum_{|\lambda|=m}\int\cdots\int_G |D^\lambda f|^p x_n^\alpha dx_1\cdots dx_n < \infty$$

for any bounded open set $G\subset R_+^n$, where $1<p<\infty$, $-\infty<\alpha<\infty$ and $D^\lambda=(\partial/\partial x_1)^{\lambda_1}\cdots(\partial/\partial x_n)^{\lambda_n}$ for a multi-index $\lambda=(\lambda_1,\dots,\lambda_n)$ with length $|\lambda|=\lambda_1+\cdots+\lambda_n$. In case $m=1$, the existence of perpendicular boundary limits of f was given by [11; Theorem 1] as a generalization of [19; Theorem 1]. The existence of non-tangential limits of f with an additional condition that f is polyharmonic of order m in R_+^n was discussed in [13; Theorem 1] for $m=1$ and in [16; Theorem 1] for general m. Our aim is, therefore, to improve [11; Theorem 1] and [16; Theorem 1].

We shall show first that if f is a function defined on R_+^n and satisfying (c), then f has mc limits at points of ∂R_+^n with an exceptional set whose size is well evaluated by the Bessel capacity. (For the definition and properties of Bessel capacity, one may refer to [8].) Next we prove, with the aid of a property of polyharmonic functions given by Edenhofer [5], that if in addition f is polyharmonic of order $m+1$ in R_+^n, then non-tangential limits of f exist. This gives a generalization of [16; Theorem 1]. We include this theorem here as an example of application of Theorem 1 although it is a special case of Theorem 2′ which will be proved later in Section 9. We shall also discuss the existence of perpendicular boundary limits of f in order to obtain an improvement of [11; Theorem 1].

In case $n=2$, Gavrilof [7; Theorem 3] showed that given a function f on U satisfying (a), for $\xi\in\partial U$ except those in a set whose Riesz capacity of order $2-\alpha$ is zero, there exists a constant c_ξ satisfying

$$c_\xi = \lim_{r\downarrow 0} f(\xi+(r\cos\theta,\ r\sin\theta)) \qquad \text{for almost every} \quad \theta\in(0,\pi).$$

By using [14; Theorem 1], we can generalize this result to the case where f is an (m, p)-quasi continuous function on R_+^n (see [10; p. 379]) satisfying (c). In Section 6 we shall discuss the fine limits with respect to a suitable capacity for functions described above.

For such a function f satisfying (c), we shall study in Section 7 when the equality

$$\lim_{r\downarrow 0}\frac{1}{r^n}\int_{B(\xi,r)\cap R_+^n}|f(x)-\ell|^q dx = 0, \qquad q>1,$$

holds. Clearly, Hölder's inequality implies that f has an mc limit ℓ at ξ at which the above equality holds. The investigation of this problem can be done in a way similar to the case of mc limits. However q depends on m, p, α and this fact complicates the matters. The main difference with the case of mc limits is that we must use a generalization of Sobolev's inequality.

In Section 9 we shall prove the existence of non-tangential limits of polyharmonic functions in a general domain. The proof, different from that of Theorem 2, will be carried out along the same lines as the proofs of [13; Theorem 1] and [16; Theorem 1]. The essential tool is an integral representation of polyharmonic functions of order m by means of their derivatives of order less than m, which is derived from a result of Edenhofer [5].

2. Preliminaries

Let R^n, $n \geqq 2$, be the n-dimensional Euclidean space. A point $x \in R^n$ will be sometimes written as $(x', x_n) \in R^{n-1} \times R^1$. We set

$$R_+^n = \{x = (x', x_n) \in R^n;\ x_n > 0\},$$

$$R_0^n = \{x = (x', x_n) \in R^n;\ x_n = 0\}.$$

For a point $x = (x_1, \dots, x_n)$ and a multi-index $\lambda = (\lambda_1, \dots, \lambda_n)$, we define

$$x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}, \quad |\lambda| = \lambda_1 + \cdots + \lambda_n,$$

$$D^\lambda = (\partial/\partial x)^\lambda = (\partial/\partial x_1)^{\lambda_1} \cdots (\partial/\partial x_n)^{\lambda_n}.$$

Given a function u whose derivatives of order m exist a.e. on an open set $G \subset R^n$, the vector valued function defined by

$$\boldsymbol{D}_m u(x) = (D^\lambda u(x))_{|\lambda| = m}$$

is called the gradient of order m of u; in particular, $\boldsymbol{D}_0$ = identity and $\boldsymbol{D}_1$ will be written sometimes as $\boldsymbol{D}$.

Following [3], we shall use the notation $BL_m(L_{loc}^p(R_+^n))$ to denote the space of all functions in $L_{loc}^p(R_+^n)$ whose (distributional) derivatives of order m are all in $L_{loc}^p(R_+^n)$. Throughout this paper, let $1 < p < \infty$. If a function $u \in L_{loc}^p(R_+^n)$ belongs to $BL_m(L_{loc}^p(R_+^n))$, then $u \in BL_k(L_{loc}^p(R_+^n))$ for any positive integer $k < m$ (cf. [3; Théorème 2.1]). For any $u \in BL_1(L_{loc}^p(R_+^n))$, there is a function which is equal to u a.e. on R_+^n and p-precise[1)] on any relatively compact open subset of R_+^n (or locally p-precise on R_+^n in the sense of Ohtsuka [17; Chap. IV]). One can show that any locally p-precise function u on R_+^n is absolutely continuous along almost every (half) line parallel to the coordinate axis and contained in R_+^n, so that u is partially differentiable a.e. on R_+^n. Moreover, for $\xi \in R_+^n$, $u(\xi + r\sigma)$ is absolutely continuous as a function of $r > 0$ for a.e. $\sigma \in \partial B(O, 1) \cap R_+^n$.

The Bessel capacity of index (β, p) is denoted by $B_{\beta,p}$. For the definition, see [6]. Denoting by C_β the Riesz capacity of order β, we have the following relations between these capacities:

1) For the definition of p-precise functions, see Ziemer [20] and Ohtsuka [17; Chap. IV].

1) If $B_{\beta,p}(E)=0$, then $C_{\beta p}(E)=0$ in case $p\leqq 2$ and $C_\gamma(E)=0$ for any $\gamma>0$ with $\gamma<\beta p$ in case $p>2$.

2) If $C_\gamma(E)=0$, then $B_{\gamma/p,p}(E)=0$ in case $p\geqq 2$ and $B_{\beta,p}(E)=0$ for any $\beta>0$ with $\beta<\gamma/p$ in case $p<2$.

These follow from Fuglede [6] together with [10; Theorems 2.4 and 3.2].

3. Mean continuous limit

We say that a function u on R_+^n has an mc limit of order $q\geqq 1$ at $\xi\in R_0^n$ if there is a number ℓ with

$$\lim_{r\downarrow 0}\frac{1}{r^n}\int_{B_+(\xi,r)}|u(x)-\ell|^q dx = 0,$$

where $B_+(\xi, r)=B(\xi, r)\cap R_+^n$. In case $q=1$, u is said to have an mc limit ℓ as defined in the introduction.

In Section 7 we shall be concerned with mc limits of general order. The statement of the result in the general case as well as its proof are rather complicated. So we start with the case $q=1$ in which the result has a simpler form.

THEOREM 1. *Let m be a positive integer, $1<p<\infty$ and $-\infty<\alpha<p-1$. Let $u\in BL_m(L_{\mathrm{loc}}^p(R_+^n))$ satisfy*

$$(1)\qquad \iint_G |\boldsymbol{D}_m u(x', x_n)|^p x_n^\alpha dx' dx_n < \infty$$

for any bounded open set $G\subset R_+^n$. Then we can find a Borel set $E\subset R_0^n$ such that $B_{m-\alpha/p,p}(E)=0$ and u has an mc limit at each point of $R_0^n\setminus E$.

Before proving this theorem, we prepare several lemmas. Let us begin with the following lemma.

LEMMA 1. *Let K be a Borel measurable function on $R^n\times R^n$ which is continuous outside the diagonal set $\{(x, x); x\in R^n\}$. Suppose there are constants $\beta>0$ and $C>0$ such that*

$$|K(x, y)|\leqq C|x-y|^{\beta-n}\qquad \text{for any}\quad x, y\in R^n.$$

For a measure μ, we set

$$U_K^\mu(x)=\int K(x, y)d\mu(y)$$

at $x\in R^n$ at which the integral has a meaning. If $\int|x^0-y|^{\beta-n}d|\mu|(y)<\infty$ and $|\mu|(\{x^0\})=0$ ($|\mu|$ denoting the total variation of μ), then

$$\lim_{r \downarrow 0} \frac{1}{r^n} \int_{B(x^0, r)} |U_K^\mu(x) - U_K^\mu(x^0)| dx = 0.$$

PROOF. Set

$$U_1(x) = \int_{|x-y|<|x-x^0|/2} K(x, y) d\mu(y),$$

$$U_2(x) = \int_{|x-y| \geqq |x-x^0|/2} K(x, y) d\mu(y).$$

Then we have

$$\begin{aligned}
&\frac{1}{r^n} \int_{B(x^0,r)} |U_1(x)| dx \\
&\leqq Cr^{-n} \int_{B(x^0,r)} \left\{ \int_{|x-y|<|x^0-x|/2} |x - y|^{\beta-n} d|\mu|(y) \right\} dx \\
&\leqq Cr^{-n} \int_{B(x^0,2r)} \left\{ \int_{|x-y|<|x^0-y|} |x - y|^{\beta-n} dx \right\} d|\mu|(y) \\
&\leqq C' r^{-n} \int_{B(x^0,2r)} |x^0 - y|^{\beta} d|\mu|(y) \\
&\leqq C'' \int_{B(x^0,2r)} |x^0 - y|^{\beta-n} d|\mu|(y) \\
&\longrightarrow 0 \quad \text{as} \quad r \downarrow 0,
\end{aligned}$$

where C' and C'' are constants. On the other hand, in case $\beta \leqq n$, since $|K(x, y)| \leqq \text{const.} |x^0 - y|^{\beta-n}$ if $|x-y| \geqq |x^0 - x|/2$, we can apply Lebesgue's dominated convergence theorem to obtain

$$\lim_{x \to x^0} U_2(x) = U_K^\mu(x^0).$$

This holds also in the case $\beta > n$, because K is continuous on $R^n \times R^n$ in this case. Hence one can show easily

$$\lim_{r \downarrow 0} \frac{1}{r^n} \int_{B(x^0,r)} |U_2(x) - U_K^\mu(x^0)| dx = 0.$$

Consequently,

$$\begin{aligned}
&\limsup_{r \downarrow 0} \frac{1}{r^n} \int_{B(x^0,r)} |U_K^\mu(x) - U_K^\mu(x^0)| dx \\
&\leqq \lim_{r \downarrow 0} r^{-n} \int_{B(x^0,r)} |U_1(x)| dx + \lim_{r \downarrow 0} r^{-n} \int_{B(x^0,r)} |U_2(x) - U_K^\mu(x^0)| dx = 0,
\end{aligned}$$

which proves the lemma.

LEMMA 2. *Let* $-1<\alpha<p-1$. *Let* f *be a non-negative function in* $L^1_{loc}(R^n)$ *which vanishes outside some compact set in* R^n *and satisfies* $\int_{R^n} f(y)^p |y_n|^\alpha dy<\infty$. *Then, for a positive integer* m *the function*

$$F(x) = \int_{R^n} |x-y|^{m-n} f(y)dy, \qquad x\in R^n_+,$$

belongs to $BL_m(L^p_{loc}(R^n_+))$ *and* $\int_{R^n_+} |\boldsymbol{D}_m F|^p x_n^\alpha dx<\infty$.

PROOF. For $\varepsilon>0$, we set

$$\kappa_\varepsilon(x) = (|x|^2+\varepsilon)^{(m-n)/2}$$

and define

$$F_\varepsilon(x) = \int_{R^n} \kappa_\varepsilon(x-y) f(y)|y_n|^{\alpha/p}dy, \qquad x\in R^n.$$

Then F_ε is infinitely differentiable on R^n and, on account of [10; Lemma 3.2 and its proof], there is a constant $c_1>0$ independent of ε such that

$$\int_{R^n} |\boldsymbol{D}_m F_\varepsilon(x)|^p dx \leqq c_1 \int_{R^n} f(y)^p |y_n|^\alpha dy.$$

Since $f\in L^1(R^n)$, $F\in L^1_{loc}(R^n_+)$ and $\kappa_\varepsilon * f$ is infinitely differentiable on R^n. Let $\lambda=(\lambda_1,\dots,\lambda_n)$ be a multi-index with length m. Then

$$|x_n^{\alpha/p} D^\lambda(\kappa_\varepsilon * f)(x) - D^\lambda F_\varepsilon(x)| \leqq c_2 \int_{R^n} |x_n^{\alpha/p} - |y_n|^{\alpha/p}|\,|x-y|^{-n} f(y)dy$$

$$= c_2 \int_{-\infty}^{\infty} K(x_n, y_n) g(x', x_n, y_n) dy_n,$$

where c_2 is a positive constant and

$$K(x_n, y_n) = \frac{|1-(x_n/|y_n|)^{\alpha/p}|}{|x_n-y_n|},$$

$$g(x', x_n, y_n) = \int_{R^{n-1}} \frac{|x_n-y_n|}{\{|x'-y'|^2+(x_n-y_n)^2\}^{n/2}} f(y)|y_n|^{\alpha/p} dy'$$

for $x=(x', x_n)$ and $y=(y', y_n)$. By a property of Poisson integral (cf. [18; Theorem 1, (a) in Chap. III and Theorem 1, (c) in Chap. I]), we can find a constant $c_3>0$ independent of x_n and y_n such that

$$\int g(x', x_n, y_n)^p dx' \leqq c_3 \int f(y)^p |y_n|^\alpha dy'.$$

Applying Appendices A.1 and A.3 in [18], we can derive that

$$\iint_{R_+^n} \left\{ \int_{-\infty}^{\infty} K(x_n, y_n) g(x', x_n, y_n) dy_n \right\}^p dx' dx_n$$

$$\leqq \int_0^\infty \left(\int_{-\infty}^{\infty} K(x_n, y_n) \left\{ \int g(x', x_n, y_n)^p dx' \right\}^{1/p} dy_n \right)^p dx_n$$

$$\leqq c_3 \int_0^\infty \left(\int_{-\infty}^{\infty} K(x_n, y_n) \left\{ \int f(y', y_n)^p |y_n|^\alpha dy' \right\}^{1/p} dy_n \right)^p dx_n$$

$$\leqq c_3 A_K^p \int_{R^n} f(y)^p |y_n|^\alpha dy,$$

where $A_K = \int_{-\infty}^{\infty} K(1, y_n) |y_n|^{-1/p} dy_n < \infty$. We thus obtain

$$\left\{ \int_{R_+^n} |D^\lambda(\kappa_\varepsilon * f)(x)|^p x_n^\alpha dx \right\}^{1/p} \leqq \|D^\lambda F_\varepsilon\|_p + c_3^{1/p} A_K \left\{ \int_{R^n} f(y)^p |y_n|^\alpha dy \right\}^{1/p}$$

$$\leqq c_4 \left\{ \int f(y)^p |y_n|^\alpha dy \right\}^{1/p}$$

with $c_4 = c_1^{1/p} + c_3^{1/p} A_K$, which is independent of ε. Now we show that

$$\int_{R_+^n} |D^\lambda F|^p x_n^\alpha dx < \infty. \tag{2}$$

Before proving this fact, we note that $F \in BL_m(L_{loc}^p(R_+^n))$, since for any $a > 0$,

$$G_a(x) = \int_{Q_a} |x - y|^{m-n} f(y) dy$$

belongs to $BL_m(L^p(R^n))$ because of [10; Lemma 3.3] and $F - G_a$ is infinitely differentiable on $Q_{a/2}$, where

$$Q_a = \{x = (x', x_n) \in R^n;\ |x'| < a,\ a^{-1} < x_n < a\}.$$

To prove (2), let $\varphi \in C_0^\infty(R_+^n)$. Then

$$\int D^\lambda F(x) \varphi(x) x_n^{\alpha/p} dx = (-1)^m \int F(x) D^\lambda(\varphi(x) x_n^{\alpha/p}) dx$$

$$= (-1)^m \lim_{\varepsilon \downarrow 0} \int (\kappa_\varepsilon * f)(x) D^\lambda(\varphi(x) x_n^{\alpha/p}) dx$$

$$= \lim_{\varepsilon \downarrow 0} \int D^\lambda(\kappa_\varepsilon * f)(x) \varphi(x) x_n^{\alpha/p} dx$$

$$\leqq \limsup_{\varepsilon\downarrow 0}\left\{\int_{R^n_+}|D^\lambda(\kappa_\varepsilon * f)(x)|^p x_n^\alpha dx\right\}^{1/p}\|\varphi\|_{p'}$$

$$\leqq c_4\left\{\int f(y)^p|y_n|^\alpha dy\right\}^{1/p}\|\varphi\|_{p'},\qquad 1/p+1/p'=1,$$

which proves (2). Since (2) is fulfilled for any λ with length m, our lemma is proved.

The following lemma can be proved in a way similar to [16; Lemma 2].

LEMMA 3. *Let $k\geqq 2$ be an integer, $a>0$ and $\xi\in R^n_0$. If $u\in BL_k(L^p_{\mathrm{loc}}(R^n_+))$ satisfies $\int_{\Gamma(\xi;a)}|\xi-y|^{k-n}|\boldsymbol{D}_k u(y)|dy<\infty$, then*

$$\int_{\Gamma(\xi;a)}|\xi-y|^{k-1-n}|\boldsymbol{D}_{k-1}u(y)|dy<\infty,$$

where $\Gamma(\xi;a)=\{x=(x',x_n)\in R^n;\ |(x',0)-\xi|<ax_n,\ |x-\xi|\leqq 1\}$.

LEMMA 4. *Let α, p and m be as in Theorem 1. Let f be a non-negative function in $L^1_{\mathrm{loc}}(R^n)$ which has compact support and satisfies $\int_{R^n} f(y)^p|y_n|^\alpha dy<\infty$. If we set*

$$E=\left\{\xi\in R^n_0;\ \int|\xi-y|^{m-n}f(y)dy=\infty\right\},$$

then $B_{m-\alpha/p,p}(E)=0$.

PROOF. First we treat the case $\alpha>0$. Consider the function

$$v(x)=\int|x-y|^{m-n}f(y)dy,\qquad x\in R^n_+.$$

Then by Lemma 2, $v\in BL_m(L^p_{\mathrm{loc}}(R^n_+))$ and $\int_{R^n_+}|\boldsymbol{D}_m v|^p x_n^\alpha dx<\infty$. Therefore, if we set

$$E'=\left\{\xi\in R^n_0;\ \int_{B(\xi,1)\cap R^n_+}|\xi-y|^{m-\alpha/p-n}[|\boldsymbol{D}_m v(y)|y_n^{\alpha/p}]dy=\infty\right\},$$

then $B_{m-\alpha/p,p}(E')=0$. It suffices to show that $E\subset E'$. Suppose $E\not\subset E'$, i.e., there is $\xi\in E-E'$. Then $v(\xi)=\infty$ and hence

(3) $$\lim_{r\downarrow 0} v(\xi+r\sigma)=\infty\qquad\text{for any }\sigma\in\partial B(O,1)\cap R^n_+$$

by the lower semicontinuity of v. On the other hand, the assumption that $\xi\notin E'$ implies

$$\int_{\Gamma(\xi;1)} |\xi - y|^{m-n} |\boldsymbol{D}_m v(y)|\, dy < \infty,$$

which gives by Lemma 3

$$\int_{\Gamma(\xi;1)} |\xi - y|^{1-n} |\boldsymbol{D} v(y)|\, dy < \infty.$$

Accordingly, setting $S = \Gamma(O, 1) \cap \partial B(O, 1)$, we have

$$\int_S \left\{ \int_0^1 |\boldsymbol{D} v(\xi + r\sigma)|\, dr \right\} dS(\sigma) < \infty.$$

From this it follows that

$$\int_0^1 |\boldsymbol{D} v(\xi + r\sigma)|\, dr < \infty \qquad \text{for a.e.} \quad \sigma \in S.$$

Since $v(\xi + r\sigma)$ is absolutely continuous on $(0, \infty)$ for a.e. $\sigma \in S$, $\lim_{r \downarrow 0} v(\xi + r\sigma)$ exists and is finite for a.e. $\sigma \in S$. This contradicts (3) and thus our lemma is proved in case $\alpha > 0$.

In case $\alpha \leqq 0$, the proof can be carried out without the aid of Lemma 2. In fact, consider the set

$$E'' = \left\{ \xi \in R_0^n;\ \int |\xi - y|^{m-\alpha/p-n} [f(y) |y_n|^{\alpha/p}] dy = \infty \right\}.$$

Then it is easy to show that $E \subset E''$ and $B_{m-\alpha/p,p}(E'') = 0$. Now our lemma is completely proved.

LEMMA 5. *Let k be a positive integer and $\beta > -1$. If $u \in BL_k(L^p_{\mathrm{loc}}(R^n_+))$ satisfies $\int_G |\boldsymbol{D}_k u|^p x_n^\beta dx < \infty$ for any bounded open set $G \subset R^n_+$, then*

$$\int_G |\boldsymbol{D}_{k-1} u|^p x_n^\gamma dx < \infty$$

for any G described above, where γ is a number such that $\gamma = \beta - p$ if $\beta > p - 1$ and $-1 < \gamma < \beta$ if $\beta \leqq p - 1$.

PROOF. We may assume that the derivatives of u of order $k-1$ are locally p-precise on R^n_+. Given a bounded open set $G \subset R^n_+$, we can find a number $a > 0$ such that $G \subset \{x = (x_1, \dots, x_n);\ |x_i| < a$ for all $i\}$ and $\int_{|x'| < a} |\boldsymbol{D}_{k-1} u(x', a)|^p dx' < \infty$. Hölder's inequality gives

$$|\boldsymbol{D}_{k-1} u(x)|^p \leqq c_1 \left(\left\{ \int_{x_n}^a |\boldsymbol{D}_k u(x', t)|\, dt \right\}^p + |\boldsymbol{D}_{k-1} u(x', a)|^p \right)$$

$$\leqq c_2\left(\int_{x_n}^a |\boldsymbol{D}_k u(x', t)|^p t^\varepsilon dt\left\{\int_{x_n}^a t^{-\varepsilon p'/p}dt\right\}^{p/p'} + |\boldsymbol{D}_{k-1}u(x', a)|^p\right)$$

for $x=(x', x_n)$ with $0<x_n<a$, where ε is a number such that $\varepsilon=\beta-\gamma-1$ if $\beta\leqq p-1$ and $p-1<\varepsilon<\beta$ if $\beta>p-1$, and c_1, c_2 are positive constants. Noting that

$$\int_{x_n}^a t^{-\varepsilon p'/p}dt \leqq \text{const.}\, x_n^{\varepsilon_0}$$

with $\varepsilon_0=\min\{0, -\varepsilon p'/p+1\}$, we obtain by Fubini's theorem

$$\int_0^a |\boldsymbol{D}_{k-1}u(x', x_n)|^p x_n^\gamma dx_n \leqq c_3\left\{\int_0^a |\boldsymbol{D}_k u(x', t)|^p t^\beta dt + |\boldsymbol{D}_{k-1}u(x', a)|^p\right\},$$

which gives easily the required inequality in our lemma.

We are now ready to prove Theorem 1.

PROOF OF THEOREM 1. Take a number q such that $q=p$ if $\alpha\leqq 0$ and $1<q<p/(\alpha+1)$ if $\alpha>0$. Then Hölder's inequality yields

$$\int_G |\boldsymbol{D}_m u|^q dx < \infty$$

for any bounded open set $G\subset R_+^n$. From Lemma 5 it follows that $\int_G |\boldsymbol{D}_k u|^q dx <\infty$ for any bounded open set $G\subset R_+^n$ and any integer k with $0\leqq k\leqq m$.

Given $N>1$, let us consider the existence of mc limits of u at points of $R_0^n \cap B(O, N)$. Take $\varphi\in C_0^\infty(R^n)$ which is equal to 1 on $B(O, 2N)$, and define

$$v(x) = u(x)\varphi(x), \qquad x\in R_+^n.$$

Then $\sum_{k=0}^m \int_{R_+^n} |\boldsymbol{D}_k v|^q dx<\infty$. In view of Theorem 5 and its proof in [18; Chap. VI], we can find a function $w\in BL_m(L^q(R^n))$ which is equal to v a.e. on R_+^n, vanishes outside some compact set in R^n and satisfies

$$\int_{B(O,N)} |\boldsymbol{D}_m w(x)|^p |x_n|^\alpha dx < \infty.$$

Hence, with the aid of [10; Theorem 3.1] we have the following integral representation:

$$w(x) = \sum_{|\lambda|=m} a_\lambda \int \frac{(x-y)^\lambda}{|x-y|^n} D^\lambda w(y)dy \qquad \text{a.e. on} \quad R^n, \tag{4}$$

where a_λ are constants independent of x. Consider the set

$$E_N = \left\{\xi\in R_0^n;\ \int_{B(O,N)} |\xi-y|^{m-n}|\boldsymbol{D}_m w(y)|dy = \infty\right\}.$$

Then $B_{m-\alpha/p,p}(E_N)=0$ on account of Lemma 4. It remains to prove that w (and hence u) has an mc limit at each point of $R_0^n \cap B(O, N)\setminus E_N$. Let $\xi \in R_0^n \cap B(O, N)\setminus E_N$. Then

$$\int |\xi - y|^{m-n} |\boldsymbol{D}_m w(y)| dy < \infty,$$

so that Lemma 1 shows that the right-hand side of (4) has an mc limit at ξ. This leads to the fact that u has an mc limit at ξ. Hence $\cup_{N=1}^{\infty} E_N$ is the required exceptional set, and we conclude the proof.

4. Non-tangential limit

A function u on R_+^n is said to have a non-tangential limit at $\xi \in R_0^n$ if

$$\lim_{x \to \xi,\, x \in \Gamma(\xi;\, a)} u(x)$$

exists and is finite for any $a>0$.

THEOREM 2. *Let m, p and α be as in Theorem 1. Let u be a function polyharmonic of order $m+1$ in R_+^n. Suppose u satisfies* (1) *for any bounded open set $G \subset R_+^n$. Then there exists a Borel set E with $B_{m-\alpha/p,p}(E)=0$ such that u has a non-tangential limit at each point of $R_0^n\setminus E$.*

PROOF. First we note the following formula:

$$u(x) = \sum_{i=0}^{m} c_i x_n^{-n} \int_{B(x, x_n/2)} |x - y|^i \left(\frac{\partial}{\partial v}\right)^i u(y) dy,$$

where $x=(x', x_n) \in R_+^n$, c_i $(0 \leqq i \leqq m)$ are constants depending only on n, m and i, and v denotes the outward normal to $\partial B(x, x_n/2)$, i.e.,

$$\frac{\partial u}{\partial v}(y) = \sum_{i=1}^{n} \frac{y_i - x_i}{|y - x|} \frac{\partial u}{\partial y_i}(y) .$$

This can be proved by the aid of [5; (15)] (cf. [16; (3)]). By Theorem 1, there is a Borel set $E_1 \subset R_0^n$ such that $B_{m-\alpha/p,p}(E_1)=0$ and u has an mc limit at each point of $R_0^n\setminus E_1$. We set

$$E_2 = \left\{\xi \in R_0^n;\ \int_{B(\xi,1) \cap R_+^n} |\xi - y|^{m-n} |\boldsymbol{D}_m u(y)| dy = \infty\right\},$$

$$E = E_1 \cup E_2.$$

Then $B_{m-\alpha/p,p}(E_2)=0$ by Lemma 4 and thus $B_{m-\alpha/p,p}(E)=0$. If $\xi \in R_0^n\setminus E$, then u has an mc limit ℓ at ξ, so that

$$
\begin{aligned}
|u(x) - \ell| &\leqq |c_0| x_n^{-n} \int_{B(x,x_n/2)} |u(y) - \ell| dy \\
&\quad + \sum_{i=1}^{m} |c_i| x_n^{i-n} \int_{B(x,x_n/2)} \left| \left(\frac{\partial}{\partial v}\right)^i u(y) \right| dy \\
&\leqq \text{const.} \left\{ |x - \xi|^{-n} \int_{B(\xi, 2|x-\xi|)} |u(y) - \ell| dy \right. \\
&\quad \left. + \sum_{i=1}^{m} \int_{B(x,x_n/2)} |\xi - y|^{i-n} |\boldsymbol{D}_i u(y)| dy \right\} \\
&\longrightarrow 0 \quad \text{as} \quad x \longrightarrow \xi, \ x = (x', x_n) \in \Gamma(\xi; a),
\end{aligned}
$$

for any $a>0$ on account of Lemma 3, since $\xi \notin E_2$ and we can find $b>0$ such that $B(x, x_n/2) \subset \Gamma(\xi; b)$ whenever $x \in \Gamma(\xi; a)$ and $|x-\xi|<1/2$. The proof is now complete.

PROPOSITION 1. *Let $\alpha < p-1$ and $mp>n$. Let K be a function on $R^n \times R^n$ which is continuous outside the diagonal set and satisfies $|K(x, y)| \leqq |x-y|^{m-n}$ for all $x, y \in R^n$. For a non-negative function $f \in L^p(R^n)$, we set*

$$
u(x) = \int K(x, y) f(y) |y_n|^{-\alpha/p} dy .
$$

If $\int |x-y|^{m-n} f(y) |y_n|^{-\alpha/p} dy \not\equiv \infty$, then u is continuous on R_+^n and one can find a set $E \subset R_0^n$ with $B_{m-\alpha/p,p}(E)=0$ such that u has a non-tangential limit at every $\xi \in R_0^n \backslash E$.

COROLLARY. *Let α, p and u be given as in Theorem 1. If $mp>n$ and u is continuous on R_+^n, then there exists $E \subset R_0^n$ such that $B_{m-\alpha/p,p}(E)=0$ and u has a non-tangential limit at every $\xi \in R_0^n \backslash E$.*

The corollary follows from the fact that by Proposition 1, the right-hand side of (4) is continuous on R_+^n and has a non-tangential limit at every $\xi \in R_0^n$ except those in a set E with $B_{m-\alpha/p,p}(E)=0$.

REMARK. If $mp>n$ and $u \in BL_m(L_{\mathrm{loc}}^p(R_+^n))$, then there is a function which is continuous on R_+^n and equal to u a.e. on R_+^n (cf. [10; Lemma 2.3 and Proposition 3.1]).

PROOF OF PROPOSITION 1. Let $z=(z', z_n) \in R_+^n$ and $\delta < z_n/4$. For $x \in B(z, \delta)$, we have by Hölder's inequality

$$
\int_{|x-y| \leqq \delta} |x - y|^{m-n} f(y) |y_n|^{-\alpha/p} dy
$$

$$\leqq \left\{\int_{|x-y|\leqq\delta} |x-y|^{p'(m-n)} |y_n|^{-\alpha p'/p} dy\right\}^{1/p'} \left\{\int f(y)^p dy\right\}^{1/p}$$

$$\leqq \text{const. } z_n^{-\alpha/p} \delta^{m-n/p} \left\{\int f(y)^p dy\right\}^{1/p}, \qquad 1/p + 1/p' = 1.$$

Since $\lim_{x\to z}\int_{|x-y|>\delta} K(x, y)f(y)|y_n|^{-\alpha/p}dy = \int_{|z-y|>\delta} K(x, y)f(y)|y_n|^{-\alpha/p}dy$ by Lebesgue's dominated convergence theorem,

$$\limsup_{x\to z} |u(x)-u(z)| \leqq \text{const. } z_n^{-\alpha/p}\delta^{m-n/p}\left\{\int f(y)^p dy\right\}^{1/p},$$

which implies that u is continuous at z, and hence on R_+^n.

Let $a>0$ and $\xi \in R_0^n$. If $\int |\xi-y|^{m-n} f(y)|y_n|^{-\alpha/p}dy<\infty$, then Lebesgue's dominated convergence theorem gives

$$\lim_{x\to\xi, x\in\Gamma(\xi;a)} \int_{|x-y|>x_n/2} K(x, y)f(y)|y_n|^{-\alpha/p}dy = u(\xi).$$

If in addition $\lim_{r\downarrow 0} r^{mp-\alpha-n}\int_{B(\xi,r)} f(y)^p dy=0$, then

$$\left|\int_{|x-y|\leqq x_n/2} K(x, y)f(y)|y_n|^{-\alpha/p}dy\right|$$

$$\leqq \text{const. } \left\{x_n^{mp-\alpha-n}\int_{|\xi-y|<(a+2)x_n} f(y)^p dy\right\}^{1/p}$$

$$\longrightarrow 0 \quad \text{as} \quad x \longrightarrow \xi,\ x\in\Gamma(\xi;a),$$

so that u has a non-tangential limit at ξ. Proposition 1 follows from Lemma 4 and the next lemma.

LEMMA 6. *Let $\beta>0$ and f be a non-negative function in $L^p(R^n)$. If we set*

$$E = \left\{x\in R^n;\ \limsup_{r\downarrow 0} r^{\beta p-n}\int_{B(x,r)} f(y)^p dy > 0\right\},$$

then $B_{\beta,p}(E)=0$.

PROOF. If $\beta p\geqq n$, then E is empty and the conclusion is trivial. Let $\beta p<n$ and consider

$$E' = \left\{x\in R^n;\ \int_0^1 \left\{r^{\beta p-n}\int_{B(x,r)} f(y)^p dy\right\}^{1/(p-1)} \frac{dr}{r} = \infty\right\}.$$

Then $B_{\beta,p}(E')=0$ by [9; Theorem 2.1]. If $x\notin E'$, then

$$\int_r^{2r}\left\{s^{\beta p-n}\int_{B(x,s)} f(y)^p dy\right\}^{1/(p-1)}\frac{ds}{s} \geqq \text{const.}\left\{r^{\beta p-n}\int_{B(x,r)} f(y)^p dy\right\}^{1/(p-1)}$$

and the last term tends to zero as $r \downarrow 0$. This implies that $E \subset E'$. Our lemma is thus proved.

PROPOSITION 2. *Let m, p and α be as in Theorem 1. If $mp \leqq n$, then there is a function $u \in C^\infty(R_+^n)$ which satisfies (1) but does not have a non-tangential limit at any point of R_0^n.*

PROOF. Let $E_j=\{(i_1 j^{-2},\dots, i_{n-1} j^{-2}, j^{-1});\ i_k=0, \pm 1,\dots, \pm j^3$ for $1 \leqq k \leqq n-1\}$ for each positive integer j. Then $B_{m,p}(E_j)=0$ and hence we can find a function $\varphi_j \in C_0^\infty(R^n)$ such that $\varphi_j \geqq 1$ on E_j, $=0$ outside $\{(x', x_n) \in R_+^n;\ 2^{-1}(j^{-1}+(j+1)^{-1}) < x_n < 2^{-1}((j-1)^{-1}+j^{-1})\}$ and satisfies $\int |D_m \varphi_j|^p |x_n|^\alpha dx < 2^{-j}$, on account of [10; Theorem 2.3]. It is easy to see that the function $u=\sum_{j=1}^\infty \varphi_j$ satisfies all the conditions in our proposition.

5. Perpendicular and radial limits

In [11], we discussed the existence of perpendicular boundary limits of locally p-precise functions u on R_+^n satisfying

$$\int_{R_+^n} |Du|^p x_n^\alpha dx < \infty, \qquad 0 \leqq \alpha < p-1.$$

Here we shall generalize the result obtained in [11].

We say that a function u on R_+^n is (m, p)-quasi continuous if given $\varepsilon>0$, there is an open set $G \subset R_+^n$ such that $B_{m,p}(G)<\varepsilon$ and u is continuous as a function on $R_+^n \backslash G$. If $u \in BL_m(L_{\mathrm{loc}}^p(R_+^n))$, then we can find an (m, p)-quasi continuous function on R_+^n which is equal to u a.e. on R_+^n (cf. [10; Lemma 2.3]).

THEOREM 3. *Let $0 \leqq \alpha < p-1$ and let u be an (m, p)-quasi continuous function in $BL_m(L_{\mathrm{loc}}^p(R_+^n))$ which satisfies (1) for any bounded open set $G \subset R_+^n$. Then there exists a Borel set $E \subset R_0^n$ such that $B_{m-\alpha/p,p}(E)=0$ and $\lim_{x_n \downarrow 0} u(x', x_n)$ exists and is finite for every $x' \in R^{n-1}$ with $(x', 0) \notin E$.*

PROOF. As in the proof of Theorem 1, we may suppose that $u \in BL_m(L^q(R^n))$ for some $q>1$, vanishes outside some compact set in R^n and satisfies (4) for a.e. $x \in R^n$. Since each term of the right-hand side of (4) is (m, p)-quasi continuous on R_+^n because of [10; Lemma 3.3], it is enough to prove the assertion of the theorem for a function u for which

$$u(x) = \int \frac{(x-y)^\lambda}{|x-y|^n} f(y) dy$$

holds on R^n_+ except for a set $E^{(1)} \subset R^n_+$ with $B_{m,p}(E^{(1)})=0$, where λ is a multi-index with length m and f is a non-negative function with compact support which satisfies $\int f(y)^p |y_n|^\alpha dy < \infty$.

Let $E^{(2)} = \left\{\xi \in R^n_0;\ \int |\xi - y|^{m-n} f(y) dy = \infty\right\}$. Then $B_{m-\alpha/p,p}(E^{(2)})=0$ and for $\xi = (\xi', 0) \in R^n_0 \setminus E^{(2)}$

$$\lim_{x_n \downarrow 0} \int_{|x-y| \geqq x_n/2} \frac{(x-y)^\lambda}{|x-y|^n} f(y) dy = \int \frac{(\xi - y)^\lambda}{|\xi - y|^n} f(y) dy$$

by Lebesgue's dominated convergence theorem, where $x = (\xi', x_n)$.

Consider the set

$$E_k = \left\{x = (x', x_n) \in R^n;\ 2^{-k} \leqq x_n < 2^{-k+1},\right.$$
$$\left.\int_{|x-y| < x_n/2} |x-y|^{m-n} f(y) dy \geqq b_k^{-1/p}\right\}$$

for each positive integer k. Here $\{b_k\}$ is chosen so that $\lim_{k\to\infty} b_k = \infty$ and $\sum_{k=1}^\infty b_k \int_{2^{-k-1} < y_n < 2^{-k+2}} f(y)^p y_n^\alpha dy < \infty$. To evaluate the size of sets E_k, it is convenient to use the following capacity: Letting $\beta > 0$ and G be an open set in R^n, we define

$$C_{\beta,p}(A; G) = \inf \|g\|_p^p, \qquad A \subset R^n,$$

where the infimum is taken over all non-negative functions $g \in L^p(R^n)$ such that $g=0$ outside G and $\int |x-y|^{\beta-n} g(y) dy \geqq 1$ for all $x \in A$. Let $a > 0$ be a number such that the support of f is contained in $B(O, a)$. If $x \in E_k$, then

$$\int_{2^{-k-1} < y_n < 2^{-k+2}} |x-y|^{m-\alpha/p-n} [f(y) y_n^{\alpha/p}] dy \geqq b_k^{-1/p}.$$

Hence we have by definition

$$C_{m-\alpha/p,p}(E_k; B(O, a)) \leqq b_k \int_{2^{-k-1} < y_n < 2^{-k+2}} f(y)^p y_n^\alpha dy.$$

In general, denote by A^* the projection of a set A to the hyperplane R^n_0. We set

$$E = (E^{(1)})^* \cup E^{(2)} \cup \left(\bigcap_{j=1}^\infty \left(\bigcup_{k=j}^\infty E_k\right)^*\right).$$

If $\xi \in R^n_0 \setminus E$, then $\lim_{x_n \downarrow 0} u(x) = \int (\xi - y)^\lambda |\xi - y|^{-n} f(y) dy$, where $x = \xi + (O, x_n)$. Thus the following lemma establishes our theorem.

LEMMA 7. *Let $\beta>0$. For a set $E\subset R^n$ and $a>0$, we have*

$$C_{\beta,p}(E^*;\, B(O, a)) \leqq C_{\beta,p}(E;\, B(O, a)).$$

Moreover, $B_{\beta,p}(E)=0$ if and only if $C_{\beta,p}(E\cap G;\, G)=0$ for any bounded open set $G\subset R^n$. Consequently, if $B_{\beta,p}(E)=0$, then $B_{\beta,p}(E^*)=0$.

The first part of this lemma can be proved in the same way as [12; Lemma 1]. The second part follows from the property of Bessel kernel and the definition of capacities.

REMARK. In case $\alpha<0$, for any compact set $E\subset R_0^n$ with $B_{m,p}(E)=0$, there exists a function $u\in C^\infty(R_+^n)$ such that $\int_{R_+^n}|\boldsymbol{D}_m u|^p x_n^\alpha dx<\infty$ and $\lim_{x_n\downarrow 0} u(x', x_n)$ does not exist for any $x'\in R^{n-1}$ with $(x', 0)\in E$.

To show this fact, let $E_k=\{x=(x', x_n)\in R^n;\ x_n=4^{-k},\ (x', 0)\in E\}$ for each positive integer k. Since $B_{m,p}(E_k)=0$, by [10; Theorem 2.3] we can find $\varphi_k\in C_0^\infty(R^n)$ such that $\varphi_k\geqq 1$ on E_k, vanishes outside $\{x=(x', x_n)\in R^n;\ 2^{-2k-1}<x_n<2^{-2k+1}\}$ and satisfies $\int|\boldsymbol{D}_m\varphi_k|^p dx\leqq 2^{2k\alpha-k}$. We have only to take $u=\sum_{k=1}^\infty \varphi_k$.

We next prove the following theorem, which is an improvement of [7; Theorem 3].

THEOREM 4. *Let α, p, m and u be as in Theorem 1. Suppose u is (m, p)-quasi continuous on R_+^n. Then there exist $E_1, E_2\subset R_0^n$ such that $C_{mp-\alpha}(E_1)=0$ in case $mp-\alpha<n$, $E_1=\emptyset$ (the empty set) in case $mp-\alpha\geqq n$, $B_{m-\alpha/p,p}(E_2)=0$ and to each $\xi\in R_0^n\backslash(E_1\cup E_2)$, there correspond a number c_ξ and a set A_ξ with the following properties:*

i) $B_{m,p}(A_\xi)=0$; ii) $A_\xi\subset\partial B(\xi, 1)$;

iii) $\lim_{r\downarrow 0} u(\xi+r(z-\xi))=c_\xi$ *for every* $z\in\partial B(\xi, 1)\cap R_+^n\backslash A_\xi$.

PROOF. As in the proof of Theorem 3, we may suppose

$$u(x)=\int\frac{(x-y)^\lambda}{|x-y|^n}f(y)dy$$

for $x\in R_+^n$ except possibly in a set $E_3\subset R_+^n$ with $B_{m,p}(E_3)=0$, where λ is a multi-index with length m and f is a non-negative function with compact support such that $\int f(y)^p|y_n|^\alpha dy<\infty$. Set

$$E_1=\left\{\xi\in R_0^n;\ \int|\xi-y|^{mp-\alpha-n}[f(y)^p|y_n|^\alpha]dy=\infty\right\},$$

$$E_2=\left\{\xi\in R_0^n;\ \int|\xi-y|^{m-n}f(y)dy=\infty\right\}.$$

Then $B_{m-\alpha/p,p}(E_2)=0$ by Lemma 4, $C_{mp-\alpha}(E_1)=0$ if $mp-\alpha<n$ and $E_1=\emptyset$ if $mp-\alpha\geqq n$.

Let $\xi\in R_0^n\setminus(E_1\cup E_2)$ be fixed. For a set E, we denote by $\tilde{E}$ the set of all points $x\in\partial B(\xi, 1)$ such that $\xi+r(x-\xi)\in E$ for some $r>0$ with $r<1$. By Lemma 5 in [14], $B_{m,p}(\tilde{E}_3)=0$. Since $\int|\xi-y|^{m-n}f(y)dy<\infty$,

$$\int_{|x-y|\geqq x_n/2}\frac{(x-y)^\lambda}{|x-y|^n}f(y)dy$$

tends to $c_\xi=\int(\xi-y)^\lambda|\xi-y|^{-n}f(y)dy$ as $x\to\xi$, $x\in\Gamma(\xi; a)$, for any $a>0$.

For $a>0$, we can find $b>0$ such that $B(x, x_n/2)\subset\Gamma(\xi; b)$ whenever $x=(x', x_n)\in\Gamma(\xi; a)$ and $|x-\xi|<1/2$. Define the function

$$g(x)=\begin{cases} f(x) & \text{if } x\in\Gamma(\xi; b), \\ 0 & \text{otherwise.}\end{cases}$$

Then $\int|\xi-y|^{mp-n}g(y)^pdy<\infty$ and for any $x\in\Gamma(\xi; a)$,

$$\left|\int_{|x-y|<x_n/2}\frac{(x-y)^\lambda}{|x-y|^n}f(y)dy\right|\leqq\int_{|x-y|<|x-\xi|/2}|x-y|^{m-n}g(y)dy.$$

By [14; Lemma 6], there exists a set $E(a)\subset\partial B(\xi, 1)$ such that $B_{m,p}(E(a))=0$ and

$$\lim_{r\downarrow 0}\int_{|z(r)-y|<|z(r)-\xi|/2}|z(r)-y|^{m-n}g(y)dy=0$$

for every $z\in\partial B(\xi, 1)\setminus E(a)$, where $z(r)=\xi+r(z-\xi)$. Thus, if $z\in\partial B(\xi, 1)\cap R_+^n\setminus(\tilde{E}_3\cup(\cup_{k=1}^\infty E(k)))$, then

$$\lim_{r\downarrow 0}u(\xi+r(z-\xi))=c_\xi.$$

Now our theorem is proved with $A_\xi=\tilde{E}_3\cup(\cup_{k=1}^\infty E(k))$.

REMARK 1. In case $mp-\alpha\geqq n$ or $p\geqq 2$, $B_{m-\alpha/p,p}(E_1\cup E_2)=0$. In case $mp-\alpha<n$ and $p<2$, we have the following theorem:

THEOREM 4′. *Let α, p, m and u be given as in Theorem 4. Then we can find $E\subset R_0^n$ such that $B_{m-\alpha/p,p}(E)=0$ and to each $\xi\in R_0^n\setminus E$, there correspond a number c_ξ and a set A_ξ with the following properties:*

i) $A_\xi\subset\partial B(\xi, 1)$;

ii) $B_{m,q}(A_\xi)=0$ *for any* q, $1<q<p$;

iii) $\lim_{r\downarrow 0}u(\xi+r(z-\xi))=c_\xi$ *for all* $z\in\partial B(\xi, 1)\cap R_+^n\setminus A_\xi$.

To prove this theorem, replace E_1 in the proof of Theorem 4 by

$$E' = \left\{\xi \in R_0^n;\ \int_0^1 \left[r^{mp-\alpha-n} \int_{B(\xi,r)} f(y)^p |y_n|^\alpha dy \right]^{1/(p-1)} \frac{dr}{r} = \infty \right\}$$

and note $B_{m-\alpha/p,p}(E')=0$ in view of [9; Theorem 2.1]. If $\xi \in R_0^n \setminus E'$, then Hölder's inequality gives

$$\int_{\Gamma(\xi;a)} |\xi - y|^{mq-n} f(y)^q dy < \infty$$

for any $a>0$ and any q, $1<q<p$. We now apply [14; Lemma 6] and obtain the desired result in the same way as the proof of the previous theorem.

REMARK 2. In view of the proofs of Theorems 3, 4 and 4′, we may take $c_\xi = \lim_{x_n \downarrow 0} u(\xi + (O, x_n))$ in Theorems 4 and 4′ if $0 \leqq \alpha < p-1$.

6. Non-tangential fine limit

We say that a set $E \subset R^n$ is (β, p)-thin at $x^0 \in R^n$ if

$$\int_0^1 [r^{\beta p-n} C_{\beta,p}(E \cap B(x^0, r) \setminus B(x^0, r/2);\ B(x^0, 2))]^{1/(p-1)} \frac{dr}{r} < \infty.$$

In case $\beta p < n$, by [15; Appendix] this is equivalent to

$$\int_0^1 [r^{\beta p-n} B_{\beta,p}(E \cap B(x^0, r))]^{1/(p-1)} \frac{dr}{r} < \infty,$$

which is given by Meyers [9] (see also [1]).

LEMMA 8 ([9; Proposition 3.1, (v)]). *If $\{E_k\}$ is a sequence of sets each of which is (β, p)-thin at x^0, then there is a sequence $\{r_k\}$ of positive numbers such that $\bigcup_{k=1}^\infty (E_k \cap B(x^0, r_k))$ is (β, p)-thin at x^0.*

THEOREM 5. *Let α, p, m and u be as in Theorem 4. Then there exists a set $E \subset R_0^n$ such that $B_{m-\alpha/p,p}(E)=0$ and to each $\xi \in R_0^n \setminus E$, there corresponds a set E_ξ with the following properties:*

i) *E_ξ is (m, p)-thin at ξ;*

ii) *$\lim_{x \to \xi,\, x \in \Gamma(\xi;a) \setminus E_\xi} u(x)$ exists and is finite for any $a>0$.*

PROOF. As in the proof of Theorem 4, we may suppose that u is a function of the form

$$u(x) = \int \frac{(x-y)^\lambda}{|x-y|^n} f(y) dy.$$

Consider the sets

$$E_1 = \left\{\xi \in R_0^n;\ \int |\xi - y|^{m-n} f(y)\, dy = \infty\right\},$$

$$E_2 = \left\{\xi \in R_0^n;\ \int_0^1 \left[r^{(mp-\alpha)-n} \int_{B(\xi,r)} f(y)^p |y_n|^\alpha dy \right]^{1/(p-1)} \frac{dr}{r} = \infty.$$

Then $B_{m-\alpha/p,p}(E_1 \cup E_2)=0$ by Lemma 4 and [9; Theorem 2.1]. Let $\xi \in R_0^n \setminus (E_1 \cup E_2)$. Since $\xi \notin E_1$,

$$\lim_{x\to\xi, x\in\Gamma(\xi;a)} \int_{|x-y|\geqq x_n/2} \frac{(x-y)^\lambda}{|x-y|^n} f(y)dy = \int \frac{(\xi-y)^\lambda}{|\xi-y|^n} f(y)dy$$

for $a>0$. Let

$$E_{k,j} = \Big\{x = (x', x_n) \in \Gamma(\xi; j);\ 2^{-k} \leqq |x-\xi| < 2^{-k+1},$$

$$\int_{|x-y|<x_n/2} |x-y|^{m-n} f(y)dy \geq b_k^{-1/p}\Big\}.$$

Here $\{b_k\}$ is a sequence of positive numbers such that $\lim_{k\to\infty} b_k = \infty$ and

$$\sum_{k=1}^{\infty} \left[b_k 2^{k(n-mp+\alpha)} \int_{B(\xi,2^{-k+2})} f(y)^p |y_n|^\alpha dy \right]^{1/(p-1)} < \infty.$$

If we set $F_j = \bigcup_{k=1}^\infty E_{k,j}$, then F_j is seen to be (m, p)-thin at ξ. By Lemma 8, we can find a sequence $\{r_j\}$ of positive numbers such that $E_\xi = \bigcup_{j=1}^\infty (F_j \cap B(\xi, r_j))$ is (m, p)-thin at ξ. One sees readily that

$$\lim_{x\to\xi, x\in\Gamma(\xi;j)\setminus E_\xi} u(x) = \int \frac{(\xi-y)^\lambda}{|\xi-y|^n} f(y)dy$$

and obtains the theorem.

REMARK. In case $\alpha \leqq 0$, one may replace $\Gamma(\xi; a)$ by R_+^n in ii) of the theorem.

7. Mean continuous limits of general order

Let us recall that a function u on R_+^n is said to have an mc limit of order $q \geqq 1$ at $\xi \in R_0^n$ if there is a number ℓ with

$$\lim_{r\downarrow 0} \frac{1}{r^n} \int_{B_+(\xi,r)} |u(x) - \ell|^q dx = 0,$$

where $B_+(\xi, r) = B(\xi, r) \cap R_+^n$.

Define p^* and p^{**} by

$$\frac{1}{p^*} = \frac{1}{p} - \frac{m}{n}, \quad \frac{1}{p^{**}} = \frac{1}{p} - \frac{1}{n}\left(m - \frac{\alpha}{p}\right).$$

THEOREM 6. *Let m, p, α and u be as in Theorem 1. Then there is $E \subset R_0^n$ with $B_{m-\alpha/p,p}(E)=0$ such that at each point of $R_0^n \backslash E$, u has an mc limit of the following order q:*

i)	$q = p^{**}$	*if* $\alpha \geqq 0$ *and* $mp - \alpha < n$;
ii)	*any* q, $1 \leqq q < \infty$,	*if* $\alpha \geqq 0$ *and* $mp - \alpha = n$;
iii)	$q = \infty$,	*if* $\alpha \geqq 0$ *and* $mp - \alpha > n$;
iv)	$q = p^*$,	*if* $\alpha < 0$ *and* $mp < n$;
v)	*any* q, $1 \leqq q < \infty$,	*if* $\alpha < 0$ *and* $mp = n$;
vi)	$q = \infty$,	*if* $\alpha < 0$ *and* $mp > n$.

To prove this theorem, we need the next lemma whose proof will be given in the appendix.

LEMMA 9. *Let α, β, p and q be given as follows:*

$$0 \leqq \beta < 1, \quad \frac{\beta}{p'} < a < n, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}\left(a - \frac{\beta}{p'}\right) > 0,$$

where $1/p+1/p'=1$. For a non-negative function $f \in L^p(R^n)$, we set

$$F(x) = \int |x - y|^{a-n} f(y) |y_n|^{-\beta/p'} dy.$$

Then there is a positive constant M independent of f such that

$$\|F\|_q \leqq M \|f\|_p.$$

PROOF OF THEOREM 6. We may suppose that u is of the form

$$u(x) = \int \frac{(x-y)^\lambda}{|x-y|^n} f(y) |y_n|^{-\alpha/p} dy,$$

where $|\lambda| = m$ and $f \in L^p(R^n)$ has compact support. Define

$$E_1 = \left\{\xi \in R_0^n;\ \int |\xi - y|^{m-n} |f(y)| |y_n|^{-\alpha/p} dy = \infty\right\},$$

$$E_2 = \left\{\xi \in R_0^n;\ \limsup_{r \downarrow 0} r^{mp-\alpha-n} \int_{B(\xi,r)} |f(y)|^p dy > 0\right\},$$

$$E = E_1 \cup E_2.$$

Then $B_{m-\alpha/p,p}(E)=0$ by Lemmas 4 and 6. Let $\xi \in R_0^n \backslash E$ be fixed. As in the proof of Theorem 1, we set

$$u_1(x) = \int_{|x-y|<|\xi-x|/2} \frac{(x-y)^\lambda}{|x-y|^n} f(y) |y_n|^{-\alpha/p} dy,$$

$$u_2(x) = \int_{|x-y| \geqq |\xi-x|/2} \frac{(x-y)^\lambda}{|x-y|^n} f(y)|y_n|^{-\alpha/p} dy.$$

Since $\lim_{x \to \xi} u_2(x) = \int (\xi - y)^\lambda |\xi - y|^{-n} f(y) |y_n|^{-\alpha/p} dy$ (cf. the proof of Lemma 1), it suffices to prove

$$\lim_{r \downarrow 0} \frac{1}{r^n} \int_{B_+(\xi, r)} |u_1(x)|^q dx = 0. \tag{5}$$

In this proof, M denotes a various constant.

Case 1: $\alpha \geqq 0$ and $mp - \alpha < n$. In this case we have by Lemma 9

$$\begin{aligned} &\frac{1}{r^n} \int_{B_+(\xi, r)} |u_1(x)|^{p^{**}} dx \\ &\leqq \frac{1}{r^n} \int \left(\int_{B(\xi, 2r)} |x - y|^{m-n} |f(y)| \, |y_n|^{-\alpha/p} dy \right)^{p^{**}} dx \\ &\leqq \left(M r^{mp-\alpha-n} \int_{B(\xi, 2r)} |f(y)|^p dy \right)^{p^{**}/p} \longrightarrow 0 \quad \text{as} \quad r \downarrow 0, \end{aligned}$$

since $\xi \notin E_2$.

Case 2: $\alpha \geqq 0$ and $mp - \alpha = n$. For q, $1 < q < \infty$, let $\varepsilon = pnq^{-1}$. If q is so large that $\alpha + \varepsilon < p - 1$, then we obtain by Lemma 9

$$\begin{aligned} \frac{1}{r^n} \int_{B_+(\xi, r)} |u_1(x)|^q dx &\leqq \frac{1}{r^n} \left(M \int_{B(\xi, 2r)} |f(y)|^p |y_n|^\varepsilon dy \right)^{q/p} \\ &\leqq \left(M 2^\varepsilon \int_{B(\xi, 2r)} |f(y)|^p dy \right)^{q/p} \longrightarrow 0 \quad \text{as} \quad r \downarrow 0. \end{aligned}$$

Case 3: $\alpha \geqq 0$ and $mp - \alpha > n$. For $\delta > 0$, it follows from Hölder's inequality that

$$\begin{aligned} &\int_{|x-y| < \delta} |x - y|^{m-n} |f(y)| \, |y_n|^{-\alpha/p} dy \\ &\leqq \left(\int_{|x-y| < \delta} |x - y|^{p'(m-n)} |y_n|^{-\alpha p'/p} dy \right)^{1/p'} \|f\|_p \\ &\leqq \left(\int_{|y| < \delta} |y|^{p'(m-n)} |y_n|^{-\alpha p'/p} dy \right)^{1/p'} \|f\|_p \\ &= \text{const.}\, \delta^{(mp-\alpha-n)/p} \|f\|_p, \end{aligned}$$

which implies that u is continuous on R^n.

Case 4: $\alpha < 0$ and $mp < n$. Lemma 9 yields

$$\frac{1}{r^n} \int_{B_+(\xi, r)} |u_1(x)|^{p^*} dx \leqq \frac{1}{r^n} \left(M \int_{B(\xi, 2r)} |f(y)|^p |y_n|^{-\alpha} dy \right)^{p^*/p}$$

$$\leqq \left(M2^{-\alpha}r^{mp-\alpha-n}\int_{B(\xi,2r)}|f(y)|^p dy\right)^{p^*/p} \longrightarrow 0 \quad \text{as} \quad r \downarrow 0.$$

Case 5: $\alpha<0$ and $mp=n$. For q, $p<q<\infty$, let β be a number such that $1/q=1/p-\beta/n$. Then we have by Lemma 9

$$\frac{1}{r^n}\int_{B_+(\xi,r)}|u_1(x)|^q dx \leqq \frac{1}{r^n}\int\left(r^{m-\beta}\int_{B(\xi,2r)}|x-y|^{\beta-n}|f(y)|\,|y_n|^{-\alpha/p}dy\right)^q dx$$

$$\leqq \left(M2^{-\alpha}r^{mp-\alpha-n}\int_{B(\xi,2r)}|f(y)|^p dy\right)^{q/p} \longrightarrow 0 \quad \text{as} \quad r \downarrow 0.$$

Case 6: $\alpha<0$ and $mp>n$. As in Case 3, we have for $\delta>0$

$$\int_{|x-y|<\delta}|x-y|^{m-n}|f(y)|\,|y_n|^{-\alpha/p}dy$$

$$\leqq \left(\int_{|x-y|<\delta}|x-y|^{p'(m-n)}|y_n|^{-\alpha p'/p}dy\right)^{1/p'}\|f\|_p$$

$$\leqq M(|x_n|+\delta)^{-\alpha/p}\delta^{(mp-n)/p}\|f\|_p,$$

which implies that u is continuous on R^n.

Thus (5) holds in all cases and our theorem is proved.

8. Remarks

We collect several remarks to Theorem 1; similar remarks to Theorems 2–6 must be made.

REMARK 1. In case $-1<\alpha<p-1$, Theorem 1 is the best possible as to the size of the exceptional sets. In fact, for any $E\subset R_0^n$ with $B_{m-\alpha/p,p}(E)=0$, we can find a function u harmonic in R_+^n and satisfying (1) with G replaced by R_+^n such that

$$\lim_{x\to\xi,\,x\in R_+^n} u(x)=\infty \qquad \text{whenever} \quad \xi\in E$$

(cf. [16; Theorem 2]).

REMARK 2. Let $\alpha<mp-1$ and let u satisfy (1) for any bounded open set $G\subset R_+^n$.

1) If there is a positive integer k such that $kp-1<\alpha<(k+1)p-1$, then the same conclusion as Theorem 1 holds.

2) If $\alpha=kp-1$ for some positive integer k, then there is a set $E\subset R_0^n$ such that $B_{\beta,p}(E)=0$ for any $\beta>0$ with $\beta<m-\alpha/p$ and u has an mc limit at each point of $R_0^n\backslash E$.

This follows from Theorem 1 and Lemma 5.

REMARK 3. If $\alpha \geqq mp-1$, then there is a function $u \in C^\infty(R_+^n)$ such that u satisfies (1) with $G=R_+^n$ and $\lim_{x\to\xi, x\in R_+^n} u(x)=\infty$ for every $\xi \in R_0^n$.

For this, consider the function

$$u(x) = \{(\log x_n)^2 + 1\}^{\varepsilon/2} \exp(-|x|^2), \qquad x = (x', x_n) \in R_+^n,$$

where $0<\varepsilon<1-1/p$. By elementary computations, one sees that

$$|\boldsymbol{D}_m u(x)| \leqq \text{const.}\ \{(\log x_n)^2 + 1\}^{(\varepsilon-1)/2} x_n^{-m} \exp(-|x|^2/2)$$

and thus that u satisfies the required conditions.

9. Extension of Theorem 2 to a general domain

Let Ω be a domain of R^n and denote by $\partial\Omega$ the boundary of Ω. We say that Ω has the cone property at $\xi \in \partial\Omega$ if there is a finite (open) cone with vertex at ξ and contained in Ω. A function u on Ω is said to have a non-tangential limit at $\xi \in \partial\Omega$ if for any finite cone Γ with vertex at ξ such that there is a finite cone Γ' with $\bar{\Gamma}\backslash\{\xi\} \subset \Gamma' \subset \Omega$ ($\bar{\Gamma}$ denoting the closure of Γ),

$$\lim_{x\to\xi, x\in\Gamma} u(x)$$

exists and is finite.

We let $\rho(x)$ represent the distance of a point x from $R^n\backslash\Omega$.

Our aim is to prove the following theorem.

THEOREM 2′. *Let Ω be a domain of R^n, m a positive integer and $\alpha<mp$. Let u be a function which is polyharmonic of order $m+1$ in Ω and satisfies*

$$\int_\Omega |\boldsymbol{D}_m u(x)|^p \rho(x)^\alpha dx < \infty.$$

Then there exists a Borel set $E \subset \partial\Omega$ with $B_{m-\alpha/p,p}(E)=0$ such that u has a non-tangential limit at each $\xi \in \partial\Omega\backslash E$ at which Ω has the cone property.

Our proof below will give another proof of Theorem 2.

PROOF OF THEOREM 2′. Consider the set

$$E = \left\{\xi \in \partial\Omega;\ \int_{B(\xi,1)\cap\Omega} |\xi - y|^{m-\alpha/p-n} |\boldsymbol{D}_m u(y)|\, \rho(y)^{\alpha/p}\, dy = \infty\right\}.$$

Then $B_{m-\alpha/p,p}(E)=0$. Let $\xi \in \partial\Omega\backslash E$ and Γ, Γ' be finite cones with vertex at ξ

such that $\bar{\Gamma}\backslash\{\xi\}\subset\Gamma'\subset\Omega$. Our purpose is to prove that $\lim_{x\to\xi, x\in\Gamma} u(x)$ exists and is finite.

First we find a constant $c>0$ such that for any $x\in\Gamma$,

$$c|x-\xi| \leqq \rho(x) \leqq |x-\xi|,$$

$$B(x, c|x-\xi|) \subset \Gamma'.$$

Then we have

$$\int_\Gamma |\xi - y|^{m-n}|\boldsymbol{D}_m u(y)|\,dy < \infty,$$

which together with Lemma 3 gives

$$\int_\Gamma |\xi - y|^{i-n}|\boldsymbol{D}_i u(y)|\,dy < \infty \qquad \text{for} \quad i = 1, 2,\dots, m-1.$$

Next we recall the following formula:

$$u(x) = \sum_{i=0}^{m} c_i r^{-n}\int_{B(x,r)} |x-y|^i\left(\frac{\partial}{\partial v}\right)^i u(y)\,dy \tag{6}$$

for $x\in\Gamma$ and $r<c|x-\xi|$. By induction we see that $(\partial/\partial v)^i u$ is of the form

$$\left(\frac{\partial}{\partial v}\right)^i u(y) = |x-y|^{-i}\sum_{j_1,j_2,\dots,j_i=1}^{n}(y_{j_1}-x_{j_1})\cdots(y_{j_i}-x_{j_i})\left(\frac{\partial^i u}{\partial y_{j_1}\cdots\partial y_{j_i}}\right)(y).$$

Therefore (6) can be written as

$$u(x) = \sum_{|\lambda|\leqq m} a_\lambda r^{-n}\int_{B(x,r)}(y-x)^\lambda(D^\lambda u(y))\,dy$$

with constants a_λ. Replacing u by $\partial u/\partial x_j$, we obtain by Green's formula,

$$\begin{aligned}\frac{\partial u}{\partial x_j}(x) &= a_0 r^{-n}\int_{B(x,r)}\frac{\partial u}{\partial y_j}(y)\,dy \\ &+ \sum_{1\leqq|\lambda|\leqq m} a_\lambda r^{-n}\int_{\partial B(x,r)}(y-x)^\lambda(D^\lambda u(y))\frac{y_j-x_j}{|y-x|}\,dS(y) \\ &- \sum_{1\leqq|\lambda|\leqq m} a_\lambda r^{-n}\int_{B(x,r)}\left(\frac{\partial}{\partial y_j}(y-x)^\lambda\right)(D^\lambda u(y))\,dy.\end{aligned}$$

Multiplying both sides by r^n and integrating them on the interval $(0, c|x-\xi|)$, we find

$$\frac{\partial u}{\partial x_j}(x) = (n+1)a_0(c|x-\xi|)^{-n-1}\int_0^{c|x-\xi|}dr\int_{B(x,r)}\frac{\partial u}{\partial y_j}\,dy$$

$$- (n+1) \sum_{1 \leq |\lambda| \leq m} a_\lambda (c|x-\xi|)^{-n-1} \int_0^{c|x-\xi|} dr \int_{B(x,r)} \left(\frac{\partial}{\partial y_j}(y-x)^\lambda\right)(D^\lambda u(y))\, dy$$

$$+ (n+1) \sum_{1 \leq |\lambda| \leq m} a_\lambda (c|x-\xi|)^{-n-1} \int_{B(x,\, c|x-\xi|)} (y-x)^\lambda (D^\lambda u(y)) \frac{y_j - x_j}{|y-x|} dy.$$

From this it follows that

$$|x-\xi| \cdot |\boldsymbol{D}u(x)| \leq C \sum_{i=1}^m \int_{B(x,\, c|x-\xi|)} |\xi - y|^{i-n} |\boldsymbol{D}_i u(y)| dy$$

with some constant $C > 0$ independent of $x \in \Gamma$.

Since $\int_\Gamma |\xi - y|^{1-n} |\boldsymbol{D}u(y)| dy < \infty$, there is a line $\ell \cap \Gamma \neq \emptyset$ such that $\lim_{x \to \xi, x \in \ell} u(x)$ exists and is finite (cf. the proof of Lemma 4). Denote the limit by a. For $x \in \Gamma$, let $x^* \in \ell$ be the point with $|x^* - \xi| = |x - \xi|$, and ℓ_{x,x^*} the line segment between x and x^*. For the sake of simplicity we assume that the aperture of Γ is smaller than 1. Then it is seen that $\ell_{x,x^*} \subset \Gamma$ and $|x - x^*|/2 < |y - \xi|$ for $x \in \Gamma$ and $y \in \ell_{x,x^*}$. By the mean value theorem we have

$$|u(x) - u(x^*)| \leq |x - x^*| \sup_{y \in \ell_{x,x^*}} |\boldsymbol{D}u(y)|$$

$$\leq 2C \sup_{y \in \ell_{x,x^*}} \sum_{i=1}^m \int_{B(y,\, c|y-\xi|)} |\xi - z|^{i-n} |\boldsymbol{D}_i u(z)| dz$$

$$\longrightarrow 0 \quad \text{as} \quad x \longrightarrow \xi,\ x \in \Gamma.$$

Thus $\lim_{x \to \xi, x \in \Gamma} u(x) = a$ and our theorem is proved.

Appendix

We now prove Lemma 9. We consider an operator T defined for $f \in L^p(R^n)$ as follows:

$$Tf(x) = \int |x - y|^{\alpha-n} f(y) |y_n|^{-\beta/p'} dy.$$

The required inequality is then expressed as $\|Tf\|_q \leq M \|f\|_p$. This means that T is of (strong) type (p, q). Let us prove this fact. Decompose $K(x) \equiv |x|^{\alpha-n}$ as $K_1 + K_\infty$, where

$$K_1(x) = K(x) \quad \text{if} \quad |x| \leq \eta, \qquad = 0 \quad \text{if} \quad |x| > \eta,$$

$$K_\infty(x) = K(x) \quad \text{if} \quad |x| > \eta, \qquad = 0 \quad \text{if} \quad |x| \leq \eta$$

and set

$$T_1 f(x) = \int K_1(x-y) f(y) |y_n|^{-\beta/p'} dy,$$

$$T_\infty f(x) = \int K_\infty(x-y) f(y) |y_n|^{-\beta/p'} dy.$$

The positive number η will be determined later. We shall show that $T_1 f \in L^p(R^n)$, $T_\infty f \in L^\infty(R^n)$ and the mapping $f \to Tf$ is of weak type (p, q), in the sense that

$$m\{x;\ |Tf(x)| > \lambda\} \leqq \left(\frac{M\|f\|_p}{\lambda}\right)^q \qquad \text{for any} \quad \lambda > 0,$$

where m is the Lebesgue measure and M is a constant independent of f and λ.

In this proof $M_1, M_2, \ldots$ denote constants independent of x, η and f. Let t be a number such that $0<t<1$. By using Hölder's inequality we have

$$(7) \quad \int |T_1 f|^p dx \leqq \int \left\{\int K_1(x-y)^{tp'} |y_n|^{-\beta} dy\right\}^{p/p'} \int K_1(x-y)^{p(1-t)} |f(y)|^p dy dx.$$

If we take $t=(n-\beta)/(np'-\beta)$, then $A \equiv (a-n)tp'+n>\beta$ and $B \equiv p(a-n)(1-t)+n>0$. First we show

$$(8) \qquad \int K_1(x-y)^{tp'} |y_n|^{-\beta} dy \leqq M_1 \eta^{A-\beta}.$$

We may assume that $x=(O, x_n)$, $x_n \geqq 0$. We divide the domain of integration into three parts, that is, (i) $2|x| \leqq |y|$, $|x-y| \leqq \eta$, (ii) $2|x|>|y|$, $y_n \leqq x_n/2$, $|x-y| \leqq \eta$, (iii) $2|x|>|y|$, $y_n>x_n/2$, $|x-y| \leqq \eta$. The corresponding integrals are denoted by I_1, I_2 and I_3 respectively. We consider polar coordinates (r, θ, ω) in R^n, where $r=|x|$, θ is the angle between x and the x_n-axis and ω denotes the other variables if (r, θ, ω) represents $x \in R^n$. Since $|y| \geqq 2|x|$ implies $|x-y| \geqq |y|/2$, we have

$$I_1 \leqq 2^{A-n} \int_0^{2\eta} r^{A-\beta-1} dr \iint_{S^{n-1}} |\cos\theta|^{-\beta} d\theta d\omega = M_2 \eta^{A-\beta}.$$

If $y_n \leqq x_n/2$, then $|x-y| \geqq |y|$. Hence

$$I_2 \leqq \int_0^{\eta} r^{A-\beta-1} dr \iint_{S^{n-1}} |\cos\theta|^{-\beta} d\theta d\omega = M_3 \eta^{A-\beta}.$$

In the case (iii) we note that $|x-y| \leqq 3x_n < 6y_n$, and hence we have

$$I_3 \leqq 6^\beta \int_{|x-y| \leqq \eta} |x-y|^{A-\beta-n} dy = M_4 \eta^{A-\beta}.$$

Thus the inequality (8) is obtained. From (7) and (8) it follows that

$$(9) \qquad \int |T_1 f|^p dx \leqq (M_1 \eta^{A-\beta})^{p/p'} \int K_1(x)^{p(1-t)} dx \int |f|^p dy$$

$$\leqq M_5 \eta^{p(A-\beta)/p'+B} \int |f|^p dy$$

$$= M_5 \eta^{p(a-\beta/p')} \int |f|^p dy.$$

Next, we show that

$$(10) \qquad |T_\infty f(x)| = \left| \int K_\infty(x-y) f(y) |y_n|^{-\beta/p'} dy \right| \leqq M_6 \eta^{-n/q} \|f\|_p$$

for all $x \in R^n$. By Hölder's inequality we have

$$\left| \int K_\infty(x-y) f(y) |y_n|^{-\beta/p'} dy \right| \leqq \left\{ \int K_\infty(x-y)^{p'} |y_n|^{-\beta} dy \right\}^{1/p'} \|f\|_p.$$

Hence it suffices to show that

$$(11) \qquad \int K_\infty(x-y)^{p'} |y_n|^{-\beta} dy \leqq M_7 \eta^{-p'n/q}.$$

Again we may assume that $x=(O, x_n)$, $x_n \geqq 0$. We break up the domain of integration into four parts, that is, (i) $2|x| \leqq |y|$, $|x-y| \geqq \eta$, (ii) $2|x| > |y|$, $y_n > x_n/2$, $|x-y| \geqq \eta$, (iii) $2|x| > |y|$, $y_n \leqq x_n/2$, $|y| \leqq \eta$, $|x-y| \geqq \eta$, (iv) $2|x| > |y|$, $y_n \leqq x_n/2$, $|y| > \eta$, $|x-y| \geqq \eta$. The integrals on these domains are denoted by I'_1, I'_2, I'_3 and I'_4 respectively. If $2|x| \leqq |y|$, then $|y|/2 \leqq |x-y| \leqq 3|y|/2$. Hence

$$I'_1 \leqq 2^{p'(n-a)} \int_{|y| \geqq 2\eta/3} |y|^{p'(a-n)} |y_n|^{-\beta} dy = M_8 \eta^{p'(a-n)-\beta+n} = M_8 \eta^{-p'n/q}.$$

In the case (ii) we see that $|x-y|^\beta |y_n|^{-\beta} \leqq 6^\beta$, and hence

$$I'_2 \leqq 6^\beta \int_{|x-y| \geqq \eta} |x-y|^{p'(a-n)-\beta} dy = M_9 \eta^{-p'n/q}.$$

For I'_3 we have

$$I'_3 \leqq \eta^{p'(a-n)} \int_{|y| \leqq \eta} |y_n|^{-\beta} dy = M_{10} \eta^{-p'n/q}.$$

Since $|x-y| \geqq |y|$ holds in the case (iv),

$$I'_4 \leqq \int_{|y| > \eta} |y|^{p'(a-n)} |y_n|^{-\beta} dy = M_{11} \eta^{-p'n/q}.$$

Therefore (11), and then (10) is obtained.

By (9) we have that for any positive number λ and $f \in L^p(R^n)$

$$m\{x;\, |Tf(x)| > 2\lambda\} \leqq m\{x;\, |T_1 f(x)| > \lambda\} + m\{x;\, |T_\infty f(y)| > \lambda\}$$

$$\leqq M_5\left\{\frac{\eta^{a-\beta/p'}\|f\|_p}{\lambda}\right\}^p + m\{x;\, |T_\infty f(x)| > \lambda\}.$$

Now we determine η so that the right-hand side of (10) is equal to λ, i.e. $\eta=(M_6\|f\|_p/\lambda)^{q/n}$. Then $T_\infty f \leqq \lambda$ in R^n and hence $m\{x;\, |T_\infty f(x)|>\lambda\}=0$. Therefore

$$m\{x;\, |Tf(x)| > 2\lambda\} \leqq \left\{\frac{M\|f\|_p}{\lambda}\right\}^q$$

for a suitable constant $M>0$ independent of λ and f. This shows that the operator T is of weak type (p, q). We take p_1 and p_2 sufficiently close to p so that $p_1<p<p_2$ and for $i=1$ and 2

$$\frac{1}{q_i} \equiv \frac{1}{p_i} - \frac{1}{n}\left(a - \frac{\beta}{p_i'}\right) > 0,$$

where $p_i'=p_i/(p_i-1)$. By the above argument we see that T is of weak type (p_i, q_i), $i=1, 2$. Hence the Marcinkiewicz interpolation theorem (A. Zygmund [21; Theorem 1]) shows that T is of (strong) type (p, q), which means the conclusion of the lemma.

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