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Semigroups of nonlinear operators and invariant sets

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Let X be a Banach space and A be an operator in X such that $A-\omega I$ is dissipative for some real number ω. Let $D_a(A)$ be the set of those $x \in \overline{D(A)}$ for which there exists a sequence $\{x_n\}$ in $D(A)$ such that $\lim_{n\to\infty} x_n = x$ and $\limsup_{n\to\infty} |||Ax_n||| < +\infty$ (see [9]). In this paper we are concerned with the set $D_a(A)$.

This work is motivated by the papers of Crandall [2] and Bénilan [1]. Assuming that $R(I-\lambda A) \supset \overline{D(A)}$ for sufficiently small positive numbers λ, Crandall defined a set $\hat{D}(A)$, which is called a generalized domain, and showed that $\hat{D}(A)$ coincides with the set of those $x \in \overline{D(A)}$ for which $T(t)x$ is Lipschitz continuous in t on compact t-sets and therefore is invariant under $T(t)$. Here the semigroup $\{T(t); t \geqq 0\}$ on $\overline{D(A)}$ is defined by $T(t)x = \lim_{n\to\infty} (I-(t/n)A)^{-n}x$ for $t \geqq 0$ and $x \in \overline{D(A)}$. An extension of the result was obtained by Bénilan [1]. He defined the set $\hat{D}(A)$ by considering an extension $\hat{A}$ of A and showed, among others, that if $-A$ is a pseudo-generator then a result of Crandall's type holds for the set $\hat{D}(A)$.

In this paper we establish some sufficient conditions in order that A generates a nonlinear semigroup $\{T(t); t \geqq 0\}$ on $\overline{D(A)}$ such that $D_a(A)$ as well as $\hat{D}(A)$ is invariant under $T(t)$ (Theorem 4.3). The set $D_a(A)$ is in general a subset of $\hat{D}(A)$ and it is shown that if $\liminf_{h\to 0+} h^{-1} d(R(I-hA), x) < +\infty$ for every $x \in \hat{D}(A)$, then $D_a(A) = \hat{D}(A)$ (Proposition 2.3). Thus Theorem 4.3 extends some results in [1] and [2]. The set $D_a(A)$ also possesses some interesting properties. We observe, for instance, that $D_a(A)$ is invariant under certain perturbations. The results in sections 2 and 4 will be used in the final section to prove a result on m-dissipativity.

1. Preliminaries

Let X be a real Banach space with norm $\|\cdot\|$. For a subset S of X, we denote by $\bar{S}$ its closure and by $d(S, x)$ the distance from $x \in X$ to S. Let A be an operator in X. By this we mean a multi-valued operator with domain $D(A)$ and range $R(A)$ both contained in X. We often identify A with its graph $\{[x, y] \in X \times X; x \in D(A), y \in Ax\}$. We denote by $\bar{A}$ the closure of A and we say that A is closed if $A = \bar{A}$. For each $x \in D(A)$, we write

$$|||Ax||| = \inf\{\|y\|; y \in Ax\}.$$

We say that A is *locally bounded on* $S\subset X$ if for each $x\in S$ there are a neighborhood U_x of x and a positive constant M such that $|||Az|||\leqq M$ for every $z\in D(A)\cap U_x$.

Let X^* denote the dual space of X. We denote by (x, f) the pairing between $x\in X$ and $f\in X^*$. For each $[x, y]\in X\times X$, we set

$$[y, x]_s = \sup\{(y, f); f\in Fx\};$$

$$[y, x]_i = \inf\{(y, f); f\in Fx\},$$

where F is the duality map from X into X^*, i.e., F is defined by

$$Fx = \{f\in X^*; (x, f) = \|x\|^2 = \|f\|^2\} \qquad \text{for} \quad x\in X.$$

The symbol $[\cdot, \cdot]_s$ will be used also to denote the corresponding functional on $X^{**}\times X^{**}$.

An operator A in X is said to be dissipative if for every $[x, y]$, $[u, v]\in A$,

$$[y - v, x - u]_i \leqq 0,$$

or equivalently if for every $\lambda>0$ and for every $[x, y]$, $[u, v]\in A$,

$$\|x - u\| \leqq \|(x - \lambda y) - (u - \lambda v)\|.$$

If A is dissipative and $R(I-\lambda A)=X$ for all (or for some) $\lambda>0$, then A is said to be m-dissipative. Here I denotes the identity operator on X.

Let ω be a real number and let A be an operator in X such that $A-\omega I$ is dissipative. Following Bénilan [1], we define an operator $\hat{A}$ in X^{**} by

$$\hat{A} = \{[x, y]\in \overline{D(A)} \times X^{**}; \forall[u, v]\in A, \exists f\in F(x - u),$$
$$(f, y) - (v, f) \leqq \omega\|x - u\|^2\}$$

and a set $\hat{D}(A)$ by

$$\hat{D}(A) = \{x\in \overline{D(A)}; \exists\{x_n\} \subset D(\hat{A}), \lim_{n\to\infty} x_n = x, \limsup_{n\to\infty} |||\hat{A}x_n||| < +\infty\}.$$

It is clear that $A\subset\hat{A}$ if A is regarded as an operator in X^{**}. Also, we note that if $[x, y]\in\hat{A}$, then

$$\|x - u\| \leqq (1 - \lambda\omega)^{-1}\|(x - \lambda y) - (u - \lambda v)\|$$

for every $[u, v]\in A$ and for every $\lambda>0$ such that $\lambda\omega<1$.

Let C be a closed subset of X. A one-parameter family $\{T(t); t\geqq 0\}$ of nonlinear operators from C into itself is called a (*nonlinear*) *semigroup on* C if it has the following properties:

(i) $T(0)x = x$ for $x\in C$;

(ii) $T(t+s)x = T(t)T(s)x$ for $t, s \geqq 0$ and $x \in C$;

(iii) $\lim_{t\to 0+} T(t)x = x$ for $x \in C$.

If in addition there is a real number ω such that

(iv) $\|T(t)x - T(t)y\| \leqq e^{\omega t}\|x-y\|$ for $t \geqq 0$ and $x, y \in C$,

then we write $\{T(t);\, t \geqq 0\} \in Q_\omega(C)$. Let $\{T(t);\, t \geqq 0\}$ be a semigroup on C and let S be a subset of C. If each $T(t)$ maps S into itself, then S is said to be *invariant under* $T(t)$.

2. Definition and properties of $D_a(A)$

DEFINITION 2.1 (cf. [9]). *Let A be an operator in X and set*
$D_a(A; M) = \{x \in \overline{D(A)};\, \exists \{x_n\} \subset D(A),\, \lim_{n\to\infty} x_n = x,\, \limsup_{n\to\infty} |||Ax_n||| \leqq M\}$
for $M>0$. We define a set $D_a(A)$ by

$$D_a(A) = \bigcup_{M>0} D_a(A; M)$$

and for each $x \in D_a(A)$, we write

$$|Ax| = \inf\{M;\, x \in D_a(A; M)\}.$$

The following result presents some elementary properties of $D_a(A)$.

PROPOSITION 2.2. *Let A be an operator in X. Then*:

(i) *$D(A) \subset D_a(A) \subset \overline{D(A)}$ and $|Ax| \leqq |||Ax|||$ for $x \in D(A)$.*

(ii) *$D_a(A) = D_a(\bar{A})$ and $|Ax| = |\bar{A}x|$ for $x \in D_a(A)$.*

(iii) *If $x \in D_a(A)$ and $|Ax| = 0$, then $x \in D(\bar{A})$ and $0 \in \bar{A}x$.*

(iv) *If $x_n \in D_a(A)$, $\lim_{n\to\infty} x_n = x$ and $\liminf_{n\to\infty} |Ax_n| < +\infty$, then $x \in D_a(A)$ and $|Ax| \leqq \liminf_{n\to\infty} |Ax_n|$.*

PROOF. (i) is clear. To prove (ii), it suffices to show that $D_a(\bar{A}) \subset D_a(A)$ and $|Ax| \leqq |\bar{A}x|$ for $x \in D_a(\bar{A})$. Let $x \in D_a(\bar{A})$. Then there exists a sequence $\{[x_n, y_n]\}$ in $\bar{A}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \|y_n\| = |\bar{A}x|$, which ensures the existence of a sequence $\{[u_n, v_n]\}$ in A such that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \|v_n\| = \lim_{n\to\infty} \|y_n\| = |\bar{A}x|$. Thus it is shown that $x \in D_a(A)$ and $|Ax| \leqq |\bar{A}x|$.

The property (iii) follows from the fact that if $x \in D_a(A)$ and $|Ax| = 0$, then there exists a sequence $\{[x_n, y_n]\}$ in A such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \|y_n\| = 0$.

Finally, let $\{x_n\}$ be any sequence in $D_a(A)$. Then there exists a sequence $\{[u_n, v_n]\}$ in A such that $\|u_n - x_n\| \leqq 1/n$ and $\|v_n\| \leqq |Ax_n| + 1/n$ for all $n \geqq 1$. Therefore, if $\lim_{n\to\infty} x_n = x$ and $\liminf_{n\to\infty} |Ax_n| < +\infty$, then $\lim_{n\to\infty} u_n = x$ and

$\liminf_{n\to\infty} \|v_n\| \leqq \liminf_{n\to\infty} |Ax_n|$. This implies that $x \in D_a(A)$ and $|Ax| \leqq \liminf_{n\to\infty} |Ax_n|$, so (iv) is proved. Q.E.D.

Now let ω be a real number and let A be an operator in X such that $A-\omega I$ is dissipative. Then $D_a(A) \subset \hat{D}(A)$, where $\hat{D}(A)$ is the set defined in section 1. In fact, since $A \subset \hat{A}$ and $\hat{D}(A) = D_a(\hat{A})$ under our notation, it follows that $D_a(A) \subset D_a(\hat{A}) = \hat{D}(A)$ and $|\hat{A}x| \leqq |Ax|$ for $x \in D_a(A)$.

The next result gives a condition under which $D_a(A) = \hat{D}(A)$.

PROPOSITION 2.3. *Let A be an operator in X such that $A-\omega I$ is dissipative and let $x \in \hat{D}(A)$. Then $x \in D_a(A)$ if and only if* $\liminf_{h\to 0+} h^{-1}d(R(I-hA), x) < +\infty$. *If in particular* $\liminf_{h\to 0+} h^{-1}d(R(I-hA), x) = 0$, *then $x \in D_a(A)$ and $|Ax| = |\hat{A}x|$.*

PROOF. If $x \in D_a(A)$, then it is easy to see that $\limsup_{h\to 0+} h^{-1}d(R(I-hA), x) \leqq |Ax|$. Next suppose that $M(x) = \liminf_{h\to 0+} h^{-1}d(R(I-hA), x) < +\infty$. Then for any $\varepsilon > 0$ such that $\varepsilon\omega < 1$, there exists a triplet (h, x_h, y_h) such that $0 < h \leqq \varepsilon$, $[x_h, y_h] \in A$ and

$$\|x_h - hy_h - x\| \leqq h(M(x) + \varepsilon). \tag{2.1}$$

Since

$$\|x_h - u\| \leqq (1 - h\omega)^{-1} \|(x_h - hy_h) - (u - hv)\|$$

for every $[u, v] \in \hat{A}$, (2.1) implies

$$\begin{aligned} \|x_h - x\| &\leqq (1 - h\omega)^{-1}(\|x_h - hy_h - x\| + h|\hat{A}x|) \\ &\leqq h(1 - h\omega)^{-1}(|\hat{A}x| + M(x) + \varepsilon). \end{aligned} \tag{2.2}$$

Moreover, (2.1) together with (2.2) implies

$$\|y_h\| \leqq (1 - h\omega)^{-1}(|\hat{A}x| + M(x) + \varepsilon) + M(x) + \varepsilon. \tag{2.3}$$

Clearly (2.2) and (2.3) show that $x \in D_a(A)$ and $|Ax| \leqq |\hat{A}x| + 2M(x)$, and the proof is completed. Q.E.D.

COROLLARY 2.4. *Let A be an operator in X such that $A-\omega I$ is dissipative and*

$$R(I - \lambda A) \supset \overline{D(A)} \qquad \textit{for all sufficiently small} \quad \lambda > 0. \tag{2.4}$$

Then $D_a(A) = \hat{D}(A)$ and $|Ax| = |\hat{A}x|$ for $x \in D_a(A)$.

REMARK 2.5. Let A be an operator as in Corollary 2.4 and let D denote the *generalized domain* which was introduced in [2]; it is by definition the set of those $x \in \overline{D(A)}$ for which $\lim_{\lambda\to 0+} \|A_\lambda x\| < +\infty$, where $A_\lambda = \lambda^{-1}[(I-\lambda A)^{-1} - I]$

for $\lambda>0$. Then we obtain $D_a(A)=D$ and $|Ax|=\lim_{\lambda\to 0+}\|A_\lambda x\|$ for $x\in D_a(A)$. Thus we see that if (2.4) holds, then $D_a(A)=\hat{D}(A)=D$ and $|Ax|=|\hat{A}x|=\lim_{\lambda\to 0+}\|A_\lambda x\|$ for $x\in D_a(A)$. (The fact that $\hat{D}(A)=D$ was already remarked by Bénilan [1] in the case $\omega=0$.)

Finally, we remark that $D_a(A)$ is invariant under certain perturbations. For example, let A be an operator in X and B be a single-valued operator in X such that $D(A)\subset D(B)$, and suppose that for each $x\in\overline{D(A)}$ there are a neighborhood U_x of x and positive constants K and $L<1$ such that

$$\|Bz\| \leqq K + L|||Az||| \qquad \text{for every} \quad z\in D(A)\cap U_x.$$

Then it is easy to check that $D_a(A+B)=D_a(A)$. But it seems to be not easy to verify that $\hat{D}(A+B)=\hat{D}(A)$ (of course, we assume here that both $\hat{D}(A)$ and $\hat{D}(A+B)$ are defined). Thus $D_a(A)$ will be useful in perturbation problems, especially in nonreflexive Banach spaces. See [7].

3. Remarks on evolution equations and semigroups

Throughout this and the next sections A denotes an operator in X such that $A-\omega I$ is dissipative for some real number ω.

Let $z\in\overline{D(A)}$ be given and consider the Cauchy problem

$$(CP;\, z) \qquad \begin{cases} (d/dt)u(t)\in Au(t) \\ u(0) = z. \end{cases}$$

This problem has been treated in recent years by means of the difference scheme:

$$(DS;\, z) \qquad \begin{cases} (t_k^n - t_{k-1}^n)^{-1}(x_k^n - x_{k-1}^n) - \varepsilon_k^n \in Ax_k^n, & k = 1, 2,\dots, N(n) \\ x_0^n = z, \end{cases}$$

where $0=t_0^n<t_1^n<\cdots<t_{N(n)-1}^n<T\leqq t_{N(n)}^n=T_n$ for some fixed $T>0$ and for all $n\geqq 1$. See [3], [5], [6], [8] and [9].

Let $\{u_n(t)\}$ be a sequence of step functions defined by

$$u_n(t) = \begin{cases} z & \text{for} \quad t = 0 \\ x_k^n & \text{for} \quad t\in(t_{k-1}^n, t_k^n], \quad k = 1, 2,\dots, N(n). \end{cases}$$

The sequence $\{u_n(t)\}$ is called a *full sequence of backward approximate solutions to* $(CP;\, z)$ *on* $[0, T]$ if the following conditions are satisfied:

(i) $\max\{t_k^n - t_{k-1}^n;\, 1\leqq k\leqq N(n)\}\longrightarrow 0$ as $n\longrightarrow\infty$;

(ii) $\sum_{k=1}^{N(n)}(t_k^n - t_{k-1}^n)\|\varepsilon_k^n\|\longrightarrow 0$ as $n\longrightarrow\infty$.

It has been proved (see [3] and [6]) that if $\{u_n(t)\}$ is a full sequence of backward approximate solutions to $(CP; z)$ on $[0, T]$, then $u_n(t)$ converges to a continuous function $u(t)$ uniformly on $[0, T]$. The limit function $u(t)$ is called a *backward (DS)-limit solution to $(CP; z)$ on* $[0, T]$ (see [6]). It should be noted that the backward (DS)-limit solution is unique whenever it exists.

We refer to the papers cited above for the properties of backward (DS)-limit solutions. We note here the following facts:

REMARK 3.1 (On changes of partition). Once we obtain the uniform convergence of the sequence $\{u_n(t)\}$, we are able to take $T_n = T$ for all $n \geqq 1$ in the above argument; one has only to consider the term

$$\varepsilon^n_{N(n)} + [(T - t^n_{N(n)-1})(T_n - t^n_{N(n)-1})]^{-1}(T_n - T)(x^n_{N(n)} - x^n_{N(n)-1})$$

instead of $\varepsilon^n_{N(n)}$.

REMARK 3.2 (On continuation of solutions). Let $z, w \in \overline{D(A)}$ and $T_1, T_2 > 0$. Let $v_1(t)$ and $v_2(t)$ be backward (DS)-limit solutions to $(CP; z)$ on $[0, T_1]$ and to $(CP; w)$ on $[0, T_2]$, respectively. If $w = v_1(T_1)$, then the function $u(t)$ defined by

$$u(t) = \begin{cases} v_1(t) & \text{for} \quad t \in [0, T_1] \\ v_2(t - T_1) & \text{for} \quad t \in (T_1, T_1 + T_2] \end{cases}$$

is necessarily a backward (DS)-limit solution to $(CP; z)$ on $[0, T_1 + T_2]$.

REMARK 3.3 (On dependence of solutions on initial values). Let $T > 0$ and $\{z_m\}$ be a sequence in $\overline{D(A)}$. For each m, let $v_m(t)$ be a backward (DS)-limit solution to $(CP; z_m)$ on $[0, T]$. If $\lim_{m\to\infty} z_m = z$, then there exists a backward (DS)-limit solution $u(t)$ to $(CP; z)$ on $[0, T]$ and $\lim_{m\to\infty} v_m(t) = u(t)$ uniformly on $[0, T]$ (see [9]).

REMARK 3.4 (The case A is continuous). Let $z \in \overline{D(A)}$ and $T > 0$. If A is continuous and the domain $D(A)$ is closed, then the backward (DS)-limit solution $u(t)$ to $(CP; z)$ on $[0, T]$ satisfies the integral equation

$$u(t) = z + \int_0^t Au(s)ds \qquad \text{for} \quad t \in [0, T],$$

that is, $u(t)$ is an exact solution of $(CP; z)$ on $[0, T]$.

Next let $\{T(t); t \geqq 0\} \in Q_\omega(\overline{D(A)})$. We say that $\{T(t); t \geqq 0\}$ is *generated by* A(or *A generates* $\{T(t); t \geqq 0\}$) if for each $x \in \overline{D(A)}$, $T(t)x$ satisfies the inequality

$$e^{-2\omega t}\|T(t)x - u\|^2 - \|x - u\|^2 \leqq 2\int_0^t e^{-2\omega\tau}[v, T(\tau)x - u]_s d\tau$$

for every $t \geqq 0$ and for every $[u, v] \in \hat{A}$.

REMARK 3.5. If $-A$ is a pseudo-generator (see [1] for the definition), then A generates a semigroup $\{T(t);\ t \geqq 0\} \in Q_0(\overline{D(A)})$.

REMARK 3.6. If for any $x \in \overline{D(A)}$ and for any $T>0$, there exists a backward (DS)-limit solution $u(t;\ x)$ to $(CP;\ x)$ on $[0, T]$, then A generates a semigroup $\{T(t);\ t \geqq 0\} \in Q_\omega(\overline{D(A)})$. Here $\{T(t);\ t \geqq 0\}$ is defined by $T(t)x = u(t;\ x)$ for $t \geqq 0$ and $x \in \overline{D(A)}$. See [6].

The following result is proved in a manner similar to [1] and we omit the proof.

PROPOSITION 3.7. *Let $\{T(t);\ t \geqq 0\} \in Q_\omega(\overline{D(A)})$ be a semigroup generated by A. Then we have*

(i) $\hat{D}(A) = D(\hat{A}) = \{x \in \overline{D(A)};\ T(t)x$ *is Lipschitz continuous in* t *on any bounded subinterval of* $[0, +\infty)\}$;

(ii) *if* $x \in \hat{D}(A)$, *then* $\|T(t)x - x\| \leqq |\hat{A}x| \int_0^t e^{\omega\tau} d\tau$ *for all* $t \geqq 0$ *and* $\lim_{t \to 0+} t^{-1} \|T(t)x - x\| = |\hat{A}x|$;

(iii) *if* $x \in \hat{D}(A)$, *then* $T(t)x \in \hat{D}(A)$ *and* $|\hat{A}T(t)x| \leqq e^{\omega t} |\hat{A}x|$ *for all* $t \geqq 0$.

This result shows that if A generates a semigroup $\{T(t);\ t \geqq 0\} \in Q_\omega(\overline{D(A)})$, then the set $\hat{D}(A)$ is invariant under $T(t)$.

In connection with Propositions 2.3 and 3.7, there is a question: Let A generate a semigroup $\{T(t);\ t \geqq 0\} \in Q_\omega(\overline{D(A)})$. Then does it follow that $D_a(A) = \hat{D}(A)$? We know already that $D_a(A) = \hat{D}(A)$ under the condition (2.4). Also, it is well-known that if X is reflexive and if A generates a semigroup $\{T(t);\ t \geqq 0\} \in Q_\omega(\overline{D(A)})$, then A has an extension B such that $B - \omega I$ is dissipative and $D(B) = D_a(B) = \hat{D}(B) = \hat{D}(A)$. But we do not know whether the above question is affirmative in general.

4. Invariance of $D_a(A)$

In this section we give some conditions under which A generates a semigroup $\{T(t);\ t \geqq 0\} \in Q_\omega(\overline{D(A)})$ such that $D_a(A)$ as well as $\hat{D}(A)$ is invariant under $T(t)$.

PROPOSITION 4.1. *Assume that the following conditions are satisfied:*

(A1) $\liminf_{h \to 0+} h^{-1} d(R(I - hA), x) = 0$ *for each* $x \in D_a(A)$.

(A2) *For each* $x \in D_a(A)$, *there is a number* $T_x > 0$ *such that on* $[0, T_x]$ *there exists a backward (DS)-limit solution to $(CP;\ x)$ whose values belong to* $D_a(A)$. *Then A generates a semigroup* $\{T(t);\ t \geqq 0\} \in Q_\omega(\overline{D(A)})$ *such that*

(i) *for each* $x \in \overline{D(A)}$, $u(t) = T(t)x$ *is a backward (DS)-limit solution to* $(CP\colon x)$ *on every finite interval* $[0, T]$;

(ii) *if* $x \in D_a(A)$, *then* $\|T(t)x - x\| \leqq |Ax|\int_0^t e^{\omega\tau}d\tau$ *for all* $t \geqq 0$ *and* $\lim_{t\to 0+} t^{-1}\|T(t)x - x\| = |Ax|$;

(iii) *if* $x \in D_a(A)$, *then* $T(t)x \in D_a(A)$ *and* $|AT(t)x| \leqq e^{\omega t}|Ax|$ *for all* $t \geqq 0$.

PROOF. By Propositions 2.3, 3.7 and Remarks 3.3, 3.6, it suffices to show that for each $x \in D_a(A)$, there exists a continuous function $u(t)$ defined on $[0, +\infty)$ such that $u(t) \in D_a(A)$ for all $t \geqq 0$ and $u(t)$ restricted to any finite interval $[0, T]$ is a backward (DS)-limit solution to $(CP; x)$ on $[0, T]$.

Let $x \in D_a(A)$ and let J denote the set of all numbers $c \in (0, +\infty)$ such that the statement in (A2) holds for c instead of T_x. Set $T^* = \sup\{c; c \in J\}$; note that $J \neq \emptyset$ by (A2). Now we shall show that $T^* = +\infty$. To do this, assume that $T^* < +\infty$. Then there exists a momotone increasing sequence $\{c_i\} \subset J$ such that $\lim_{i\to\infty} c_i = T^*$. For each i, let $v_i(t)$ be a backward (DS)-limit solution to $(CP; x)$ on $[0, c_i]$ such that $v_i(t) \in D_a(A)$ for all $t \in [0, c_i]$ and let $\{v_{i,n}(t)\}$ be a full sequence of backward approximate solutions to $(CP; x)$ on $[0, c_i]$ such that

$$\delta_{i,n} = \sup\{\|v_{i,n}(t) - v_i(t)\|; t \in [0, c_i]\} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Note that $|Av_i(t)| \leqq e^{\omega t}|Ax|$ for all $t \in [0, c_i]$ and for all i (cf. [6] and [7]). Next let $\{\varepsilon_m\}$ be a sequence of positive numbers such that $\lim_{m\to\infty} \varepsilon_m = 0$. For each m, choose an integer $k = k(m) \geqq 1$ such that $T^* - c_k \leqq \varepsilon_m$ and then an element $[x_m, y_m] \in A$ such that $\|x_m - v_k(c_k)\| \leqq \varepsilon_m$ and $\|y_m\| \leqq |Av_k(c_k)| + \varepsilon_m$. With this choice, define step functions $u_{m,n}(t)$ on $[0, T^*]$ by

$$u_{m,n}(t) = \begin{cases} v_{k,n}(t) & \text{for } t \in [0, c_k] \\ x_m & \text{for } t \in (c_k, T^*]. \end{cases}$$

Then we have

$$\begin{aligned} &\|x_m - v_{k,n}(c_k) - (T^* - c_k)y_m\| \\ &\quad \leqq \|x_m - v_k(c_k)\| + \|v_k(c_k) - v_{k,n}(c_k)\| + (T^* - c_k)\|y_m\| \\ &\quad \leqq \varepsilon_m + \delta_{k,n} + \varepsilon_m(e^{|\omega|T^*}|Ax| + \varepsilon_m) \end{aligned}$$

for all m and n. Therefore, taking Remark 3.1 into account, we see that a suitable subsequence of $\{u_{m,n}(t)\}$ becomes a full sequence of backward approximate solutions to $(CP; x)$ on $[0, T^*]$, that is, there exists a backward (DS)-limit solution $u(t)$ to $(CP; x)$ on $[0, T^*]$. Since $u(t) = v_i(t)$ on $[0, c_i]$ and $|Av_i(t)| \leqq e^{\omega t}|Ax|$ for $t \in [0, c_i]$, it is clear that $u(t) \in D_a(A)$ for all $t \in [0, T^*]$. Thus $T^* \in J$. In view of (A2) and Remark 3.2, this contradicts the definition of T^*. So we conclude that $T^* = +\infty$, and the proof is completed. Q.E.D.

REMARK 4.2. If A is locally bounded on $D_a(A)$, then (A2) follows from

(A1). See Remark 4.5 below.

We next exhibit a condition under which (A1) and (A2) in Proposition 4.1 are satisfied.

Let $\mathscr{L}$ denote the collection of all nonnegative functions L defined on $[0, +\infty)\times[0, +\infty)$ such that

(L_1) L is nondecreasing with respect to each of variables;

(L_2) $L(\sigma_1, L(\sigma_2, \tau))\leqq L(\sigma_1+\sigma_2, \tau)$ for every (σ_1, τ) and (σ_2, τ),

and consider the following condition:

$(R_a)_{loc}$ For each $x\in D_a(A)$, there are a neighborhood U_x of x and a function $L\in\mathscr{L}$ such that for any $z\in D_a(A)\cap U_x$, there exist a sequence $\{h_n\}$ of positive numbers and a sequence $\{z_n\}$ in $D(A)$ satisfying the properties

$$(a_1)\quad h_n \longrightarrow 0 \text{ and } h_n^{-1}d(z_n - h_nAz_n, z)\longrightarrow 0 \text{ as } n\longrightarrow\infty;$$

$$(a_2)\quad |Az_n|\leqq L(h_n, |Az|) \text{ for all } n.$$

We now obtain

THEOREM 4.3. *Suppose that A satisfies the condition $(R_a)_{loc}$. Then for each $x\in D_a(A)$, there exist a number $T_x>0$ and a backward (DS)-limit solution $u(t)$ to $(CP; x)$ on $[0, T_x]$ such that $u(t)\in D_a(A)$ for all $t\in[0, T_x]$. Consequently, the conclusion of Proposition* 4.1 *holds true.*

PROOF. Let $x\in D_a(A)$. Let $r>0$ and $L\in\mathscr{L}$ be such that $(R_a)_{loc}$ is satisfied with $U_x=S_r(x)$ and with the function L, where $S_r(x)=\{y\in X;\ \|y-x\|\leqq r\}$. Set $M_x=L(2, |Ax|)$ and $T_x=\min\{1, r/2(M_x+1)\}$. Note that by $(R_a)_{loc}$, $|Ax|\leqq L(\sigma, |Ax|)$ for all $\sigma>0$ and hence $|Ax|\leqq M_x$. In the following, we shall show that there exists a full sequence $\{u_n(t)\}$ of backward approximate solutions to $(CP; x)$ on $[0, T_x]$ such that $|Au_n(t)|\leqq M_x$ for all $t\in[0, T_x]$ and for all n, from which the theorem follows.

Let $\{\varepsilon_n\}$ be a sequence in $(0, 1]$ such that $\varepsilon_n|\omega|<1/2$ for all n and $\varepsilon_n\to 0$ as $n\to\infty$. For each n, set $t_0^n=0$ and $x_0^n=x$, and define inductively $t_{k+1}^n>0$ and $x_{k+1}^n\in D(A)$ in the following manner: If $x_k^n\in D_a(A)\cap S_r(x)$, let h_k^n denote the supremum of all numbers $h\in[0, \varepsilon_n]$ for which there exists an element $x_h\in D(A)$ such that

$$d(x_h - hAx_h, x_k^n)\leqq h\varepsilon_n/2; \tag{4.1}$$

$$|Ax_h|\leqq L(h, |Ax_k^n|). \tag{4.2}$$

By $(R_a)_{loc}$, it is clear that $h_k^n>0$. We define $t_{k+1}^n=t_k^n+h_k^n$ and x_{k+1}^n as the element in $D(A)$ such that

$$\|x_{k+1}^n - h_k^n y_{k+1}^n - x_k^n\|\leqq h_k^n\varepsilon_n \qquad \text{for some } y_{k+1}^n\in Ax_{k+1}^n; \tag{4.3}$$

$$(4.4) \qquad |Ax_{k+1}^n| \leqq L(h_k^n, |Ax_k^n|).$$

Note that (4.1) and (4.2) ensure the existence of such an element x_{k+1}^n.

Now assume that t_k^n and x_k^n are defined and that $t_k^n < T_x$. Then by (4.3), (4.4) and (L_2), we have

$$(4.5) \qquad \begin{aligned} \|x_{i+1}^n - x_i^n\| &\leqq h_i^n(1 - h_i^n\omega)^{-1}(|Ax_i^n| + \varepsilon_n) \\ &\leqq 2h_i^n(L(h_{i-1}^n, |Ax_{i-1}^n|) + \varepsilon_n) \\ &\leqq 2h_i^n(L(t_i^n, |Ax|) + \varepsilon_n) \\ &\leqq 2h_i^n(M_x + 1) \end{aligned}$$

for $1 \leqq i \leqq k-1$. Here we used the fact that $\varepsilon_n|\omega| < 1/2$. Since $|Ax| \leqq M_x$, (4.5) holds also for $i=0$. Therefore, we obtain

$$\|x_k^n - x\| \leqq 2(M_x + 1)\textstyle\sum_{i=0}^{k-1} h_i^n = 2t_k^n(M_x + 1) < 2T_x(M_x + 1) \leqq r,$$

that is, $x_k^n \in D_a(A) \cap S_r(x)$. This fact implies that the above induction can be continued so long as $t_k^n < T_x$.

In order to complete the proof, we assert that there exists an integer $N(n) \geqq 1$ such that $t_{N(n)-1}^n < T_x \leqq t_{N(n)}^n$. Assume that $t_k^n < T_x$ for all $k \geqq 0$. Then there exists a sequence $\{x_k^n\}$ in $D_a(A) \cap S_r(x)$ satisfying the properties

$$(4.6) \qquad |Ax_{k+1}^n| \leqq L(h_k^n, |Ax_k^n|) \leqq L(t_{k+1}^n, |Ax|) \leqq M_x, \quad k \geqq 0;$$

$$(4.7) \qquad \|x_{k+1}^n - x_k^n\| \leqq 2(t_{k+1}^n - t_k^n)(M_x + 1), \quad k \geqq 0.$$

Since $c = \lim_{k\to\infty} t_k^n$ exists in $(0, T_x]$, it follows from (4.6) and (4.7) that $z = \lim_{k\to\infty} x_k^n$ exists in $D_a(A) \cap S_r(x)$ and $|Az| \leqq L(c - t_k^n, |Ax_k^n|)$ for all $k \geqq 0$. Therefore, by $(R_a)_{loc}$, there exist a number $h \in (0, \varepsilon_n/2]$ and an element $z_h \in D(A)$ such that

$$(4.8) \qquad d(z_h - hAz_h, z) \leqq h\varepsilon_n/3;$$

$$(4.9) \qquad |Az_h| \leqq L(h, |Az|).$$

Choose an integer $k_0 \geqq 1$ so that $c - t_k^n \leqq \varepsilon_n/2$ for all $k \geqq k_0$. Since

$$|Az_h| \leqq L(h, |Az|) \leqq L(h, L(c - t_k^n, |Ax_k^n|)) \leqq L(h + c - t_k^n, |Ax_k^n|)$$

for all $k \geqq 0$, it follows from the definition of h_k^n that

$$(4.10) \qquad d(z_h - (h + c - t_k^n)Az_h, x_k^n) > (h + c - t_k^n)\varepsilon_n/2$$

for all $k \geqq k_0$, which implies

$$d(z_h - hAz_h, z) = \lim_{k\to\infty} d(z_h - (h + c - t_k^n)Az_h, x_k^n) \geqq h\varepsilon_n/2.$$

This contradicts (4.8), so the assertion follows.

In this way, we see that for each n, there exist an integer $N(n)\geqq 1$ and a set $\{(t_k^n, x_k^n); 0\leqq k\leqq N(n)\}$ of pairs such that $0=t_1^n<t_0^n<\cdots<t_{N(n)-1}^n<T_x\leqq t_{N(n)}^n$, $\max_{1\leqq k\leqq N(n)}(t_k^n-t_{k-1}^n)\leqq\varepsilon_n$, $x_0^n=x$ and such that (4.3) and (4.4) hold true for $0\leqq k\leqq N(n)-1$. Define a sequence $\{u_n(t)\}$ of step functions by

$$u_n(t)=\begin{cases} x & \text{for } t=0 \\ x_k^n & \text{for } t\in(t_{k-1}^n, t_k^n], \quad 1\leqq k\leqq N(n);\ n\geqq 1. \end{cases}$$

Then it is evident that $\{u_n(t)\}$ is the desired full sequence. Q. E. D.

REMARK 4.4. It remains open whether $D_a(A)=\hat{D}(A)$, even if A satisfies the condition $(R_a)_{loc}$.

REMARK 4.5. The condition $(R_a)_{loc}$ is a generalization of the condition (R_a) treated in [9], so that it includes the following condition

(R_1) for each $x\in D_a(A)$, there exists a sequence $\{h_n\}$ of positive numbers such that $h_n\to 0$ as $n\to\infty$ and $\bigcap_{n\geqq 1}\overline{R(I-h_nA)}\supset\{x\}$.

The condition $(R_a)_{loc}$ also includes the following condition

(R_2) A is locally bounded on $D_a(A)$ and $\liminf_{h\to 0+} h^{-1}d(R(I-hA), x)=0$ for each $x\in D_a(A)$.

In view of this, Theorem 4.3 may be applied to the cases: (i) A is continuous, and (ii) A is demicontinuous.

REMARK 4.6. The above results are easily extended to the case in which A is ω-quasi-dissipative (see [6] for the definition).

5. A criterion for *m*-dissipativity

Let A be a dissipative operator in X and S be a subset of X. A is said to be *locally m-dissipative on* S (see [4]) if for each $x\in S$, there exist a neighborhood U_x of x and a sequence $\{h_n\}$ of positive numbers such that $h_n\to 0$ as $n\to\infty$ and

$$\bigcap_{n\geqq 1} R(I-h_nA)\supset U_x. \tag{5.1}$$

Among others, Kato [4] proved that if A is almost demiclosed and locally m-dissipative on $D(A)$, then A is m-dissipative, provided that X^* is uniformly convex, and Bénilan [1] proved that if A is locally m-dissipative on $\hat{D}(A)$, then A is m-dissipative even if X is nonreflexive.

By applying the foregoing results, we give here a proof of the following result which is also found in Kobayashi-Kobayasi [7].

THEOREM 5.1. *If A is locally m-dissipative on $D_a(A)$, then A is m-dissipative.*

For the proof, we prepare the following lemma.

LEMMA 5.2. *Suppose that A is locally m-dissipative on $D_a(A)$. Let $w \in X$ and set $B=A-I+w$. Then $D_a(B)=D_a(A)$ and B is locally m-dissipative on $D_a(B)$.*

PROOF. It is evident that $B+I$ is dissipative. Also, it is easy to check that $D_a(B)=D_a(A)$.

Let $x \in D_a(B)$. Since $x \in D_a(A)$, we can choose a positive number r and a sequence $\{h_n\}$ of positive numbers convergent to zero so that (5.1) holds with $U_x=S_r(x)$ and with the sequence $\{h_n\}$. Set $\rho=r/4$ and $\delta=\min\{1/2, r/4(\|x\| + \|w\| + |Ax| + 1)\}$. Now we shall show that

$$R(I - h_nB) \supset S_\rho(x) \qquad \text{for all} \quad h_n \in (0, \delta]. \tag{5.2}$$

To this end, for each $z \in S_\rho(x)$ and $h_n \in (0, \delta]$, define a transformation G_n with $D(G_n)=S_r(x)$ by

$$G_n y = z - h_n(J_n y - w) \qquad \text{for} \quad y \in S_r(x),$$

where $J_n=(I-h_nA)^{-1}$. Then we have

$$\|G_n y_1 - G_n y_2\| = h_n\|J_n y_1 - J_n y_2\| \leqq h_n\|y_1 - y_2\| \leqq \frac{1}{2}\|y_1 - y_2\|$$

for $y_1, y_2 \in S_r(x)$, and

$$\begin{aligned}\|G_n y - x\| &\leqq \|G_n y - G_n x\| + \|G_n x - x\| \\ &\leqq h_n\|y - x\| + \|z - x\| + h_n(\|x\| + \|w\| + \|J_n x - x\|) \\ &\leqq h_n r + r/4 + h_n(\|x\| + \|w\| + h_n|Ax|) \\ &\leqq r\end{aligned}$$

for $y \in S_r(x)$. Therefore, by applying the fixed point theorem for a contraction map, we see that there exists an element $x_n \in S_r(x)$ such that $G_n x_n = x_n$, i.e.,

$$z - h_n(J_n x_n - w) = x_n. \tag{5.3}$$

Set $z_n=J_n x_n$. Then $z_n \in D(A)$ and $x_n \in (I-h_nA)z_n$, so (5.3) yields

$$z \in (I - h_nA)z_n + h_n(z_n - w) = (I - h_nB)z_n.$$

Thus (5.2) is proved, and the proof is completed. Q.E.D.

We are now in a position to prove Theorem 5.1.

PROOF OF THEOREM 5.1. Let $w \in X$ and set $B=A-I+w$. Then by Lemma

5.2 and Remark 4.5, B satisfies the condition $(R_a)_{loc}$. Therefore, according to Theorem 4.3, B generates a semigroup $\{T(t);\ t\geqq 0\}\in Q_{-1}(\overline{D(B)})$ such that if $x\in D_a(B)$, then $T(t)x\in D_a(B)$ for all $t\geqq 0$,

$$(5.4)\qquad \|T(t+s)x-T(t)x\|\leqq e^{-t}\|T(s)x-x\|\leqq(e^{-t}-e^{-(t+s)})\,|Bx|$$

for all t, $s\geqq 0$ and

$$(5.5)\qquad |BT(t)x|\leqq e^{-t}|Bx| \qquad \text{for all}\quad t\geqq 0.$$

Now let $x\in D_a(B)$. Then (5.4) and (5.5) imply that $z=\lim_{t\to+\infty}T(t)x$ exists in $D_a(B)$ and $|Bz|=0$. Noting that B is closed (see [4]), we see from Proposition 2.2 that $z\in D(B)$ and $0\in Bz$, i.e., $w\in(I-A)z$. Since $w\in X$ is arbitrary, it is proved that $R(I-A)=X$, so A is m-dissipative. Q.E.D.

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