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On null sets for extremal distances of order 2 and harmonic functions

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Introduction

L. Ahlfors and A. Beurling [1] gave a characterization of the removable singularities for the class of analytic functions with finite Dirichlet integral, in terms of extremal distances on the complex plane. In the N-dimensional euclidean space R^N, Väisälä [9] introduced the notion of null sets for extremal distances of order N, namely, NED-sets, and gave measure-theoretic conditions for NED-sets.

In the present paper, we are concerned with the extremal distances of order 2 in the N-dimensional space R^N with $N \geqq 3$ and give several characterizations of null sets for these extremal distances. More precisely, we shall consider the following three kinds of null sets. Given a compact set E in R^N, denote by $(\hat{E}^c)_Q$ the Kerékjártó-Stoïlow compactification of E^c $(=R^N-E)$ and by $(\hat{E}^c)_I$ the Aleksandrov compactification of E^c. Let B_0, B_1 be two disjoint closed balls in E^c, λ be the extremal distance of order 2 between B_0 and B_1 and λ_0 (resp. λ_Q, λ_I) be the extremal distance of order 2 between B_0 and B_1 relative to $E^c-(B_0 \cup B_1)$ (resp. $(\hat{E}^c)_Q-(B_0 \cup B_1)$, $(\hat{E}^c)_I-(B_0 \cup B_1)$). If $\lambda_0=\lambda$ (resp. $\lambda_Q=\lambda$, $\lambda_Q=\lambda_I$) for every choice of B_0 and B_1, then we call E an NED_2-set (resp. NED_2^Q-set, $NED_2^{Q,I}$-set).

Corresponding to these extremal distances, there are notions of 2-capacities of condenser, which were studied by many authors (for example, see [12], [5], [11]); and there are also notions of principal functions (see [8]). We shall give characterizations of NED_2-sets, NED_2^Q-sets and $NED_2^{Q,I}$-sets in terms of corresponding 2-capacities of condensers and principal functions.

Another characterizations will be given by the removability for certain classes of harmonic functions (cf. [5], [8], [11] for related results). Let G be a bounded domain containing E and let $HD^2(G)$ (resp. $HD^2(G-E)$) be the class of all harmonic functions with finite Dirichlet integrals on G (resp. on $G-E$). We shall say that E is removable for $\widetilde{HD}^2$ (resp. KD^2; $\widetilde{KD}^2$; HD^2) if every $u \in HD^2(G-E)$ with "vanishing normal derivative along E" (resp. with no flux; with no flux and "constant value along each component of E"; with no additional condition) can be extended to a function in $HD^2(G)$. (For precise definitions of $\widetilde{HD}^2$, KD^2, $\widetilde{KD}^2$, see § 3, as well as the references cited above.) It is well known (see [3]) that E is removable for HD^2 if and only if the Newtonian capacity of E is zero. We shall show that E is an NED_2-set (resp. NED_2^Q-set; $NED_2^{Q,I}$-set) if and only if it is removable for $\widetilde{HD}^2$ (resp. for KD^2 as well as for $\widetilde{KD}^2$; for HD^2). From the relations $\widetilde{HD}^2 \subset KD^2 \subset HD^2$, it follows that every $NED_2^{Q,I}$-set is an NED_2^Q-set and every NED_2^Q-set is an NED_2-set. In the final remark (Remark 7.3), we give examples which show that these three classes of null sets are actually different. [Here we note that in the two dimensional case, the classes of NED_2^Q-sets and NED_2-sets coincide (see [1], [8]).]

§1. Preliminaries

We shall denote by $x=(x_1, x_2,\dots, x_N)$ a point in R^N, and set $|x|=(x_1^2+x_2^2+\cdots+x_N^2)^{1/2}$. The inner product of x and $y\in R^N$ will be denoted by (x, y). For a set E in R^N, we let ∂E denote its boundary, $\bar{E}$ its closure and E^c its complement. The N-dimensional Lebesgue measure of E will be written as $V(E)$. For an open set G in R^N, let $L^2(G)$ be the family of real valued measurable functions f on G for which $|f|^2$ is integrable. We denote by $C^\infty(G)$ the family of infinitely differentiable functions on G and by $C_0^\infty(G)$ the subfamily consisting of functions with compact support in G. For a function u defined in G, we let ∇u denote the gradient of u in case it exists.

Let τ be a C^1-surface which divides R^N into a bounded domain and an unbounded domain. In this paper, when we consider the normal derivative $\partial/\partial\nu$ at a point of τ, the normal is drawn in the direction of the unbounded domain.

Let G be a domain in R^N. By a locally rectifiable chain in G we mean a countable formal sum $\gamma=\sum\gamma_i$, where each γ_i is a locally rectifiable curve in G. If f is a non-negative Borel measurable function defined in G and $\gamma=\sum\gamma_i$ is a locally rectifiable chain in G, then we set $\int_\gamma fds=\sum\int_{\gamma_i} fds$, where ds is the line element. Let Γ be a family of locally rectifiable chains in G. A non-negative Borel measurable function f defined in G is called admissible in association with Γ if $\int_\gamma fds\geqq 1$ for each $\gamma\in\Gamma$. The 2-module $M_2(\Gamma)$ is defined by $\inf_f\int_G f^2dx$, where the infimum is taken over all functions f admissible in association with Γ and dx is the volume element. The following properties are well known (see, e.g., [7, Chapter I]):

(1.1) If $\Gamma_1\subset\Gamma_2$, then $M_2(\Gamma_1)\leqq M_2(\Gamma_2)$.

(1.2) If each $\gamma_1\in\Gamma_1$ contains a $\gamma_2\in\Gamma_2$, then $M_2(\Gamma_1)\leqq M_2(\Gamma_2)$.

A property will be said to hold 2-almost everywhere (=2-a.e.) on Γ if the 2-module of the subfamily of exceptional chains is zero. Denote by $\hat{G}_Q$ (resp. $\hat{G}_I$) the Kerékjártó-Stoïlow compactification (resp. the Aleksandrov compactification) of G. In case $\hat{\Gamma}$ is a family of curves in $\hat{G}_Q$ or in $\hat{G}_I$ such that the restriction $\gamma|_G$ is a locally rectifiable chain in G for each $\gamma\in\hat{\Gamma}$, we denote by $M_2(\hat{\Gamma})$ the 2-module of $\{\gamma|_G;\ \gamma\in\hat{\Gamma}\}$. Hereafter, by a curve we shall mean a locally rectifiable curve.

A real valued function u defined in a domain G in R^N is called 2-precise (or *BLD*), if it is absolutely continuous along 2-a.e. curve in G and $|\nabla u|$, which is defined a.e., belongs to $L^2(G)$. For a compact subset α of ∂G, let $\tilde{\Gamma}_G(\alpha)$ be the family of all curves in G each of which starts from some point of G and tends to α. A 2-precise function u on G has a finite curvilinear limit $u(\gamma)$ along 2-a.e. curve γ in $\tilde{\Gamma}_G(\alpha)$ (see [7, Theorem 5.4]).

§2. Extremal distances and capacities of condenser

Let G be a domain in R^N and α_0, α_1 be non-empty compact subsets of ∂G such that $\alpha_0\cap\alpha_1=\emptyset$. We denote by $\Gamma(\alpha_0, \alpha_1; G)$ the family of curves in G each of which connects α_0 and α_1. The reciprocal of $M_2(\Gamma(\alpha_0, \alpha_1; G))$ will be called the extremal distance of order 2 between α_0 and α_1 relative to G. Denote by $\mathscr{D}(\alpha_0, \alpha_1; G)$ the family of all 2-precise functions u on G such that $u(\gamma)=0$ (resp. 1) for 2-a.e. $\gamma\in\tilde{\Gamma}_G(\alpha_0)$ (resp. $\tilde{\Gamma}_G(\alpha_1)$). We define the 2-capacity of the condenser $(\alpha_0, \alpha_1; G)$ as

$$C_2(\alpha_0, \alpha_1; G) = \inf\left\{\int_G|\nabla u|^2dx;\ u\in\mathscr{D}(\alpha_0, \alpha_1; G)\right\}.$$

LEMMA 2.1 ([11, Theorem 4]). *In case $M_2(\tilde{\Gamma}_G(\alpha_0)\cup\tilde{\Gamma}_G(\alpha_1))>0$, there exists a unique harmonic function $u_0\in\mathscr{D}(\alpha_0, \alpha_1; G)$ for which*

$$C_2(\alpha_0, \alpha_1; G) = \int_G|\nabla u_0|^2dx.$$

It is characterized by the condition that $u_0\in\mathscr{D}(\alpha_0, \alpha_1; G)$ and

$$\int_G(\nabla u_0, \nabla v)dx=0$$

for every 2-precise function v on G such that $v(\gamma)=0$ for 2-a.e. $\gamma\in\tilde{\Gamma}_G(\alpha_0)\cup\tilde{\Gamma}_G(\alpha_1)$.

We call u_0 the extremal function for $C_2(\alpha_0, \alpha_1; G)$.

The following property is known (see [7, Theorem 6.10] or [12, Theorem 3.8]):

$$(2.1)\quad C_2(\alpha_0, \alpha_1; G) = M_2(\Gamma(\alpha_0, \alpha_1; G)).$$

Let E be a compact set contained in a domain G such that E^c is a domain. Let α_0, α_1 be as above. We denote by $\Gamma_Q(\alpha_0, \alpha_1; G-E)$ (resp. $\Gamma_I(\alpha_0, \alpha_1; G-E)$) the family of curves in $(\hat{E}^c)_Q-G^c$ (resp. $(\hat{E}^c)_I-G^c$) each of which connects α_0 and α_1.

Let $\{G_n\}_{n=1}^{\infty}$ be an approximation of $G-E$ towards E such that each G_n is a subdomain of $G-E$, each ∂G_n consists of ∂G and a finite number of compact C^1-surfaces $\beta_1^{(n)},\dots,\beta_{j(n)}^{(n)}$, $\bar{G}_n \subset G_{n+1} \cup \partial G$ $(n=1, 2,\dots)$ and $\cup_{n=1}^{\infty} G_n = G-E$. Denote by $\mathscr{D}^*(\alpha_0, \alpha_1; G_n, \{\beta_j^{(n)}\})$ the family of all $u \in \mathscr{D}(\alpha_0, \alpha_1; G_n)$ such that $u(\gamma) = a_j$ for 2-a.e. $\gamma \in \tilde{\Gamma}_{G_n}(\beta_j^{(n)})$ $(j=1,\dots, j(n))$, where each a_j is a constant depending on u. We set

$$C_2^*(\alpha_0, \alpha_1; G_n, \{\beta_j^{(n)}\}) = \inf\left\{\int_{G_n} |\nabla u|^2 dx;\ u \in \mathscr{D}^*(\alpha_0, \alpha_1; G_n, \{\beta_j^{(n)}\})\right\}.$$

In the same way as in [11, Theorem 1], we see that if $M_2(\tilde{\Gamma}_{G_n}(\alpha_0) \cup \tilde{\Gamma}_{G_n}(\alpha_1)) > 0$, then there exists a unique harmonic function $u_{n,Q}$ in $\mathscr{D}^*(\alpha_0, \alpha_1; G_n, \{\beta_j^{(n)}\})$ for which

$$C_2^*(\alpha_0, \alpha_1; G_n, \{\beta_j^{(n)}\}) = \int_{G_n} |\nabla u_{n,Q}|^2 dx.$$

We call $u_{n,Q}$ the extremal function for $C_2^*(\alpha_0, \alpha_1; G_n, \{\beta_j^{(n)}\})$. It is characterized by the condition that $u_{n,Q} \in \mathscr{D}^*(\alpha_0, \alpha_1; G_n, \{\beta_j^{(n)}\})$ and

$$\int_{G_n} (\nabla u_{n,Q}, \nabla v) dx = 0$$

for every 2-precise function v on G_n such that $v(\gamma)=0$ for 2-a.e. $\gamma \in \tilde{\Gamma}_{G_n}(\alpha_0) \cup \tilde{\Gamma}_{G_n}(\alpha_1)$ and $v(\gamma) = a_j$ for 2-a.e. $\gamma \in \tilde{\Gamma}_{G_n}(\beta_j^{(n)})$ $(j=1,\dots, j(n))$. Note that $C_2^*(\alpha_0, \alpha_1; G_n, \{\beta_j^{(n)}\}) \geqq C_2^*(\alpha_0, \alpha_1; G_{n+1}, \{\beta_j^{(n+1)}\})$ (cf. [11, § 1]). Therefore the limit

$$C_2^{**}(\alpha_0, \alpha_1; G-E, \tilde{\beta}_Q) = \lim_{n\to\infty} C_2^*(\alpha_0, \alpha_1; G_n, \{\beta_j^{(n)}\})$$

exists and does not depend on the choice of approximation.

LEMMA 2.2. *In case* $M_2(\tilde{\Gamma}_G(\alpha_0) \cup \tilde{\Gamma}_G(\alpha_1)) > 0$, *there exists a unique harmonic function* u_Q *on* $G-E$ *such that*

$$C_2^{**}(\alpha_0, \alpha_1; G-E, \tilde{\beta}_Q) = \int_{G-E} |\nabla u_Q|^2 dx$$

and $u_Q(\gamma)=0$ (*resp.* 1) *for 2-a.e.* $\gamma \in \tilde{\Gamma}_G(\alpha_0)$ (*resp.* $\tilde{\Gamma}_G(\alpha_1)$). *It satisfies the condition that*

$$\int_{G-E} (\nabla u_Q, \nabla \phi) dx = 0$$

for every $\phi \in C^\infty(G-E)$ *such that* $|\nabla \phi| \in L^2(G-E)$, $\nabla \phi$ *vanishes on some neighborhood of* E *and* $\phi(\gamma)=0$ *for 2-a.e.* $\gamma \in \tilde{\Gamma}_G(\alpha_0) \cup \tilde{\Gamma}_G(\alpha_1)$.

PROOF. Let $\{G_n\}_{n=1}^{\infty}$ be an approximation of $G-E$ towards E as above. Let $u_{n,Q}$ be the harmonic function which is extremal for $C_2^*(\alpha_0, \alpha_1; G_n, \{\beta_j^{(n)}\})$.

As in the proof of [11, Theorems 1 and 2] we see that there exists a unique harmonic function u_Q on $G-E$ such that

$$C_2^{**}(\alpha_0, \alpha_1; G-E, \tilde{\beta}_Q) = \int_{G-E} |\nabla u_Q|^2 dx$$

and

$$\lim_{n\to\infty} \int_{G_n} |\nabla(u_{n,Q}-u_Q)|^2 dx = 0.$$

Let ϕ be a function of $C^\infty(G-E)$ such that $|\nabla\phi| \in L^2(G-E)$, $\nabla\phi$ vanishes on an open neighborhood U of E and $\phi(\gamma)=0$ for 2-a.e. $\gamma \in \tilde{\Gamma}_G(\alpha_0) \cup \tilde{\Gamma}_G(\alpha_1)$. For n such that $\partial G_n - \partial G \subset U$, we have

$$\int_{G_n} (\nabla u_{n,Q}, \nabla\phi) dx = 0.$$

By letting $n\to\infty$ we see that $\int_{G-E} (\nabla u_Q, \nabla\phi) dx = 0$.

We call u_Q the extremal function for $C_2^{**}(\alpha_0, \alpha_1; G-E, \tilde{\beta}_Q)$.

In case G is an unbounded domain such that ∂G is compact, we define the following capacities of the condenser of E. Let $\{G_n\}_{n=1}^\infty$ be an approximation of $G-E$ towards $E \cup \{\infty\}$ such that each G_n is a bounded subdomain of $G-E$, each ∂G_n consists of ∂G and a finite number of compact C^1-surfaces $\beta_1^{(n)}, \dots, \beta_{j(n)}^{(n)}$, $\bar{G}_n \subset G_{n+1} \cup \partial G$ $(n=1, 2, \dots)$ and $\bigcup_{n=1}^\infty G_n = G-E$, where ∞ means the point at infinity of R^N. Denote by $\mathscr{D}^*(\alpha_0, \alpha_1; G_n, \{\beta_j^{(n)}\})$ the family of all $u \in \mathscr{D}(\alpha_0, \alpha_1; G_n)$ such that $u(\gamma) = a_j$ for 2-a.e. $\gamma \in \tilde{\Gamma}_{G_n}(\beta_j^{(n)})$ $(j=1, \dots, j(n))$, where each a_j is a constant depending on u. Denote by $\mathscr{D}^*(\alpha_0, \alpha_1; G_n, \beta^{(n)})$ the family of all $u \in \mathscr{D}^*(\alpha_0, \alpha_1; G_n, \{\beta_j^{(n)}\})$ such that $u(\gamma) = \text{const.}$ for 2-a.e. $\gamma \in \bigcup_{j=1}^{j(n)} \tilde{\Gamma}_{G_n}(\beta_j^{(n)})$. We set

$$C_2^*(\alpha_0, \alpha_1; G_n, \{\beta_j^{(n)}\}) = \inf\left\{\int_{G_n} |\nabla u|^2 dx;\ u \in \mathscr{D}^*(\alpha_0, \alpha_1; G_n, \{\beta_j^{(n)}\})\right\},$$

$$C_2^*(\alpha_0, \alpha_1; G_n, \beta^{(n)}) = \inf\left\{\int_{G_n} |\nabla u|^2 dx;\ u \in \mathscr{D}^*(\alpha_0, \alpha_1; G_n, \beta^{(n)})\right\}.$$

We know (cf. [11, § 1]) that

$$C_2^{**}(\alpha_0, \alpha_1; G-E, \beta_Q) = \lim_{n\to\infty} C_2^*(\alpha_0, \alpha_1; G_n, \{\beta_j^{(n)}\}),$$

$$C_2^{**}(\alpha_0, \alpha_1; G-E, \beta_I) = \lim_{n\to\infty} C_2^*(\alpha_0, \alpha_1; G_n, \beta^{(n)})$$

exist and these capacities of condenser do not depend on the choice of approximation $\{G_n\}$.

LEMMA 2.3 ([11, Theorem 2]). (a) *In case* $M_2(\tilde{\Gamma}_G(\alpha_0)\cup\tilde{\Gamma}_G(\alpha_1))>0$, *there exists a unique function* u_Q *harmonic on* $G-E$ *which satisfies*

$$C_2^{**}(\alpha_0, \alpha_1; G-E, \beta_Q) = \int_{G-E}|\nabla u_Q|^2 dx.$$

It satisfies the condition that

$$\int_{G-E}(\nabla u_Q, \nabla\phi)dx = 0$$

for every $\phi\in C^\infty(G-E)$ *such that the support of* $|\nabla\phi|$ *is compact in* $G-E$ *and* $\phi=0$ *on* $U\cap(G-E)$ *for some neighborhood* U *of* $\alpha_0\cup\alpha_1$.

(b) *In case* $M_2(\tilde{\Gamma}_G(\alpha_0)\cup\tilde{\Gamma}_G(\alpha_1))>0$, *there exists a unique function* u_I *harmonic on* $G-E$ *which satisfies*

$$C_2^{**}(\alpha_0, \alpha_1; G-E, \beta_I) = \int_{G-E}|\nabla u_I|^2 dx.$$

It satisfies the condition that

$$\int_{G-E}(\nabla u_I, \nabla\phi)dx = 0$$

for every $\phi\in C^\infty(G-E)$ *such that* $\phi=const.$ *on* $U'\cap(G-E)$ *for some neighborhood* U' *of* $E\cup\{\infty\}$ *and* $\phi=0$ *on* $U\cap(G-E)$ *for some neighborhood* U *of* ∂G.

We call u_Q (resp. u_I) the extremal function for $C_2^{**}(\alpha_0, \alpha_1; G-E, \beta_Q)$ (resp. $C_2^{**}(\alpha_0, \alpha_1; G-E, \beta_I)$).

The following property is known:

(2.2) $C_2^{**}(\alpha_0, \alpha_1; G-E, \beta_Q)=M_2(\Gamma_Q(\alpha_0, \alpha_1; G-E))$

and $C_2^{**}(\alpha_0, \alpha_1; G-E, \beta_I)=M_2(\Gamma_I(\alpha_0, \alpha_1; G-E))$ ([11, Theorem 6]).

Hereafter we shall always assume that E is a compact set whose complement is a domain. Suppose that B_0 and B_1 are two disjoint closed balls in E^c. Set $D=R^N-(B_0\cup B_1)$ and $\alpha_i=\partial B_i$ $(i=0, 1)$. We consider the families $\Gamma(\alpha_0, \alpha_1; D)$ and $\Gamma(\alpha_0, \alpha_1; D-E)$. Following Väisälä [9], we define the following null set for extremal distances.

DEFINITION 1. A compact set E is called an NED_2-set if $M_2(\Gamma(\alpha_0, \alpha_1; D-E))=M_2(\Gamma(\alpha_0, \alpha_1; D))$ for all pairs of α_0 and α_1.

Moreover, we consider the families $\Gamma_Q(\alpha_0, \alpha_1; D-E)$ and $\Gamma_I(\alpha_0, \alpha_1; D-E)$. By (1.1) and (1.2), we see

$$M_2(\Gamma(\alpha_0, \alpha_1; D-E)) \leqq M_2(\Gamma(\alpha_0, \alpha_1; D)) \leqq M_2(\Gamma_Q(\alpha_0, \alpha_1; D-E))$$

$$\leqq M_2(\Gamma_I(\alpha_0, \alpha_1; D-E)).$$

DEFINITION 2. We say that E is an NED_2^Q-set (resp. $NED_2^{Q,I}$-set) if $M_2(\Gamma(\alpha_0, \alpha_1; D)) = M_2(\Gamma_Q(\alpha_0, \alpha_1; D-E))$ (resp. $M_2(\Gamma_Q(\alpha_0, \alpha_1; D-E)) = M_2(\Gamma_I(\alpha_0, \alpha_1; D-E)))$ for all pairs of α_0 and α_1.

The property (2.1) allows us to replace the equality $M_2(\Gamma(\alpha_0, \alpha_1; D-E)) = M_2(\Gamma(\alpha_0, \alpha_1; D))$ by the equivalent equality $C_2(\alpha_0, \alpha_1; D-E) = C_2(\alpha_0, \alpha_1; D)$. Thus we have the following definition which is equivalent to Definition 1.

DEFINITION 1′. A compact set E is called an NED_2-set if $C_2(\alpha_0, \alpha_1; D-E) = C_2(\alpha_0, \alpha_1; D)$ for all pairs of α_0 and α_1.

Similarly, by virtue of (2.2) we have

DEFINITION 2′. A compact set E is called an NED_2^Q-set (resp. $NED_2^{Q,I}$-set) if $C_2(\alpha_0, \alpha_1; D) = C_2^{**}(\alpha_0, \alpha_1; D-E, \beta_Q)$ (resp. $C_2^{**}(\alpha_0, \alpha_1; D-E, \beta_Q) = C_2^{**}(\alpha_0, \alpha_1; D-E, \beta_I))$ for all pairs of α_0 and α_1.

§3. Classes of Dirichlet-finite harmonic functions

Let E be a compact set such that E^c is a domain and G be a domain containing E. We denote by $HD^2(G)$ the class of all functions u harmonic on G such that its Dirichlet integral $\int_G |\nabla u|^2 dx$ is finite.

Let $\{\Omega_n\}_{n=1}^{\infty}$ be an approximation of E^c towards E, that is, each Ω_n is an unbounded subdomain of E^c, each $\partial\Omega_n$ consists of a finite number of compact C^1-surfaces such that the interior of each surface of $\partial\Omega_n$ contains at least one point of E, $\bar{\Omega}_n \subset \Omega_{n+1}$ $(n=1, 2,\dots)$ and $\cup_{n=1}^{\infty} \Omega_n = E^c$. Let G be a domain such that $G \supset E$ and $G \supset \partial\Omega_n$ for all n. Let g and u be harmonic functions in $HD^2(G-E)$. Then the limit of $\int_{\partial\Omega_n} g(\partial u/\partial\nu)dS$ exists and does not depend on the choice of approximation $\{\Omega_n\}$, where dS is the surface element. Therefore we use the symbolic expression

$$\int_{\partial E} g\frac{\partial u}{\partial \nu}dS = \lim_{n\to\infty}\int_{\partial\Omega_n} g\frac{\partial u}{\partial \nu}dS.$$

We denote by $\widetilde{HD}^2(G-E; E)$ (resp. $KD^2(G-E; E)$) the class consisting of all $u \in HD^2(G-E)$ each of which satisfies

$$\int_{G-E} (\nabla u, \nabla \phi)dx = 0$$

for every $\phi \in C^{\infty}(G-E)$ such that $\phi = 0$ outside some compact set contained in G and $|\nabla\phi| \in L^2(G-E)$ (resp. for every $\phi \in C_0^{\infty}(G)$ such that $\nabla\phi$ vanishes on some open neighborhood of E). We denote by $\widetilde{KD}^2(G-E; E)$ the class consisting of all $u \in KD^2(G-E; E)$ each of which satisfies

$$\int_{\partial E} u \frac{\partial g}{\partial \nu} dS = 0$$

for every $g \in KD^2(G-E; E)$. We note that $\widetilde{HD}^2(G-E; E) \subset KD^2(G-E; E)$ and $\int_\tau (\partial u/\partial \nu) dS = 0$ for any $u \in KD^2(G-E; E)$ and for any compact C^1-surface τ in $G-E$ which is homologous to zero in G. In case $G=R^N$, we write $KD^2(E^c)$ for $KD^2(R^N-E; E)$.

DEFINITION. We say that a compact set E is removable for HD^2 (resp. $\widetilde{HD}^2$, KD^2, $\widetilde{KD}^2$) if for some bounded domain G containing E every function in $HD^2(G-E)$ (resp. $\widetilde{HD}^2(G-E; E)$, $KD^2(G-E; E)$, $\widetilde{KD}^2(G-E; E)$) can be extended to a function in $HD^2(G)$.

The following lemma which relates the removable sets for KD^2 to the extremal distances is known.

LEMMA 3.1 ([11, Theorems 7 and 13]). *For E to be removable for KD^2 it is necessary and sufficient that $M_2(\Gamma(\alpha_0, \alpha_1; D-E)) = M_2(\Gamma_Q(\alpha_0, \alpha_1; D-E))$ for all pairs of α_0 and α_1, where $D=R^N-(B_0 \cup B_1)$ and $\alpha_i=\partial B_i$ $(i=0, 1)$ for two disjoint closed balls B_0, B_1 contained in E^c.*

LEMMA 3.2. *$\int_{\partial E} g(\partial u/\partial \nu) dS = 0$ for any $g \in HD^2(G-E)$ and for any $u \in \widetilde{HD}^2(G-E; E)$.*

PROOF. Take any $\psi \in C_0^\infty(G)$ such that $\psi=1$ on some neighborhood of E. By Green's formula we have

$$\int_{\partial E} g \frac{\partial u}{\partial \nu} dS = -\int_{G-E} (\nabla u, \nabla(\psi g)) dx = 0.$$

PROPOSITION 3.1. *Let G be a domain containing E and α_0, α_1 be non-empty compact subsets of ∂G such that $\alpha_0 \cap \alpha_1 = \emptyset$ and $M_2(\tilde{\Gamma}_G(\alpha_0) \cup \tilde{\Gamma}_G(\alpha_1)) > 0$. Let u_0 and u_Q be extremal functions for $C_2(\alpha_0, \alpha_1; G-E)$ and $C_2^{**}(\alpha_0, \alpha_1; G-E, \tilde{\beta}_Q)$ respectively. Then u_0 (resp. u_Q) belongs to $\widetilde{HD}^2(G-E; E)$ (resp. $\widetilde{KD}^2(G-E; E)$).*

PROOF. The fact that u_0 belongs to $\widetilde{HD}^2(G-E; E)$ is a simple consequence of Lemma 2.1. By Lemma 2.2, $u_Q \in KD^2(G-E; E)$. Let $\{G_n\}_{n=1}^\infty$ be an approximation of $G-E$ towards E as in §2. Let $u_{n,Q}$ be the harmonic function which is extremal for $C_2^*(\alpha_0, \alpha_1; G_n, \{\beta_j^{(n)}\})$. We shall show that $u_{n,Q}$=const. on each $\beta_j^{(n)}$. Take a bounded domain $\tilde{G}$ such that $E \subset \tilde{G} \subset G$ hold, $\partial \tilde{G}$ is a compact

C^1-surface and $\cup_{j=1}^{j(n)} \beta_j^{(n)} \subset \tilde{G}$ for all n. We know (cf. [8, p. 239]) that there exists a unique harmonic function $\tilde{u}_{n,Q}$ on $\tilde{G} \cap G_n$ such that $\tilde{u}_{n,Q} = u_{n,Q}$ on $\partial\tilde{G}$, $\tilde{u}_{n,Q}$=const. on each $\beta_j^{(n)}$ and $\int_{\beta_j^{(n)}} (\partial \tilde{u}_{n,Q}/\partial\nu) dS = 0$ for each $j = 1, \dots, j(n)$. By using Green's formula, we obtain

$$\int_{\tilde{G}\cap G_n} |\nabla \tilde{u}_{n,Q}|^2 dx \leqq \int_{\tilde{G}\cap G_n} |\nabla u_{n,Q}|^2 dx.$$

The function $\hat{u}_{n,Q}$ which is equal to $\tilde{u}_{n,Q}$ in $\tilde{G} \cap G_n$ and to $u_{n,Q}$ on $G - \tilde{G}$ belongs to $\mathscr{D}^*(\alpha_0, \alpha_1; G_n, \{\beta_j^{(n)}\})$. By the uniqueness of $u_{n,Q}$, $u_{n,Q} = \hat{u}_{n,Q}$ in G_n. Hence $u_{n,Q}$=const. on each $\beta_j^{(n)}$. As stated in the proof of Lemma 2.2, we see that

$$\lim_{n\to\infty} \int_{G_n} |\nabla (u_{n,Q} - u_Q)|^2 dx = 0.$$

Therefore, $u_{n,Q}$ converges to u_Q uniformly on every compact subset of $G - E$.

Take any $g \in KD^2(G-E; E)$. We have

$$\int_{\partial G_n - \partial G} u_Q \frac{\partial g}{\partial \nu} dS = \lim_{m\to\infty} \int_{\partial G_n - \partial G} u_{m,Q} \frac{\partial g}{\partial \nu} dS$$

$$= \lim_{m\to\infty} \int_{\partial (G_m - G_n)} u_{m,Q} \frac{\partial g}{\partial \nu} dS,$$

because $u_{m,Q}$=const. on each $\beta_j^{(m)}$ and $\int_{\beta_j^{(m)}} (\partial g/\partial \nu) dS = 0$. Hence

$$\int_{\partial G_n - \partial G} u_Q \frac{\partial g}{\partial \nu} dS = \lim_{m\to\infty} \int_{G_m - G_n} (\nabla u_{m,Q}, \nabla g) dx$$

$$= \int_{G-E-G_n} (\nabla u_Q, \nabla g) dx.$$

Letting $n \to \infty$ we conclude that $\int_{\partial E} u_Q (\partial g/\partial \nu) dS = 0$. Accordingly $u_Q \in \widetilde{KD}^2(G-E; E)$ and thus our proposition is proved.

REMARK. Let G be an unbounded domain such that ∂G is compact. Let u_Q be the extremal function for $C_2^{**}(\alpha_0, \alpha_1; G-E, \beta_Q)$. By Lemma 2.3 (a), we have $u_Q \in KD^2(G-E; E)$. Moreover, as in the proof of Proposition 3.1 we have $u_Q \in \widetilde{KD}^2(G-E; E)$.

We take two bounded domains Ω_0 and Ω_1 such that $E \subset \Omega_0 \subset \bar{\Omega}_0 \subset \Omega_1 \subset \bar{\Omega}_1 \subset G$ and each of $\partial\Omega_0$ and $\partial\Omega_1$ consists of a single compact C^1-surface. For any u in $HD^2(G-E)$, we set

$$u_i^*(x) = \frac{1}{\sigma(N-2)} \int_{\partial\Omega_i} \left(\frac{1}{r^{N-2}} \frac{\partial u}{\partial \nu} - u \frac{\partial}{\partial \nu} \left(\frac{1}{r^{N-2}} \right) \right) dS, \qquad (i = 0, 1),$$

where r denotes the distance from a point x to the variable on $\partial\Omega_i$, σ is the surface area of the unit sphere in R^N. Each u_i^* is harmonic in $R^N-\partial\Omega_i$ $(i=0, 1)$. When x lies in the domain $\Omega_1-\bar{\Omega}_0$, the equality $u(x)=u_1^*(x)-u_0^*(x)$ holds. Moreover, if we define a harmonic function H_u by

$$H_u(x)=\begin{cases} u_0^*(x) & \text{if } x\in R^N-\bar{\Omega}_0 \\ u_1^*(x)-u(x) & \text{if } x\in \bar{\Omega}_0-E, \end{cases}$$

then it is easy to see that H_u is a harmonic function with finite Dirichlet integral in E^c and regular at infinity, that is, $\lim_{|x|\to\infty} H_u(x)=0$. We note that if $u\in KD^2(G-E;\,E)$, then $H_u\in KD^2(E^c)$.

Let G be an unbounded domain such that ∂G is compact and let $u\in HD^2(G)$. Then there is a constant c such that $u+c$ is regular at infinity. Moreover, we know that $|x|(u+c)$ and $|x|^{N-1}(\partial u/\partial x_i)$ $(i=1,\dots, N)$ are bounded as $|x|\to\infty$. Hence we have

$$\int_{\partial G_n}(u+c)\frac{\partial v}{\partial \nu}dS \longrightarrow 0 \qquad (\text{as } n\longrightarrow\infty), \tag{3.1}$$

where $v\in HD^2(G)$ and $\{G_n\}_{n=1}^{\infty}$ is an approximation of G towards $\{\infty\}$.

§4. Principal functions

Let E be a compact set such that E^c is a domain. Take any distinct two points x^0, x^1 in E^c and open balls V_0, V_1 centered at x^0, x^1 with disjoint closures in E^c. Let $\{D_n\}_{n=1}^{\infty}$ be an exhaustion of E^c, that is, each D_n is a bounded subdomain of E^c, each ∂D_n consists of a finite number of C^1-surfaces $\beta_j^{(n)}$ $(j=1,\dots, j(n))$, $\bar{D}_n\subset D_{n+1}$ $(n=1, 2,\dots)$ and $\cup_{n=1}^{\infty} D_n=E^c$. We may assume that D_n contains $\overline{V_0\cup V_1}$ for all n. We know (cf. [8, p. 242]) that there exist the principal functions $P_{0,n}$, $P_{1,n}^Q$ and $P_{1,n}^I$ with respect to x^0, x^1 and D_n, which are characterized by the following properties:

(1) $P_{0,n}$, $P_{1,n}^Q$ and $P_{1,n}^I$ are harmonic on $D_n-(\{x^0\}\cup\{x^1\})$ and continuous on ∂D_n;

(2) $$\begin{aligned} P_{0,n} &= \sigma^{-1}|x-x^0|^{2-N}+h_{0,n} && \text{on } V_0,\\ P_{1,n}^Q &= \sigma^{-1}|x-x^0|^{2-N}+h_{1,n}^Q && \text{on } V_0,\\ P_{1,n}^I &= \sigma^{-1}|x-x^0|^{2-N}+h_{1,n}^I && \text{on } V_0, \end{aligned}$$

where $h_{0,n}$, $h_{1,n}^Q$ and $h_{1,n}^I$ are harmonic on V_0;

(3) $$\begin{aligned} P_{0,n} &= -\sigma^{-1}|x-x^1|^{2-N}+f_{0,n} && \text{on } V_1,\\ P_{1,n}^Q &= -\sigma^{-1}|x-x^1|^{2-N}+f_{1,n}^Q && \text{on } V_1, \end{aligned}$$

$$P^I_{1,n} = -\sigma^{-1}|x-x^1|^{2-N} + f^I_{1,n} \quad \text{on} \quad V_1,$$

where $f_{0,n}$, $f^Q_{1,n}$ and $f^I_{1,n}$ are harmonic on V_1 and $f_{0,n}(x^1)=f^Q_{1,n}(x^1)=f^I_{1,n}(x^1)=0$;

(4) $\dfrac{\partial P_{0,n}}{\partial \nu} = 0$ on ∂D_n,

$$P^Q_{1,n} = \text{const. on each } \beta_j^{(n)} \text{ and } \int_{\beta_j^{(n)}} \frac{\partial P^Q_{1,n}}{\partial \nu} dS = 0$$

for $j=1,\dots, j(n)$,

$$P^I_{1,n} = \text{const. on } \partial D_n.$$

We see that the limits

$$P_0 = \lim_{n\to\infty} P_{0,n}, \quad P^Q_1 = \lim_{n\to\infty} P^Q_{1,n}, \quad P^I_1 = \lim_{n\to\infty} P^I_{1,n};$$

$$h_0 = \lim_{n\to\infty} h_{0,n}, \quad h^Q_1 = \lim_{n\to\infty} h^Q_{1,n}, \quad h^I_1 = \lim_{n\to\infty} h^I_{1,n};$$

$$f_0 = \lim_{n\to\infty} f_{0,n}, \quad f^Q_1 = \lim_{n\to\infty} f^Q_{1,n}, \quad f^I_1 = \lim_{n\to\infty} f^I_{1,n}$$

exist and the convergences are uniform on every compact subset of E^c. These limit functions do not depend on the choice of exhaustion (see [8, p. 246]).

Further we define a harmonic function $\tilde{P}$ on $R^N-(\{x^0\} \cup \{x^1\})$ by

$$\tilde{P}(x) = \sigma^{-1}(|x-x^0|^{2-N} - |x-x^1|^{2-N} - |x^0-x^1|^{2-N}).$$

Set

$$\tilde{h}(x) = -\sigma^{-1}(|x-x^1|^{2-N} + |x^0-x^1|^{2-N}),$$

$$\tilde{f}(x) = \sigma^{-1}(|x-x^0|^{2-N} - |x^0-x^1|^{2-N}).$$

Then $\tilde{h}$ (resp. $\tilde{f}$) is harmonic on some neighborhood of x^0 (resp. x^1), and $\tilde{f}(x^1)=0$.

For the purpose of obtaining a relation between null sets and principal functions, we consider the following quantities which are similar to the span (cf. [8, p. 247]):

$$\tilde{S}(x^0, x^1) = h_0(x^0) - \tilde{h}(x^0),$$

$$\tilde{S}^Q(x^0, x^1) = \tilde{h}(x^0) - h^Q_1(x^0),$$

$$S^{Q,I}(x^0, x^1) = h^Q_1(x^0) - h^I_1(x^0).$$

LEMMA 4.1.

(a) $\displaystyle\int_{\partial E} g(\partial P_0/\partial \nu)dS = 0$ *for any* $g \in HD^2(G-E)$.

(b) $\displaystyle\int_{\partial E} P^Q_1(\partial g/\partial \nu)dS = 0$ *for any* $g \in KD^2(G-E;\, E)$.

(c) $\int_{\partial E}(P_1^I+c)(\partial g/\partial \nu)dS=0$ *for any* $g\in HD^2(G-E)$, *where* c *is a constant such that* P_1^I+c *is regular at infinity.*

PROOF. (a) and (b) are seen in the same manner as Proposition 3.1 (cf. [8, p. 58]). To show (c), we consider the harmonic function H_g which is defined in § 3. By using Green's formula and (3.1) we have

$$\int_{\partial E}(P_1^I+c)\frac{\partial H_g}{\partial \nu}dS=0.$$

By the definition, $H_g=g_1^*-g$ in Ω_0-E for some domain Ω_0 containing E and g_1^* is harmonic on Ω_0. It is easy to see that

$$\int_{\partial E}(P_1^I+c)\frac{\partial g_1^*}{\partial \nu}dS=0.$$

Hence, $\int_{\partial E}(P_1^I+c)(\partial g/\partial \nu)dS=0$.

Using Green's formula and Lemma 4.1, we have the following

LEMMA 4.2.

(a) $\int_{E^c}|\nabla(P_0-\tilde{P})|^2dx=(N-2)\tilde{S}(x^0,x^1)-\int_{\partial E}\tilde{P}\frac{\partial\tilde{P}}{\partial \nu}dS.$

(b) $\int_{E^c}|\nabla(\tilde{P}-P_1^Q)|^2dx=(N-2)\tilde{S}^Q(x^0,x^1)-\int_{\partial E}\tilde{P}\frac{\partial\tilde{P}}{\partial \nu}dS.$

(c) $\int_{E^c}|\nabla(P_1^Q-P_1^I)|^2dx=(N-2)S^{Q,I}(x^0,x^1).$

PROOF. Let c_0 and $\tilde{c}$ be constants such that P_0+c_0 and $\tilde{P}+\tilde{c}$ are regular at infinity. By using Green's formula and Lemma 4.1 (a), we obtain

$$\begin{aligned}\int_{E^c}|\nabla(P_0-\tilde{P})|^2dx&=-\int_{\partial E}(P_0-\tilde{P}+c_0-\tilde{c})\frac{\partial(P_0-\tilde{P})}{\partial \nu}dS\\&=\int_{\partial E}(P_0-\tilde{P}+c_0-\tilde{c})\frac{\partial\tilde{P}}{\partial \nu}dS\\&=\int_{\partial E}P_0\frac{\partial\tilde{P}}{\partial \nu}dS-\int_{\partial E}\tilde{P}\frac{\partial\tilde{P}}{\partial \nu}dS.\end{aligned}$$

We take a sufficiently small $r>0$ such that $E\cap(B_0\cup B_1)=\emptyset$ and $B_0\cap B_1=\emptyset$, where $B_i=\{x;\ |x-x^i|\leqq r\}$ $(i=0, 1)$. Let $\alpha_i=\partial B_i$ $(i=0, 1)$. By using Green's formula and Lemma 4.1 (a) we have

$$\int_{\partial E} P_0 \frac{\partial \tilde{P}}{\partial \nu} dS = \int_{\partial E} (\tilde{P}+\tilde{c}) \frac{\partial P_0}{\partial \nu} dS + \int_{\alpha_0 \cup \alpha_1} \left((\tilde{P}+\tilde{c}) \frac{\partial P_0}{\partial \nu} - (P_0+c_0) \frac{\partial \tilde{P}}{\partial \nu} \right) dS$$

$$= \int_{\alpha_0 \cup \alpha_1} \left((\tilde{P}+\tilde{c}) \frac{\partial P_0}{\partial \nu} - (P_0+c_0) \frac{\partial \tilde{P}}{\partial \nu} \right) dS.$$

Letting $r \to 0$, we see

$$\int_{\partial E} P_0 \frac{\partial \tilde{P}}{\partial \nu} dS = (N-2)\tilde{S}(x^0, x^1).$$

Thus we obtain (a). By similar arguments we obtain (b) and (c).

From this lemma we can derive that the property $\tilde{S}(x^0, x^1)=0$ means $P_0-\tilde{P}=0$ in E^c. In fact, since $\tilde{P}$ is harmonic on E, by Green's formula we obtain

$$\int_{\partial E} \tilde{P} \frac{\partial \tilde{P}}{\partial \nu} dS = \int_E |\nabla \tilde{P}|^2 dx.$$

If $\tilde{S}(x^0, x^1)=0$, then (a) of the above lemma implies

$$0 \leqq \int_{E^c} |\nabla (P_0-\tilde{P})|^2 dx = -\int_E |\nabla \tilde{P}|^2 dx \leqq 0,$$

so that $P_0-\tilde{P}=$const. in E^c. Since $f_0(x^1)=\tilde{f}(x^1)=0$, $P_0-\tilde{P}=0$ in E^c. Moreover, since $V(\{x; |\nabla \tilde{P}(x)|=0\})=0$, it follows that $V(E)=0$. Thus we have

COROLLARY 4.1. *If $\tilde{S}(x^0, x^1)=0$, then $P_0-\tilde{P}=0$ in E^c and $V(E)=0$. If $P_0-\tilde{P}=0$ in E^c, then $\tilde{S}(x^0, x^1)=0$ and $V(E)=0$.*

Similarly, we obtain

COROLLARY 4.2. *If $\tilde{S}^Q(x^0, x^1)=0$, then $\tilde{P}-P_1^Q=0$ in E^c and $V(E)=0$. If $\tilde{P}-P_1^Q=0$ in E^c, then $\tilde{S}^Q(x^0, x^1)=0$ and $V(E)=0$.*

We prove

PROPOSITION 4.1. *If E is an NED_2-set (resp. NED_2^Q-set, $NED_2^{Q,I}$-set), then $\tilde{S}(x^0, x^1)$ (resp. $\tilde{S}^Q(x^0, x^1)$, $S^{Q,I}(x^0, x^1)$) is equal to zero for all distinct two points x^0, x^1 in E^c.*

PROOF. Take any distinct two points x^0, x^1 in E^c and mutually disjoint closed balls B_0, B_1 in E^c of radius r and with centers at x^0, x^1 respectively. Set $D=R^N-(B_0 \cup B_1)$ and $\alpha_i=\partial B_i$ $(i=0, 1)$. As in the latter half of the proof of [11, Theorem 13], for sufficiently small r we have the following inequalities:

$$\text{(4.1)} \qquad \max_{x\in\alpha_0} h_0 - \min_{x\in\alpha_1} f_0 \geqq \frac{N-2}{C_2(\alpha_0, \alpha_1; D-E)} - \frac{2}{\sigma r^{N-2}}$$

$$\geqq \min_{x\in\alpha_0} h_0 - \max_{x\in\alpha_1} f_0;$$

(4.2) $$\max_{x\in\alpha_0} \tilde{h} - \min_{x\in\alpha_1} \tilde{f} \geqq \frac{N-2}{C_2(\alpha_0, \alpha_1; D)} - \frac{2}{\sigma r^{N-2}}$$

$$\geqq \min_{x\in\alpha_0} \tilde{h} - \max_{x\in\alpha_1} \tilde{f};$$

(4.3) $$\max_{x\in\alpha_0} h_1^Q - \min_{x\in\alpha_1} f_1^Q \geqq \frac{N-2}{C_2^{**}(\alpha_0, \alpha_1; D-E, \beta_Q)} - \frac{2}{\sigma r^{N-2}}$$

$$\geqq \min_{x\in\alpha_0} h_1^Q - \max_{x\in\alpha_1} f_1^Q;$$

(4.4) $$\max_{x\in\alpha_0} h_1^I - \min_{x\in\alpha_1} f_1^I \geqq \frac{N-2}{C_2^{**}(\alpha_0, \alpha_1; D-E, \beta_I)} - \frac{2}{\sigma r^{N-2}}$$

$$\geqq \min_{x\in\alpha_0} h_1^I - \max_{x\in\alpha_1} f_1^I.$$

If E is an NED_2-set, then the equality

$$C_2(\alpha_0, \alpha_1; D-E) = C_2(\alpha_0, \alpha_1; D)$$

holds. By (4.1) and (4.2) we see that

$$\max_{x\in\alpha_0} h_0 - \min_{x\in\alpha_1} f_0 \geqq \min_{x\in\alpha_0} \tilde{h} - \max_{x\in\alpha_1} \tilde{f},$$

$$\max_{x\in\alpha_0} \tilde{h} - \min_{x\in\alpha_1} \tilde{f} \geqq \min_{x\in\alpha_0} h_0 - \max_{x\in\alpha_1} f_0.$$

Since $f_0(x^1)=\tilde{f}(x^1)=0$, letting $r\to 0$ we have $\tilde{S}(x^0, x^1)=0$. The results for NED_2^Q-set and $NED_2^{Q,I}$-set are established in the same manner.

REMARK 4.1. In the above proof we showed that if $C_2(\alpha_0, \alpha_1; R^N-(B_0 \cup B_1)-E)=C_2(\alpha_0, \alpha_1; R^N-(B_0 \cup B_1))$ for any mutually disjoint closed balls B_0 and B_1 in E^c with respective centers x^0 and x^1, then $\tilde{S}(x^0, x^1)=0$. Similar facts are true for $\tilde{S}^Q(x^0, x^1)$ and $S^{Q,I}(x^0, x^1)$.

In view of Corollaries 4.1 and 4.2 we have

COROLLARY 4.3 (cf. [9, Theorem 1]). *If E is an NED_2-set or an NED_2^Q-set, then* $V(E)=0$.

REMARK 4.2. We shall show later in Remarks 5.2 and 6.1 by examples that the converse of Corollary 4.3 is not always true.

§5. NED_2-sets and removable sets for $\widetilde{HD}^2$

Let G be a domain containing a compact set E. Let α_0, α_1 be non-empty compact subsets of ∂G such that $\alpha_0 \cap \alpha_1 = \varnothing$. We say that a compact set E is an NED_2-set with respect to G if $M_2(\Gamma(\alpha_0, \alpha_1; G))=M_2(\Gamma(\alpha_0, \alpha_1; G-E))$ for every disjoint compact subsets α_0, α_1 of ∂G. By (2.1) we see that E is an NED_2-set with respect to G if and only if $C_2(\alpha_0, \alpha_1; G)=C_2(\alpha_0, \alpha_1; G-E)$ for every α_0 and α_1.

A bounded domain R is called a ring domain if its complement consists of two components.

In this section we shall show the following theorem.

THEOREM 5.1. *For a compact set E in R^N, the following statements are equivalent to each other*:

(1) *E is an NED_2-set*;

(2) *$\tilde{S}(x^0, x^1)$ is equal to zero for all distinct two points x^0, x^1 in E^c*;

(3) *$\tilde{S}(x^0, x^1)$ is equal to zero for all x^0 in a non-empty open set in E^c and some x^1 in E^c*;

(4) *For any bounded domain G containing E, every u in $\widetilde{HD}^2(G-E; E)$ can be extended to a function in $HD^2(G)$*;

(5) *E is removable for $\widetilde{HD}^2$*;

(6) *E is an NED_2-set with respect to every domain G containing E*;

(7) *The equality $C_2(\alpha_0, \alpha_1; R-E)=C_2(\alpha_0, \alpha_1; R)$ holds for every ring domain R containing E, where α_0 and α_1 are two boundary components of R*;

(8) *The equalities $C_2(\alpha_0^i, \alpha_1^i; \Omega-E)=C_2(\alpha_0^i, \alpha_1^i; \Omega)$, $i=1,\dots, N$, hold for some N-dimensional open rectangle $\Omega\supset E$ with sides parallel to the coordinate axes, where α_0^i and α_1^i are the sides of Ω parallel to the coordinate plane $x_i=0$.*

REMARK 5.1. For a result related to the equivalence between (1), (4), (5) and (6), see [10, Theorem 3.1].

To prove this theorem we prepare some propositions.

LEMMA 5.1. *Let G be a bounded domain containing E and u be a function in $\widetilde{HD}^2(G-E; E)$. Then*

$$\int_{E^c}(\nabla H_u, \nabla(P_0-\tilde{P}))dx=(N-2)(H_u(x^0)-H_u(x^1))+\int_{\partial E}\tilde{P}\frac{\partial u_1^*}{\partial\nu}dS$$

for any distinct two points x^0, x^1 in E^c, where H_u and u_1^* are harmonic functions defined in § 3.

PROOF. By using Green's formula and Lemma 4.1 (a), we obtain

$$\int_{E^c}(\nabla H_u, \nabla(P_0-\tilde{P}))dx=-\int_{\partial E}H_u\frac{\partial(P_0-\tilde{P})}{\partial\nu}dS=\int_{\partial E}H_u\frac{\partial\tilde{P}}{\partial\nu}dS.$$

We take a sufficiently small $r>0$ such that $E\cap(B_0\cup B_1)=\emptyset$ and $B_0\cap B_1=\emptyset$, where $B_i=\{x; |x-x^i|\leqq r\}$ $(i=0, 1)$. Let $\alpha_i=\partial B_i$ $(i=0, 1)$. By using Green's formula we have

$$\int_{\partial E} H_u \frac{\partial \tilde{P}}{\partial \nu} dS = \int_{\partial E} (\tilde{P}+c)\frac{\partial H_u}{\partial \nu} dS + \int_{\alpha_0 \cup \alpha_1} \left((\tilde{P}+c)\frac{\partial H_u}{\partial \nu} - H_u \frac{\partial \tilde{P}}{\partial \nu} \right) dS,$$

where c is a constant such that $\tilde{P}+c$ is regular at infinity. Since $H_u = u_1^* - u$ in $\Omega_0 - E$ for some domain Ω_0 containing E and u_1^* is harmonic on Ω_0,

$$\int_{\partial E} (\tilde{P}+c)\frac{\partial H_u}{\partial \nu} dS = \int_{\partial E} \tilde{P} \frac{\partial u_1^*}{\partial \nu} dS - \int_{\partial E} (\tilde{P}+c)\frac{\partial u}{\partial \nu} dS.$$

Since $u \in \widetilde{HD}^2(G-E; E)$, by Lemma 3.2 we have

$$\int_{\partial E} (\tilde{P}+c)\frac{\partial u}{\partial \nu} dS = 0.$$

Thus,

$$\int_{E^c} (\nabla H_u, \nabla (P_0 - \tilde{P}))dx = \int_{\partial E} \tilde{P} \frac{\partial u_1^*}{\partial \nu} dS + \int_{\alpha_0 \cup \alpha_1} \left((\tilde{P}+c)\frac{\partial H_u}{\partial \nu} - H_u \frac{\partial \tilde{P}}{\partial \nu} \right) dS.$$

Letting $r \to 0$, we obtain the required equality.

PROPOSITION 5.1. *Let G be a bounded domain containing E and u be a function in $\widetilde{HD}^2(G-E; E)$. If $P_0 - \tilde{P} = 0$ in E^c for any x^0 in a non-empty open set Ω in E^c and some x^1 in E^c, then u can be extended to a function in $HD^2(G)$.*

PROOF. Let $x^0 \in \Omega$ and $x^1 \in E^c$. By Lemma 5.1 we have

$$(N-2)(H_u(x^1) - H_u(x^0)) = \int_{\partial E} \tilde{P} \frac{\partial u_1^*}{\partial \nu} dS.$$

Since $\tilde{P}$ and u_1^* are harmonic functions with finite Dirichlet integrals on some neighborhood of E and $V(E)=0$ by Corollary 4.1, using Green's formula we have

$$\int_{\partial E} \tilde{P} \frac{\partial u_1^*}{\partial \nu} dS = \int_E (\nabla \tilde{P}, \nabla u_1^*)dx = 0.$$

Hence, $H_u(x^1) = H_u(x^0)$. Letting x^0 vary in Ω, we obtain $H_u = \text{const.}$ in E^c. Therefore $u = u_1^* + \text{const.}$ in $\Omega_0 - E$ for some domain Ω_0 containing E. Thus we conclude that u can be extended to a function in $HD^2(G)$.

PROPOSITION 5.2. *Let G be a domain containing E and α_0, α_1 be disjoint compact subsets of ∂G. Let u_0 be an extremal function for $C_2(\alpha_0, \alpha_1; G-E)$. If $V(E)=0$ and u_0 can be extended to a harmonic function $\tilde{u}_0$ on G, then the equality*

$$C_2(\alpha_0, \alpha_1; G-E) = C_2(\alpha_0, \alpha_1; G)$$

holds.

PROOF. Since $\tilde{u}_0 \in \mathscr{D}(\alpha_0, \alpha_1; G)$ and $V(E)=0$, we have

$$C_2(\alpha_0, \alpha_1; G) \leqq \int_G |\nabla \tilde{u}_0|^2 dx = \int_{G-E} |\nabla u_0|^2 dx = C_2(\alpha_0, \alpha_1; G-E).$$

The converse inequality being valid, the equality follows.

LEMMA 5.2. *Let G be a bounded domain containing E and let R be a ring domain of the form $R=\{x;\ r_0<|x-x^0|<r_1\}$ with $R\supset \bar{G}$. Let $\tilde{u}_0$ and u_0 be extremal functions for $C_2(\alpha_0, \alpha_1; R)$ and $C_2(\alpha_0, \alpha_1; R-E)$ respectively, where $\alpha_i=\{x;\ |x-x^0|=r_i\}$ $(i=0, 1)$. Then*

$$(5.1)\qquad \int_{R-E} (\nabla H_u, \nabla(\tilde{u}_0-u_0))dx = \int_{\alpha_0-\alpha_1} H_u \frac{\partial u_0}{\partial \nu} dS - \int_{\partial E} \tilde{u}_0 \frac{\partial u_1^*}{\partial \nu} dS$$

for any u in $\widetilde{HD}^2(G-E;\ E)$, where H_u and u_1^* are harmonic functions defined in §3.

PROOF. We note that $\tilde{u}_0-u_0$ is harmonic on $R-E$ and $\tilde{u}_0-u_0=0$ on $\alpha_0 \cup \alpha_1$. By using Green's formula we have

$$(5.2)\qquad \int_{R-E} (\nabla H_u, \nabla(\tilde{u}_0-u_0))dx = -\int_{\partial E} (\tilde{u}_0-u_0)\frac{\partial H_u}{\partial \nu} dS.$$

By the definition we can take a bounded domain Ω_0 such that $G\supset\Omega_0\supset E$ and $H_u=u_1^*-u$ in Ω_0-E. Since $u\in\widetilde{HD}^2(G-E;\ E)$, by Lemma 3.2 we have

$$\int_{\partial E} \tilde{u}_0 \frac{\partial u}{\partial \nu} dS = 0.$$

Hence,

$$(5.3)\qquad \int_{\partial E} \tilde{u}_0 \frac{\partial H_u}{\partial \nu} dS = \int_{\partial E} \tilde{u}_0 \frac{\partial u_1^*}{\partial \nu} dS.$$

On the other hand, Green's formula gives

$$\int_{\alpha_1-\alpha_0-\partial E} \left(u_0 \frac{\partial H_u}{\partial \nu} - H_u \frac{\partial u_0}{\partial \nu}\right) dS = 0.$$

Since $u_0\in\widetilde{HD}^2(G-E;\ E)$, $\int_{\partial E} H_u(\partial u_0/\partial \nu)dS=0$ by Lemma 3.2. Hence,

$$\int_{\partial E} u_0 \frac{\partial H_u}{\partial \nu} dS = \int_{\alpha_1} \frac{\partial H_u}{\partial \nu} dS + \int_{\alpha_0-\alpha_1} H_u \frac{\partial u_0}{\partial \nu} dS.$$

Since α_1 is homologous in E^c to some β consisting of a finite number of C^1-

surfaces in $\Omega_0 - E$,

$$\int_{\alpha_1} \frac{\partial H_u}{\partial \nu} dS = \int_\beta \frac{\partial H_u}{\partial \nu} dS = \int_\beta \frac{\partial u_1^*}{\partial \nu} dS - \int_\beta \frac{\partial u}{\partial \nu} dS.$$

Since u_1^* is harmonic on Ω_0 and u belongs to $\widetilde{HD}^2(G-E; E)$, it follows that

$$\int_{\alpha_1} \frac{\partial H_u}{\partial \nu} dS = 0.$$

Thus,

$$\int_{\partial E} u_0 \frac{\partial H_u}{\partial \nu} dS = \int_{\alpha_0 - \alpha_1} H_u \frac{\partial u_0}{\partial \nu} dS. \tag{5.4}$$

By (5.2), (5.3) and (5.4), we obtain (5.1).

PROPOSITION 5.3. *Let G be a bounded domain containing E and u be a function in $\widetilde{HD}^2(G-E; E)$. If $C_2(\alpha_0, \alpha_1; R-E) = C_2(\alpha_0, \alpha_1; R)$ for every ring domain R of the form $R = \{x;\ r_0 < |x-x^0| < r_1\}$ with $R \supset \bar{G}$, where $\alpha_i = \{x;\ |x-x^0| = r_i\}$ $(i=0, 1)$, then u can be extended to a function in $HD^2(G)$.*

PROOF. Let $\tilde{u}_0$ and u_0 be the same as in Lemma 5.2. By assumption we see that $\tilde{u}_0 = u_0$ in $R-E$ and $V(E) = 0$. It is known that $\tilde{u}_0$ is given by

$$\tilde{u}_0(x) = (|x-x^0|^{2-N} - r_0^{2-N})/(r_1^{2-N} - r_0^{2-N}). \tag{5.5}$$

By Lemma 5.2 we have

$$\int_{\alpha_0 - \alpha_1} H_u \frac{\partial \tilde{u}_0}{\partial \nu} dS = \int_{\partial E} \tilde{u}_0 \frac{\partial u_1^*}{\partial \nu} dS.$$

Since $\tilde{u}_0$ and u_1^* are harmonic on some neighborhood of E and $V(E) = 0$, we see that

$$\int_{\partial E} \tilde{u}_0 \frac{\partial u_1^*}{\partial \nu} dS = \int_E (\nabla \tilde{u}_0, \nabla u_1^*) dx = 0.$$

Thus we have

$$\int_{\alpha_0} H_u \frac{\partial \tilde{u}_0}{\partial \nu} dS = \int_{\alpha_1} H_u \frac{\partial \tilde{u}_0}{\partial \nu} dS. \tag{5.6}$$

Since

$$\frac{\partial \tilde{u}_0}{\partial \nu} = (2-N) r_0^{1-N} (r_1^{2-N} - r_0^{2-N})^{-1} \qquad \text{on} \quad \alpha_0$$

and H_u is harmonic on $\{x;\ |x-x^0| \leqq r_0\}$, by the mean value property we have

$$(5.7) \qquad \int_{\alpha_0} H_u \frac{\partial \tilde{u}_0}{\partial \nu} dS = (2-N)\sigma(r_1^{2-N} - r_0^{2-N})^{-1}\left\{\frac{1}{\sigma r_0^{N-1}}\int_{\alpha_0} H_u dS\right\}$$
$$= (2-N)\sigma(r_1^{2-N} - r_0^{2-N})^{-1} H_u(x^0).$$

On the other hand, by a straightforward computation we have

$$\int_{A_1} |\nabla \tilde{u}_0|^2 dx = (N-2)\sigma r_1^{2-N}(r_1^{2-N} - r_0^{2-N})^{-2},$$

where $A_1 = \{x;\ |x-x^0| \geqq r_1\}$. Using Green's formula and the Schwarz inequality, we obtain

$$\left|\int_{\alpha_1} H_u \frac{\partial \tilde{u}_0}{\partial \nu} dS\right| = \left|\int_{A_1} (\nabla H_u, \nabla \tilde{u}_0) dx\right|$$
$$\leqq \left(\int_{A_1} |\nabla H_u|^2 dx\right)^{1/2}\left(\int_{A_1} |\nabla \tilde{u}_0|^2 dx\right)^{1/2}$$
$$\leqq c((N-2)\sigma r_1^{2-N})^{1/2} |r_1^{2-N} - r_0^{2-N}|^{-1},$$

where $c = \left(\int_{E^c} |\nabla H_u|^2 dx\right)^{1/2} < \infty$. By (5.6) and (5.7) we have

$$|H_u(x^0)| \leqq c((N-2)\sigma r_1^{N-2})^{-1/2}.$$

Since this is valid for arbitrarily large r_1, $H_u(x^0) = 0$. Letting x^0 vary in some open set in $(\bar{G})^c$, we conclude that $H_u = 0$ in E^c. Therefore $u = u_1^*$ in $\Omega_0 - E$ for some domain Ω_0 containing E. Thus we conclude that u can be extended to a function in $HD^2(G)$.

PROPOSITION 5.4. *If the equalities*

$$C_2(\alpha_0^i, \alpha_1^i; \Omega - E) = C_2(\alpha_0^i, \alpha_1^i; \Omega), \qquad i = 1, \dots, N,$$

hold for some open rectangle Ω containing E with sides parallel to the coordinate axes, where α_0^i and α_1^i are the sides of Ω parallel to the coordinate plane $x_i = 0$, then $\tilde{S}(x^0, x^1)$ is equal to zero for all distinct two points x^0, x^1 in $(\bar{\Omega})^c$.

PROOF. Let $\tilde{u}^i$ and u^i be extremal functions for $C_2(\alpha_0^i, \alpha_1^i; \Omega)$ and $C_2(\alpha_0^i, \alpha_1^i; \Omega - E)$ respectively. It is well known that $\tilde{u}^i$ is of the form $ax_i + b$. From assumption it follows that $V(E) = 0$ and for each i, $i = 1, \dots, N$, $u^i = a_i x_i + b_i$ in $\Omega - E$. Denote by $\mathscr{A}^i$ the class of all 2-precise functions v on $\Omega - E$ such that $v(\gamma) = 0$ for 2-a.e. $\gamma \in \tilde{\Gamma}_\Omega(\alpha_0^i) \cup \tilde{\Gamma}_\Omega(\alpha_1^i)$. By Lemma 2.1, u^i satisfies the variational condition that

$$\int_{\Omega - E} (\nabla u^i, \nabla v) dx = 0$$

for every v in $\mathscr{A}^i$. Hence we derive that

$$\int_{\Omega-E} \frac{\partial v}{\partial x_i} dx = 0 \tag{5.8}$$

for every v in $\mathscr{A}^i$.

Now, we take any two disjoint closed balls B_0, B_1 in $R^N-\bar{\Omega}$. Set $D=R^N-(B_0 \cup B_1)$ and $\alpha_i=\partial B_i$ $(i=0, 1)$. Let u be the extremal function for $C_2(\alpha_0, \alpha_1; D-E)$. We shall show that u can be extended to a 2-precise function on D. Let $\tilde{u}=u$ on $D-E$ and $\tilde{u}=0$ on E; $U_i=(\partial u/\partial x_i)$ on $D-E$ and $U_i=0$ on E, $i=1,\dots, N$. Then $\tilde{u}$ is locally integrable in D and $U_i \in L^2(D)$, $i=1,\dots, N$. For any $\phi \in C_0^\infty(\Omega)$, since $u\phi|_{\Omega-E} \in \cap_{i=1}^N \mathscr{A}^i$, (5.8) implies

$$\int_\Omega U_i \phi dx = \int_{\Omega-E} \frac{\partial u}{\partial x_i} \phi dx = -\int_{\Omega-E} u \frac{\partial \phi}{\partial x_i} dx = -\int_\Omega \tilde{u} \frac{\partial \phi}{\partial x_i} dx.$$

This means that $U_i=(\partial \tilde{u}/\partial x_i)$ on Ω in the distribution sense. By [7, Theorem 4.21], there exists a 2-precise function $\hat{u}$ on D such that $\hat{u}=\tilde{u}$ a.e. on Ω. Obviously, we may take $\hat{u}=u$ on $\Omega-E$.

Next, since $\hat{u} \in \mathscr{D}(\alpha_0, \alpha_1; D)$ and $V(E)=0$, we have

$$C_2(\alpha_0, \alpha_1; D) \leqq \int_D |\nabla \hat{u}|^2 dx = \int_{D-E} |\nabla u|^2 dx = C_2(\alpha_0, \alpha_1; D-E).$$

Since the converse inequality is trivial, we conclude that

$$C_2(\alpha_0, \alpha_1; D) = C_2(\alpha_0, \alpha_1; D-E).$$

As stated in Remark 4.1, $\tilde{S}(x^0, x^1)$ is equal to zero for all distinct two points x^0, x^1 in $R^N-\bar{\Omega}$. The proof is thus completed.

PROOF OF THEOREM 5.1. By Proposition 4.1, (1) implies (2). Clearly (2) implies (3). If (3) is true, then from Corollary 4.1 and Proposition 5.1, (4) follows. Clearly (4) implies (5).

Suppose (5) is valid. Let G be some bounded domain containing E such that every u in $\widetilde{HD}^2(G-E; E)$ can be extended to a function in $HD^2(G)$. For two points x^0, x^1 in $(\bar{G})^c$, let P_0 be the principal function with respect to x^0, x^1 and E^c. It is easy to see that the restriction of P_0 to $G-E$ belongs to $\widetilde{HD}^2(G-E; E)$. Hence, $P_0-\tilde{P}=0$ in E^c, so that (3) holds.

By Proposition 5.2 and Corollary 4.1, (3) and (4) imply (6). By Proposition 5.3, (7) implies (4). Clearly (6) implies (1), (7) and (8). Finally by Proposition 5.4, (3) follows from (8). The proof is completed.

By the equivalence between (1) and (8), we have

COROLLARY 5.1 (cf. [10, Property 3.3]). *Any compact subset of an NED_2-set is an NED_2-set.*

By the equivalence between (1) and (4), we have

COROLLARY 5.2 (cf. [10, Corollary 3.5]). *If $E_1,\dots, E_m$ are mutually disjoint NED_2-sets, then $\cup_{n=1}^{m} E_n$ is an NED_2-set.*

REMARK 5.2. It may occur that the equality $C_2(\alpha_0, \alpha_1; D-E)=C_2(\alpha_0, \alpha_1; D)$ holds for some disjoint closed balls B_0, B_1 in E^c even though E is not an NED_2-set. For example, let R^{N-1} be the hyperplane $\{x\in R^N;\ x_N=0\}$ and let $E=\{x\in R^{N-1};\ |x|\leqq 1\}$. It is easy to see that E is not an NED_2-set. If we let $B_0=\{x;\ |x-x^0|\leqq 1\}$ and $B_1=\{x;\ |x-x^1|\leqq 1\}$, where $x^0=(3, 0,\dots, 0)$ and $x^1=(-3, 0,\dots, 0)$, then $C_2(\alpha_0, \alpha_1; D-E)=C_2(\alpha_0, \alpha_1; D)$, since the extremal function for $C_2(\alpha_0, \alpha_1; D)$ is symmetric with respect to R^{N-1}, and hence it is extremal for $C_2(\alpha_0, \alpha_1; D-E)$.

Similarly, if we consider the ring domain $R=\{x;\ 1<|x-x^0|<5\}$, then $C_2(\alpha_0, \alpha_1; R-E)=C_2(\alpha_0, \alpha_1; R)$ for the above E. Thus the property that $C_2(\alpha_0, \alpha_1; R-E)=C_2(\alpha_0, \alpha_1; R)$ for some ring domain $R\supset E$ does not imply that E is an NED_2-set.

This example also shows that the converse of Corollary 4.3 is not always true, that is, there exists a compact set E with $V(E)=0$ which is not an NED_2-set.

REMARK 5.3. We can easily see that if the $(N-1)$-dimensional Hausdorff measure of E is zero, then E is an NED_2-set (cf. [9, Theorem 2] and [7, Theorem 2.13]). Now we show that the converse is not always true.

Let E be an N-dimensional symmetric generalized Cantor set such that the $(N-1)$-dimensional Hausdorff measure of E is infinite and the projection of E on each of the coordinate axes has one-dimensional measure zero (see [4, p. 375]). Let Ω be an N-dimensional open rectangle containing E with sides parallel to the coordinate axes and α_0^i, α_1^i be the sides parallel to the coordinate plane $x_i=0$. Since the projection of E on α_0^i $(i=1,\dots, N)$ has the $(N-1)$-dimensional Lebesgue measure zero, we can show that

$$M_2(\Gamma(\alpha_0^i, \alpha_1^i; \Omega-E)) = M_2(\Gamma(\alpha_0^i, \alpha_1^i; \Omega))\ (i = 1,\dots, N).$$

By Theorem 5.1 ((8)⇒(1)), we see that E is an NED_2-set.

§6. NED_2^0-sets and removable sets for KD^2

The main purpose of this section is to prove that the class of NED_2^0-sets is identical with the class of removable sets for KD^2.

THEOREM 6.1. *For a compact set E in R^N, the following statements are* *equivalent to each other:*

(1) *E is an NED_2^Q-set;*

(2) *$\tilde{S}^Q(x^0, x^1)$ is equal to zero for all distinct two points x^0, x^1 in E^c;*

(3) *$\tilde{S}^Q(x^0, x^1)$ is equal to zero for all x^0 in a non-empty open set in E^c and some x^1 in E^c;*

(4) *For any bounded domain G containing E, every u in $\widetilde{KD}^2(G-E; E)$ can be extended to a function in $HD^2(G)$;*

(5) *E is removable for $\widetilde{KD}^2$;*

(6) *The equality $C_2(\alpha_0, \alpha_1; G)=C_2^{**}(\alpha_0, \alpha_1; G-E, \tilde{\beta}_Q)$ holds for every domain G containing E and every mutually disjoint compact subsets α_0, α_1 of ∂G;*

(7) *The equality $C_2(\alpha_0, \alpha_1; R)=C_2^{**}(\alpha_0, \alpha_1; R-E, \tilde{\beta}_Q)$ holds for every ring domain R containing E, where α_0 and α_1 are two boundary components of R;*

(8) *The equalities $C_2(\alpha_0^i, \alpha_1^i; \Omega)=C_2^{**}(\alpha_0^i, \alpha_1^i; \Omega-E, \tilde{\beta}_Q)$, $i=1,\dots, N$, hold for some N-dimensional open rectangle $\Omega\supset E$ with sides parallel to the coordinate axes, where α_0^i and α_1^i are the sides of Ω parallel to the coordinate plane $x_i=0$;*

(9) *E is removable for KD^2.*

To prove this theorem we prepare some propositions.

PROPOSITION 6.1. *Let G be a domain containing E and u be a function in $\widetilde{KD}^2(G-E; E)$. If $\tilde{P}-P_1^Q=0$ in E^c for any x^0 in a non-empty open set Ω in E^c and some x^1 in E^c, then u can be extended to a function in $HD^2(G)$.*

PROOF. We may assume that the distance between Ω and E is positive. Let H_u be the harmonic function defined in §3. Let $x^0\in\Omega$, $x^1\in E^c$ and P_1^Q be the principal function with respect to x^0, x^1 and E^c. As in the proof of Lemma 4.2, we have

$$\int_{E^c}(\nabla H_u, \nabla(\tilde{P}-P_1^Q))dx \tag{6.1}$$

$$= (N-2)(H_u(x^1)-H_u(x^0)) + \int_{\partial E}\left(H_u\frac{\partial P_1^Q}{\partial \nu} - \tilde{P}\frac{\partial H_u}{\partial \nu}\right)dS.$$

By assumption, (6.1) implies

$$(N-2)(H_u(x^0)-H_u(x^1)) = \int_{\partial E}\left(H_u\frac{\partial P_1^Q}{\partial \nu} - P_1^Q\frac{\partial H_u}{\partial \nu}\right)dS. \tag{6.2}$$

Since the restriction of H_u to $G-E$ belongs to $KD^2(G-E; E)$, by Lemma 4.1 (b) we have

$$\int_{\partial E}P_1^Q\frac{\partial H_u}{\partial \nu}dS = 0.$$

By the definition, $H_u = u_1^* - u$ in $\Omega_0 - E$ for some domain Ω_0 containing E. Therefore,

$$\int_{\partial E} H_u \frac{\partial P_1^Q}{\partial \nu} dS = \int_{\partial E} u_1^* \frac{\partial \tilde{P}}{\partial \nu} dS - \int_{\partial E} u \frac{\partial P_1^Q}{\partial \nu} dS.$$

Since u_1^* and $\tilde{P}$ are harmonic on Ω_0 and $V(E)=0$ by Corollary 4.2, we see that

$$\int_{\partial E} u_1^* \frac{\partial \tilde{P}}{\partial \nu} dS = \int_E (\nabla \tilde{P}, \nabla u_1^*) dx = 0.$$

Take a bounded domain G_0 such that $E \subset G_0 \subset G$ and $\Omega \cup \{x^1\} \subset (\bar{G}_0)^c$. Since the restriction of P_1^Q (resp. u) to $G_0 - E$ belongs to $KD^2(G_0 - E; E)$ (resp. $\widetilde{KD}^2(G_0 - E; E)$),

$$\int_{\partial E} u \frac{\partial P_1^Q}{\partial \nu} dS = 0.$$

Thus (6.2) implies that $H_u(x^0) = H_u(x^1)$. Letting x^0 vary in Ω, we conclude that $H_u = \text{const.}$ in E^c. This implies that u can be extended to a function in $HD^2(G)$.

PROPOSITION 6.2. *Let G be a domain containing E and let α_0, α_1 be disjoint compact subsets of ∂G. Let u_Q be an extremal function for $C_2^{**}(\alpha_0, \alpha_1; G-E, \tilde{\beta}_Q)$. If $V(E)=0$ and u_Q can be extended to a harmonic function $\tilde{u}_Q$ on G, then the equality*

$$C_2^{**}(\alpha_0, \alpha_1; G-E, \tilde{\beta}_Q) = C_2(\alpha_0, \alpha_1; G)$$

holds.

PROOF. We may assume that $M_2(\tilde{\Gamma}_G(\alpha_0) \cup \tilde{\Gamma}_G(\alpha_1)) > 0$, for otherwise, the constant 0 is extremal for $C_2^{**}(\alpha_0, \alpha_1; G-E, \tilde{\beta}_Q)$ so that the assertion is trivial. Let $\tilde{u}$ be the extremal function for $C_2(\alpha_0, \alpha_1; G)$. Let ψ be a function of $C^\infty(G)$ such that $\psi=0$ on some neighborhood of E and $\psi=1$ outside some compact set contained in G. By Lemma 2.2 we have

$$\int_G (\nabla \tilde{u}_Q, \nabla[\psi(\tilde{u}_Q - \tilde{u})]) dx = \int_{G-E} (\nabla u_Q, \nabla[\psi(\tilde{u}_Q - \tilde{u})]) dx = 0.$$

Since $(1-\psi)(\tilde{u}_Q - \tilde{u}) \in C_0^\infty(G)$ and $\tilde{u}_Q$ is harmonic on G,

$$\int_G (\nabla \tilde{u}_Q, \nabla[(1-\psi)(\tilde{u}_Q - \tilde{u})]) dx = 0.$$

Hence,

$$\int_G (\nabla \tilde{u}_Q, \nabla(\tilde{u}_Q - \tilde{u})) dx = 0.$$

On the other hand, by Lemma 2.1 we have

$$\int_G (\nabla \tilde{u}, \nabla(\tilde{u}_Q - \tilde{u}))dx = 0.$$

Therefore

$$\int_G |\nabla(\tilde{u}_Q - \tilde{u})|^2 dx = 0.$$

It follows that $\tilde{u}_Q = \tilde{u}$ and hence $\tilde{u}_Q$ is extremal for $C_2(\alpha_0, \alpha_1; G)$. Since $V(E)=0$, we have

$$C_2(\alpha_0, \alpha_1; G) = \int_G |\nabla \tilde{u}_Q|^2 dx = \int_{G-E} |\nabla u_Q|^2 dx = C_2^{**}(\alpha_0, \alpha_1; G-E, \tilde{\beta}_Q).$$

Let Ω be an N-dimensional open rectangle with sides parallel to the coordinate axes, E be a compact set in Ω (possibly an empty set) and G be a bounded domain containing $\bar{\Omega}$. We set

$$M_2^i(\Omega - E) = \inf_\psi \int_{\Omega-E} |\nabla \psi|^2 dx \qquad (i = 1,\dots, N),$$

where the infimum is taken over all $\psi \in C_0^\infty(G)$ such that $\nabla\psi$ vanishes on some neighborhood of E, $\psi(x)=0$ on α_0^i which is one of the sides of Ω parallel to the coordinate plane $x_i=0$, and $\psi(x)=1$ on α_1^i which is the opposite side of α_0^i.

PROPOSITION 6.3 ([5, Theorem 4] and [11, Theorem 11]). *A compact set E is removable for KD^2 if and only if the equalities $M_2^i(\Omega - E) = M_2^i(\Omega)$, $i=1,\dots, N$, hold for some open rectangle $\Omega \supset E$.*

Now we shall show

PROPOSITION 6.4. *With the same notation as above, if $\Omega \supset E$, then $M_2^i(\Omega - E) = C_2^{**}(\alpha_0^i, \alpha_1^i; \Omega - E, \tilde{\beta}_Q)$ for every $i=1,\dots, N$.*

PROOF. We denote by $\mathscr{D}^{**}$ the family of all $u \in \mathscr{D}(\alpha_0^i, \alpha_1^i; \Omega)$ such that $u=$const. on each component of some neighborhood of E. We observe that

$$C_2^{**}(\alpha_0^i, \alpha_1^i; \Omega - E, \tilde{\beta}_Q) = \inf\left\{\int_{\Omega-E} |\nabla u|^2 dx;\ u \in \mathscr{D}^{**}\right\}.$$

Denote by $\hat{\mathscr{D}}$ the family of all $u \in \mathscr{D}(\alpha_0^i \cup \partial G, \alpha_1^i; G - \alpha_0^i - \alpha_1^i)$ such that $u=$ const. on each component of some neighborhood of E. As in the proof of [11, Theorem 10] we have

$$M_2^i(\Omega - E) = \inf\left\{\int_{\Omega-E} |\nabla u|^2 dx;\ u \in \hat{\mathscr{D}}\right\}.$$

Since the restriction of $u \in \hat{\mathscr{D}}$ to $\Omega - E$ belongs to $\mathscr{D}^{**}$, $C_2^{**}(\alpha_0^i, \alpha_1^i; \Omega - E, \tilde{\beta}_Q) \leqq M_2^i(\Omega - E)$. On the other hand, for each $u \in \mathscr{D}^{**}$, there exists a 2-precise function $\tilde{u}$ in R^N such that $u = \tilde{u}$ in Ω (see [7, Theorem 5.8]). Let Ω_0 and Ω_1 be bounded domains such that $\Omega \subset \bar{\Omega} \subset \Omega_0 \subset \bar{\Omega}_0 \subset \Omega_1 \subset \bar{\Omega}_1 \subset G$ and each of $\partial\Omega_0$ and $\partial\Omega_1$ consists of one compact C^1-surface. Take a function $\phi \in C^\infty(G)$ such that $\phi = 1$ on $\bar{\Omega}_0$ and $\phi = 0$ on $G - \Omega_1$. It is easy to see that $\phi\tilde{u}$ belongs to $\hat{\mathscr{D}}$. Therefore,

$$M_2^i(\Omega - E) \leqq \int_{\Omega - E} |\nabla(\phi\tilde{u})|^2 dx = \int_{\Omega - E} |\nabla u|^2 dx.$$

Since this is valid for any $u \in \mathscr{D}^{**}$, we have $M_2^i(\Omega - E) \leqq C_2^{**}(\alpha_0^i, \alpha_1^i; \Omega - E, \tilde{\beta}_Q)$. Thus we obtain the required equality.

LEMMA 6.1. *Let $R = \{x; r_0 < |x - x^0| < r_1\}$, $\alpha_i = \{x; |x - x^0| = r_i\}$ $(i = 0, 1)$ and E be a compact subset of R. Let $\tilde{u}$ and u_Q be extremal functions for $C_2(\alpha_0, \alpha_1; R)$ and $C_2^{**}(\alpha_0, \alpha_1; R - E, \tilde{\beta}_Q)$ respectively. If $C_2(\alpha_0, \alpha_1; R) = C_2^{**}(\alpha_0, \alpha_1; R - E, \tilde{\beta}_Q)$, then $\tilde{u} = u_Q$ in $R - E$ and $V(E) = 0$.*

PROOF. Note that $\tilde{u}$ is harmonic on R and u_Q belongs to $\widetilde{KD}^2(R - E; E)$ by Proposition 3.1. By using Green's formula we have

$$C_2^{**}(\alpha_0, \alpha_1; R - E, \tilde{\beta}_Q) - C_2(\alpha_0, \alpha_1; R)$$
$$= \int_{R - E} |\nabla(u_Q - \tilde{u})|^2 dx + \int_E |\nabla\tilde{u}|^2 dx.$$

From assumption it follows that

$$\int_{R - E} |\nabla(u_Q - \tilde{u})|^2 dx = 0 \tag{6.3}$$

and

$$\int_E |\nabla\tilde{u}|^2 dx = 0. \tag{6.4}$$

Therefore, $\tilde{u} = u_Q$ in $R - E$ by (6.3). On the other hand, $\tilde{u}$ is given by the right-hand side of (5.5). Hence, $|\nabla\tilde{u}| \neq 0$ on R. Thus we conclude that $V(E) = 0$ by (6.4).

By the method similar to the proof of Lemma 5.2, we can show

LEMMA 6.2. *Let G be a bounded domain containing E and let R be a ring domain of the form $R = \{x; r_0 < |x - x^0| < r_1\}$ with $R \supset \bar{G}$. Let $\tilde{u}$ and u_Q be extremal functions for $C_2(\alpha_0, \alpha_1; R)$ and $C_2^{**}(\alpha_0, \alpha_1; R - E, \tilde{\beta}_Q)$ respectively, where $\alpha_i = \{x; |x - x^0| = r_i\}$ $(i = 0, 1)$. Then*

$$\int_{R-E}(\nabla H_u, \nabla(\tilde{u}-u_Q))dx = \int_{\alpha_1-\alpha_0} H_u\frac{\partial \tilde{u}}{\partial \nu}dS - \int_{\partial E} u_1^*\frac{\partial \tilde{u}}{\partial \nu}dS$$

for any u in $\widetilde{KD}^2(G-E; E)$, where H_u and u_1^* are harmonic functions defined in §3.

PROPOSITION 6.5. *Let G be a bounded domain containing E and u be a function in $\widetilde{KD}^2(G-E; E)$. If $C_2(\alpha_0, \alpha_1; R)=C_2^{**}(\alpha_0, \alpha_1; R-E, \tilde{\beta}_Q)$ for every ring domain R of the form $R=\{x; r_0<|x-x^0|<r_1\}$ with $R\supset \bar{G}$, where $\alpha_i=\{x; |x-x^0|=r_i\}$ $(i=0, 1)$, then u can be extended to a function in $HD^2(G)$.*

PROOF. Let $\tilde{u}$ and u_Q be the same as in Lemma 6.2. By Lemma 6.1, $\tilde{u}=u_Q$ in $R-E$ and $V(E)=0$. For u in $\widetilde{KD}^2(G-E; E)$, applying Lemma 6.2 we have

$$\int_{\alpha_1-\alpha_0} H_u\frac{\partial \tilde{u}}{\partial \nu}dS = \int_{\partial E} u_1^*\frac{\partial \tilde{u}}{\partial \nu}dS.$$

Since u_1^* and $\tilde{u}$ are harmonic on some neighborhood of E and $V(E)=0$, we see that

$$\int_{\partial E} u_1^*\frac{\partial \tilde{u}}{\partial \nu}dS = \int_E(\nabla u_1^*, \nabla \tilde{u})dx = 0.$$

Hence

$$\int_{\alpha_0} H_u\frac{\partial \tilde{u}}{\partial \nu}dS = \int_{\alpha_1} H_u\frac{\partial \tilde{u}}{\partial \nu}dS.$$

As in the proof of Proposition 5.3, we see that $H_u=0$ in E^c. Therefore, u can be extended to a function in $HD^2(G)$.

PROOF OF THEOREM 6.1. By Proposition 4.1, (1) implies (2). Clearly (2) implies (3). If (3) is true, then from Corollary 4.2 and Proposition 6.1, (4) follows. Clearly (4) implies (5).

Suppose (5) is valid. Let G be some bounded domain containing E such that every u in $\widetilde{KD}^2(G-E; E)$ can be extended to a function in $HD^2(G)$. For two points x^0, x^1 in $(\bar{G})^c$, let P_1^Q be the principal function with respect to x^0, x^1 and E^c. Since the restriction of P_1^Q to $G-E$ belongs to $\widetilde{KD}^2(G-E; E)$, we have $P_1^Q-\tilde{P}=0$ in E^c. Hence we have (3) by Corollary 4.2. By Propositions 3.1 and 6.2, (4) implies (6).

Clearly (6) implies (7) and (8). By Proposition 6.5, we see that (7) implies (4). Suppose (8) is valid. Since the equality $M_2^i(\Omega)=C_2(\alpha_0^i, \alpha_1^i; \Omega)$ holds, from Propositions 6.3 and 6.4 it follows that E is removable for KD^2. Thus (9) follows. If (9) is true, then from Lemma 3.1, (1) follows. The proof is completed.

REMARK 6.1. Let R be a ring domain containing E. We shall show that if R and E are suitably chosen, then $C_2(\alpha_0, \alpha_1; R)=C_2^{**}(\alpha_0, \alpha_1; R-E, \tilde{\beta}_Q)$ even if E is not removable for KD^2. Let $R=\{x; 1<|x|<2\}$, $\alpha_0=\{x; |x|=1\}$ and $\alpha_1=\{x; |x|=2\}$. Set $a=(3/2, 0,\dots, 0)$ and

$$E = \{x; |x-a| \leqq 1/2, \quad |x| = 3/2\}.$$

Obviously, $R\supset E$. The extremal function $\tilde{u}$ for $C_2(\alpha_0, \alpha_1; R)$ is given by $\tilde{u}(x)=(|x|^{2-N}-1)/(2^{2-N}-1)$. It is easy to show that the restriction of $\tilde{u}$ to $R-E$ is extremal for $C_2^{**}(\alpha_0, \alpha_1; R-E, \tilde{\beta}_Q)$. Therefore $C_2(\alpha_0, \alpha_1; R)=C_2^{**}(\alpha_0, \alpha_1; R-E, \tilde{\beta}_Q)$.

Next, we shall show that E is not removable for KD^2. If E is removable for KD^2, then so is for $\widetilde{HD}^2$. By Theorem 5.1 ((5)⇒(7)) $C_2(\alpha_0, \alpha_1; R-E)=C_2(\alpha_0, \alpha_1; R)$. Therefore it is enough to show that $C_2(\alpha_0, \alpha_1; R-E)<C_2(\alpha_0, \alpha_1; R)$. We introduce the polar coordinates $(r, \theta_1,\dots, \theta_{N-1})$ in R^N, that is, $r=|x|$, $x_1=r\cos\theta_1,\dots, x_N=r\sin\theta_1\cdots\sin\theta_{N-2}\sin\theta_{N-1}$, for $x=(x_1,\dots, x_N)$. Let

$$\Omega = \{x; 3/2 < |x| < 2, \ \cos\theta_1 > 17/18\},$$

$$\tilde{\alpha}_0 = \{x; |x| = 2, \cos\theta_1 \geqq 17/18\} \cup \{x; 3/2 \leqq |x| \leqq 2, \cos\theta_1 = 17/18\}$$

and

$$\tilde{\alpha}_1 = \{x; |x-a| \leqq 1/4, \ |x| = 3/2\}.$$

Take $\phi\in C^\infty(\Omega)$ such that $\phi>0$ in Ω, $|\nabla\phi|\in L^2(\Omega)$ and $\lim_{x\to\tilde{x}}\phi(x)=i$ for every $\tilde{x}\in\tilde{\alpha}_i$ $(i=0, 1)$. We extend ϕ by the constant 0 to a 2-precise function on $R-E$. Then we have that

$$\int_{R-E}(\nabla\tilde{u}, \nabla\phi)dx = (2-N)(2^{2-N}-1)^{-1}\int_{|x|=1}\int_1^2\frac{\partial\phi}{\partial r}drdS$$

$$= (N-2)(2^{2-N}-1)^{-1}\int_E\phi dS < 0.$$

From Lemma 2.1, it follows that the restriction of $\tilde{u}$ to $R-E$ is not extremal for $C_2(\alpha_0, \alpha_1; R-E)$. On the other hand, the restriction of $\tilde{u}$ to $R-E$ belongs to $\mathscr{D}(\alpha_0, \alpha_1; R-E)$. Thus we conclude that $C_2(\alpha_0, \alpha_1; R-E)<C_2(\alpha_0, \alpha_1; R)$.

§7. $NED_2^{Q,I}$-sets and removable sets for HD^2

It is well known that a compact set E is removable for HD^2 if and only if the Newtonian capacity of E is equal to zero (see, e.g., [3, §VII, Theorem 1], [5, Theorem 2]). In this section we shall prove that the class of $NED_2^{Q,I}$-sets is identical with the class of removable sets for HD^2.

If any two functions in $HD^2(E^c)$ which differ by a constant are identified, then $HD^2(E^c)$ becomes a Hilbert space with norm $\|u\| = \left(\int_{E^c} |\nabla u|^2 dx\right)^{1/2}$ and inner product $(u, v) = \int_{E^c} (\nabla u, \nabla v) dx$. We note that $KD^2(E^c)$ is a closed linear subspace of $HD^2(E^c)$.

Let Ω be a regular domain in E^c, that is a bounded subdomain of E^c for which $\partial\Omega$ consists of a finite number of compact C^1-surfaces $\beta_1, \dots, \beta_k$ and no component of $E^c - \Omega$ is relatively compact in E^c. Denote by $HM^2(\Omega)$ the class of all u in $HD^2(\Omega)$ such that $u = \text{const.}$ on each β_j $(j = 1, \dots, k)$. Let $HM^2(E^c)$ be the class of all u in $HD^2(E^c)$ with the following property:

For every $\varepsilon > 0$ and every compact set K in E^c there exist a regular domain $\Omega \supset K$ and a function $u_\Omega \in HM^2(\Omega)$ such that $\int_\Omega |\nabla(u - u_\Omega)|^2 dx < \varepsilon$.

The following orthogonal decomposition of the space $HD^2(E^c)$ follows in the same manner as in [2, p. 295].

LEMMA 7.1. $HD^2(E^c) = KD^2(E^c) \oplus HM^2(E^c)$.

The following lemma is proved by using Green's formula and Lemma 4.1.

LEMMA 7.2. *Let P_0 and P_1^I be principal functions with respect to x^0, x^1 and E^c. Let u be a harmonic function in $HD^2(E^c)$. Then*

$$\int_{E^c} (\nabla u, \nabla(P_0 - P_1^I)) dx = (N-2)(u(x^0) - u(x^1)).$$

We shall give some equivalent conditions for E to be removable for HD^2.

THEOREM 7.1. *For a compact set E in R^N, the following statements are equivalent to each other*:

(1) *E is an $NED_2^{Q,I}$-set;*

(2) *$S^{Q,I}(x^0, x^1)$ is equal to zero for all distinct two points x^0, x^1 in E^c;*

(3) *$S^{Q,I}(x^0, x^1)$ is equal to zero for all x^0 in a non-empty open set Ω in E^c and some x^1 in E^c;*

(4) *The equality $HD^2(E^c) = KD^2(E^c)$ holds;*

(5) *For any domain G containing E, the equality $HD^2(G-E) = KD^2(G-E; E)$ holds;*

(6) *E is removable for HD^2;*

(7) *The equality $C_2(\alpha_0, \alpha_1; D-E) = C_2^{**}(\alpha_0, \alpha_1; D-E, \beta_I)$ holds for every two mutually disjoint closed balls B_0, B_1 in E^c, where $\alpha_i = \partial B_i$ $(i = 0, 1)$ and $D = R^N - (B_0 \cup B_1)$.*

PROOF. By Proposition 4.1, (1) implies (2). Clearly (2) implies (3). Suppose (3) is valid. Let $x^0 \in \Omega$ and $x^1 \in E^c$. Let P_0, P_1^Q and P_1^I be the principal functions with respect to x^0, x^1 and E^c. By Lemma 4.2 (c), we have $P_1^Q - P_1^I = 0$ in E^c. For any u in $HM^2(E^c)$, we shall show that $u = \text{const.}$ in E^c. We see that

$$\int_{E^c} (\nabla u, \nabla(P_1^I - P_0))dx + \int_{E^c} (\nabla u, \nabla(P_0 - P_1^Q))dx = 0.$$

From Lemma 7.2 it follows that

$$(N-2)(u(x^0) - u(x^1)) = \int_{E^c} (\nabla u, \nabla(P_0 - P_1^Q))dx.$$

Since $P_0 - P_1^Q \in KD^2(E^c)$ and $HM^2(E^c)$ and $KD^2(E^c)$ are orthogonal to each other,

$$\int_{E^c} (\nabla u, \nabla(P_0 - P_1^Q))dx = 0.$$

Hence, $u(x^0) = u(x^1)$. Letting x^0 vary in Ω, we obtain $u = \text{const.}$ in E^c. Since this is valid for any u in $HM^2(E^c)$, (4) follows from Lemma 7.1.

Suppose (4) is valid. Take any domain G containing E. Let $u \in HD^2(G - E)$. Then the function H_u defined in §3 belongs to $HD^2(E^c)$. Hence $H_u \in KD^2(E^c)$. Therefore $u \in KD^2(G - E; E)$. Thus (4) implies (5). If E is not removable for HD^2, then the Newtonian capacity of E is positive. Let μ be the equilibrium mass-distribution on E and consider the potential

$$\int_E \frac{d\mu(y)}{|x-y|^{N-2}}.$$

For any domain G containing E, this function belongs to $HD^2(G - E)$ but does not belong to $KD^2(G - E; E)$. Thus the inclusion $HD^2(G - E) \supset KD^2(G - E; E)$ is proper. Hence (5) implies (6).

Suppose (6) is valid. Since the Newtonian capacity of E is equal to zero, for any bounded domain G containing E every function in $HD^2(G - E)$ can be extended to a function in $HD^2(G)$. Take two mutually disjoint closed balls B_0, B_1 in E^c. Set $D = R^N - (B_0 \cup B_1)$ and $\alpha_i = \partial B_i$ $(i = 0, 1)$. Let u_0 and u_I be extremal functions for $C_2(\alpha_0, \alpha_1; D - E)$ and $C_2^{**}(\alpha_0, \alpha_1; D - E, \beta_I)$ respectively. Take a bounded domain G with $E \subset G \subset D$. Since the restriction of $u_0 - u_I$ to $G - E$ belongs to $HD^2(G - E)$, there exists a harmonic function h in $HD^2(G)$ such that $u_0 - u_I = h$ in $G - E$. By Lemma 2.1 and Lemma 2.3 (b), we easily see that

$$\int_{\alpha_0 \cup \alpha_1} \frac{\partial u_0}{\partial \nu} dS = \int_{\alpha_0 \cup \alpha_1} \frac{\partial u_I}{\partial \nu} dS = 0.$$

By using Green's formula we have

$$0 \leqq \int_{D-E} |\nabla(u_0 - u_I)|^2 dx$$

$$= -\int_{\alpha_0 \cup \alpha_1 \cup \partial E} (u_0 - u_I + c)\frac{\partial(u_0 - u_I)}{\partial \nu} dS$$

$$= -c\int_{\alpha_0 \cup \alpha_1} \frac{\partial(u_0 - u_I)}{\partial \nu} dS - \int_{\partial E} (h + c)\frac{\partial h}{\partial \nu} dS$$

$$= -\int_E |\nabla h|^2 dx \leqq 0,$$

where c is a constant such that $u_0 - u_I + c$ is regular at infinity. Therefore $u_0 = u_I$ and we obtain (7). Since

$$C_2(\alpha_0, \alpha_1; D-E) \leqq C_2^{**}(\alpha_0, \alpha_1; D-E, \beta_Q) \leqq C_2^{**}(\alpha_0, \alpha_1; D-E, \beta_I),$$

(7) implies (1). The proof is completed.

REMARK 7.1. The results on the equivalence between (1), (4), (6) and (7) are the euclidean space version of Minda's results on Riemann surfaces ([6, Theorem 9 and its corollary]).

REMARK 7.2. Let R be a ring domain $\{x; r_0 < |x - x^0| < r_1\}$ containing E. Let $\alpha_i = \{x; |x - x^0| = r_i\}$ $(i = 0, 1)$. If E is removable for HD^2, then $M_2(\Gamma(\alpha_0, \alpha_1; R-E)) = M_2(\Gamma_I(\alpha_0, \alpha_1; R-E))$ for every ring domain R containing E; however, we do not know whether the converse is true or not.

REMARK 7.3. We denote by $N_{\widetilde{HD}^2}$, N_{KD^2} and N_{HD^2} the classes of removable sets for $\widetilde{HD}^2$, KD^2 and HD^2, respectively. For any bounded domain G containing E, we see that $\widetilde{HD}^2(G-E; E) \subset KD^2(G-E; E) \subset HD^2(G-E)$ so that $N_{\widetilde{HD}^2} \supset N_{KD^2} \supset N_{HD^2}$. We show by examples that $N_{\widetilde{HD}^2} \supsetneqq N_{KD^2}$ and $N_{KD^2} \supsetneqq N_{HD^2}$.

Let $E_{(N)}$ be an N-dimensional symmetric generalized Cantor set such that the Newtonian capacity of $E_{(N)}$ is positive and $V(E_{(N)}) = 0$ (see [3, §IV, Theorems 3 and 4]). By [5, Corollary 2 to Theorem 4] and [11, Theorem 11] we have $E_{(N)} \in N_{KD^2}$. Since $E_{(N)} \notin N_{HD^2}$, the inclusion $N_{KD^2} \supset N_{HD^2}$ is proper.

Let $E_{(N-1)}$ be an $(N-1)$-dimensional symmetric generalized Cantor set such that the Newtonian capacity of $E_{(N-1)} \times \{0\} \subset R^N$ is positive and the $(N-1)$-dimensional Hausdorff measure of $E_{(N-1)}$ is zero (see [3, §IV, Theorems 3 and 4], [4, p. 373]). Set $E = E_{(N-1)} \times [0, 1]$. Let Ω be an N-dimensional open rectangle containing E with sides parallel to the coordinate axes and α_0^i, α_1^i be the sides of Ω parallel to the coordinate plane $x_i = 0$. Since the projection of E on α_0^i $(i = 1, \dots, N)$ has the $(N-1)$-dimensional Lebesgue measure zero, we can show that

$$M_2(\Gamma(\alpha_0^i, \alpha_1^i; \Omega - E)) = M_2(\Gamma(\alpha_0^i, \alpha_1^i; \Omega)) \quad (i = 1, \dots, N).$$

By Theorem 5.1 ((8)$\Rightarrow$(5)) we obtain $E \in N_{\widetilde{HD}^2}$.

Next, let $E_i = E_{(N-1)} \times \{i\}$ $(i=0, 1)$. Since the Newtonian capacity of E_i is positive, there exists the equilibrium mass-distribution μ_i on E_i $(i=0, 1)$. Consider the function

$$u(x) = \int_{E_0} \frac{d\mu_0(y)}{|x-y|^{N-2}} - \int_{E_1} \frac{d\mu_1(y)}{|x-y|^{N-2}}.$$

It is easy to show that u belongs to $KD^2(\Omega - E; E)$ but can not be extended to a function in $HD^2(\Omega)$. This implies that $E \notin N_{KD^2}$. Thus the inclusion $N_{\tilde{H}D^2} \supset N_{KD^2}$ is proper.

Thus in the N-dimensional space R^N $(N \geqq 3)$, the classes of NED_2-sets, NED_2^Q-sets and $NED_2^{Q,I}$-sets are actually different.

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