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The order of the canonical elememt in the J-group of the lens space

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§1. Statement of the result

The standard lens space mod m is the orbit manifold

$$L^n(m) = S^{2n+1}/Z_m \qquad (Z_m = \{z \in S^1: z^m = 1\})$$

of the $(2n+1)$-sphere $S^{2n+1}(\subset C^{n+1})$ by the diagonal action $z(z_0,\dots, z_n)=(zz_0, \dots, zz_n)$. Let η be the *canonical complex line bundle over* $L^n(m)$, i.e., the induced bundle of the canonical complex line bundle over the complex projective space $CP^n = S^{2n+1}/S^1$ by the natural projection $L^n(m)\to CP^n$.

Then, the purpose of this note is to prove the following

THEOREM 1.1. *Let p be an odd prime and r a positive integer. Then, the order of the J-image*

$$J(r\eta - 2) \in \tilde{J}(L^n(p^r))$$

of the stable class of the real restriction $r\eta$ of the canonical line bundle η is equal to

$$p^{f(n,r)}, \quad f(n,r) = \max\{s + [n/p^s(p-1)]p^s: 0 \leqq s < r \text{ and } p^s(p-1) \leqq n\},$$

where $f(n,r)=\max \emptyset = 0$ if $n<p-1$.

We notice that the above theorem is valid also for the case $p=2$ and $r\geqq 2$, by the result in the forthcoming paper [2].

It is proved by J. F. Adams [1] and D. Quillen [4] that

$$J(X) \cong KO(X)/\textstyle\sum_k (\cap_e k^e(\Psi^k - 1)KO(X))$$

(X: finite dimensional CW-complex) where Ψ^k is the Adams operation. Based on this result, we prove the theorem in §2 and study more generally the order of $Jr(\eta^i - 1)$ $(i\geqq 1)$ in §3, by using the partial results obtained in [3].

§2. Proof of Theorem 1.1

Let p be an odd prime. Consider the $2n$-skeleton

$$L_0^n(p^r) = \{[z_0, \ldots, z_n] \in L^n(p^r) : z_n \text{ is real} \geqq 0\}$$

of a CW-complex $L^n(p^r)$. Denote the restriction of the canonical line bundle η on $L_0^n(p^r)$ by the same letter η, and the stable class of η by

(2.1) $$\sigma = \eta - 1 \in \tilde{K}(L^n(p^r)) \quad \text{or} \quad \tilde{K}(L_0^n(p^r)).$$

Then, we have the following (2.2–5) in [3; Prop. 1.3, Prop. 6.3]:

(2.2) $$\tilde{J}(L^n(p^r)) \cong \begin{cases} \tilde{J}(L_0^n(p^r)) \oplus Z_2 & \textit{if} \quad n \equiv 0 \bmod 4, \\ \tilde{J}(L_0^n(p^r)) & \textit{otherwise}, \end{cases}$$

by the induced homomorphism of the inclusion.

(2.3) $$-Jr\sigma = Jr(\sigma^{p-1}) \quad \textit{in} \quad \tilde{J}(L_0^n(p^r)),$$

($K(X) \xrightarrow{r} KO(X) \xrightarrow{J} J(X)$ are the real restriction and the J-homomorphism).

(2.4) Consider the induced homomorphism

$$i^*: \tilde{J}(L_0^n(p^r)) \longrightarrow \tilde{J}(L_0^{n-1}(p^r))$$

of the natural inclusion i given by $C^n = C^n \times \{0\} \subset C^{n+1}$.

(i) *If* $n \not\equiv 0 \bmod p-1$, *then* i^* *is an isomorphism.*

(ii) *If* $n = ap^s(p-1)$ *and* $(a,p)=1$, *then* i^* *is epimorphic and* Ker i^* *is the cyclic subgroup of order* $p^{\min\{r,s+1\}}$ *generated by* $Jr(\sigma^n)$.

(2.5) *The order of* $\tilde{J}(L_0^n(p^r))$ *is equal to* p^ν, $\nu = \sum_{s=0}^{r-1} [n/p^s(p-1)]$.

Now, let $f(n,r)$ be the non-negative integer such that

(2.6) $$\# Jr\sigma = p^{f(n,r)} \quad \text{in} \quad \tilde{J}(L_0^n(p^r)) \qquad (n \geqq 0,\ r \geqq 1)$$

by (2.5), where $\#\alpha$ denotes the order of α.

Then, we can prove Theorem 1.1 by the following lemmas.

LEMMA 2.7. (i) $\# Jr\sigma = p^{f(n,r)}$ *in* $\tilde{J}(L^n(p^r))$.

(ii) $$f(n,r) = f((p-1)[n/(p-1)],r).$$

PROOF. We notice that $i^*\eta = \eta$ and hence $i^*\sigma = \sigma$ by (2.1) for the inclusion i: $L_0^n(p^r) \subset L^n(p^r)$ or $L^n(p^r) \subset L_0^{n+1}(p^r)$. Then, (i) follows immediately from (2.2) and (2.5) since p is an odd prime, and (ii) from (2.4) (i). q.e.d.

LEMMA 2.8. *If* $n = (p-1)l$ *and* $l = mp^{r-1}$, *then*

$$f(0,r) = 0, \quad f(n,r) = r - 1 + l \qquad \textit{for} \quad n > 0.$$

PROOF. The first equality is trivial since $L_0^0(p^r)$ consists of one point.

Assume $n>0$. Then $\sharp Jr(\sigma^n)=p^r$ by (2.4) (ii) and the assumption. On the other hand, we have the equality

$$p^{r-1}\sigma^n = (-1)^{l-1}p^{r-1+l-1}\sigma^{p-1} \quad \text{in} \quad \tilde{K}(L_0^n(p^r)),$$

by [3; Lemma 3.5]. Thus, we see the lemma by (2.3). q.e.d.

LEMMA 2.9. *If* $n=(p-1)l$, $l\neq mp^{r-1}$ *and* $r\geqq 2$, *then*

$$f(n,r) = \max\{f(n-p+1,r), f(n,r-1)\}.$$

PROOF. Consider the commutative diagram

$$\begin{array}{ccc} \operatorname{Ker} i^* \subset \tilde{J}(L_0^n(p^r)) & \xrightarrow{i^*} & \tilde{J}(L_0^{n-p+1}(p^r)) \\ \downarrow{\pi^*} & & \downarrow{\pi'^*} \\ \operatorname{Ker} i'^* \subset \tilde{J}(L_0^n(p^{r-1})) & \xrightarrow{i'^*} & \tilde{J}(L_0^{n-p+1}(p^{r-1})) \end{array}$$

of the induced homomorphisms, where i and i' are the inclusions and π and π' are the natural projections induced by the inclusion $Z_{p^{r-1}}\subset Z_{p^r}$.

By the assumption, $n=ap^s(p-1)$ for some a and s with $(a,p)=1$ and $0\leqq s<r-1$. Thus, (2.4) implies that $\operatorname{Ker} i^*$ and $\operatorname{Ker} i'^*$ in the above diagram are both the cyclic groups of order p^{s+1} generated by $Jr(\sigma^n)$. Therefore

(2.10) $$\pi^*\mid \operatorname{Ker} i^*\colon \operatorname{Ker} i^* \cong \operatorname{Ker} i'^*,$$

by noticing that $\pi^*\eta=\eta$ and hence $\pi^*\sigma^n=\sigma^n$.

Since $i^*\sigma=\sigma$ and $\pi^*\sigma=\sigma$, the definition (2.6) implies that

$$f(n,r) \geqq \max\{f(n-p+1, r), f(n,r-1)\}.$$

Moreover, if $f(n,r)>\max\{f(n-p+1,r), f(n,r-1)\}$, then the non-zero element $p^{f(n,r)-1}Jr\sigma$ in $\tilde{J}(L_0^n(p^r))$ is mapped to 0 by i^* and π^*. This contradicts (2.10). Thus we have the lemma. q.e.d.

PROOF OF THEOREM 1.1. By Lemma 2.7, it is sufficient to show that

(2.11) $$f(n, r) = \max\{s + [l/p^s]p^s\colon 0 \leqq s < r \text{ and } p^s \leqq l\} \quad \text{for} \quad n = (p-1)l.$$

By Lemma 2.8, (2.11) holds if $l=mp^{r-1}$ and especially if $r=1$.

For the case $r\geqq 2$ and $mp^{r-1}<l<(m+1)p^{r-1}$, assume inductively that (2.11) holds for $(n-p+1,r)$ and $(n,r-1)$ instead of (n,r). Then, we see easily that the right hand side of the equality in Lemma 2.9 is equal to

$$\begin{cases} f(n,r-1) & \text{if} \quad m=0, \\ \max\{f(n,r-1),\ r-1+[(l-1)/p^{r-1}]p^{r-1}\} & \text{if} \quad m\geqq 1, \end{cases}$$

and hence to the right hand side of (2.11). Thus Lemma 2.9 implies (2.11) by induction.

These complete the proof of the theorem. q.e.d.

REMARK 2.12. If $[n/(p-1)]=\sum_{i=1}^{k} d_i p^{s_i}$ with $0\leqq s_1<\cdots<s_k$ and $0<d_i<p$ for $1\leqq i\leqq k$, then $f(n,r)$ in Theorem 1.1 is equal to

$$f(n,r) = \max\{\textstyle\sum_{i=j}^{k} d_i p^{s_i} + \min\{s_j, r-1\} : 1 \leqq j \leqq k\}.$$

Furthermore,

$$f(n,r) = \max\{t + \min\{v_p(t), r-1\} : 1 \leqq t \leqq [n/(p-1)]\} \quad (n \geqq p-1),$$

where $v_p(t)$ is the exponent of p in the prime power decomposition of t.

§3. The order of $Jr(\eta^i-1)$

In this section, we prove the following

THEOREM 3.1. *Let p be an odd prime and r a positive integer. Then for any $i\geqq 1$ with exponent $v=v_p(i)$ of p in its prime power decomposition,*

$$\#Jr(\eta^i - 1) = \#Jr(\eta^{p^v} - 1) = p^{f(n,r;v)} \quad in \quad \tilde{J}(L^n(p^r))$$

where the exponent $f(n,r;v)$ is equal to

$$\max\{s-v+[n/p^s(p-1)]p^{s-v} : v \leqq s < r \ and \ p^s(p-1) \leqq n\} \qquad (\max\emptyset = 0).$$

The theorem for $i=1$ is Theorem 1.1.

To prove the theorem, we prepare two lemmas. Set

$$\text{(3.2)} \qquad \sigma(s) = \eta^{p^s} - 1 \in \tilde{K}(L^n(p^r)) = \tilde{K}(L_0^n(p^r)) \qquad \text{for} \quad s \geqq 0,$$

where $\sigma(0)=\sigma$ and $\sigma(s)=0$ if $s\geqq r$ (cf. [3; (1.2)]).

LEMMA 3.3. *The following equalities hold in $\tilde{J}(L_0^n(p^r))$:*

(i) $Jr(\eta^i - 1) = Jr\sigma(v_p(i))$ *for* $i \geqq 1$.

(ii) $Jr((\sigma(0)\cdots\sigma(s))^{p-1}) = -Jr\sigma(s)$ *for* $0 \leqq s < r$.

PROOF. By the proof of [3; Prop. 1.3], we notice that the kernel of Jr: $K(L_0^n(p^r))\to J(L_0^n(p^r))$ is generated additively by the elements

$$\eta^j\sigma(s) \qquad (0 \leqq s < r, \ 1 \leqq j < p^s(p-1)).$$

(i) If $p^s\leqq i<p^{s+1}$, then $\eta^i-1=\eta^j\sigma(s)+\eta^j-1$ where $j=i-p^s$ by (3.2). If $j>0$ in addition, then $Jr(\eta^i-1)=Jr(\eta^j-1)$ by the above notice and $\sigma(s)=0$

($s \geqq r$). By continuing this, we see easily (i).

(ii) In $\{(\sigma(0)\cdots\sigma(s-1))^{p-1}\sigma(s)^{p-2}\}\sigma(s)$, (3.2) shows that $\{\ \}$ is an integral polynomial in η of degree $p^s(p-1)-1$ with constant term -1. Thus, (ii) follows immediately from the above notice. q.e.d.

LEMMA 3.4. *If $n=(p-1)l$ and $l=mp^{r-1}>0$, then for $0\leqq s<r$,*

$$\sharp Jr\sigma(s) = p^{r-s-1+l/p^s} \quad in \quad \tilde{J}(L_0^n(p^r)).$$

PROOF. Set $\sigma'(s)=(\sigma(0)\cdots\sigma(s))^{p-1}$ and $\sigma'(-1)=1$. Then, under the assumption of the lemma, [3; Prop. 3.2 and Lemma 3.5] implies the following equalities in $\tilde{K}(L_0^n(p^r))$:

$$(3.5)\quad \begin{cases} p^{r-s-i}\sigma'(s-1)\sigma(s)^{n(s)+i}=0 & \text{for}\quad 0<i\leqq r-s, \\ p^{r-s+[i/(p-1)]}\sigma'(s-1)\sigma(s)^{n(s)-i}=0 & \text{for}\quad 0\leqq i<n(s); \end{cases}$$

$$(3.6)\quad p^{r-s-1}\sigma'(s-1)\sigma(s)^{n(s)} = (-1)^i p^{r-s-1+i}\sigma'(s-1)\sigma(s)^{n(s)-(p-1)i}$$
$$= -(-1)^{l(s)}p^{r-s-2+l(s)}\sigma'(s) \quad \text{for} \quad 0\leqq i<l(s),$$

where $n(s)=n/p^s$ and $l(s)=l/p^s$. On the other hand, (3.2) implies

$$(3.7)\qquad \sigma(s) = (1+\sigma(t))^p-1 = pA(t)\sigma(t)+\sigma(t)^p \qquad (t=s-1\geqq 0)$$

where $A(t)=\sum_{i=1}^{p-1}\frac{1}{p}\binom{p}{i}\sigma(t)^{i-1}$ is an integral polynomial in $\sigma(t)$ of degree $p-2$ with constant term 1. Therefore, we see for $t=s-1$ that

$$p^{r-s-2+l(s)}\sigma'(s) = -(-1)^{l(s)}p^{r-s-1}\sigma'(t)\sigma(s)^{n(s)} \qquad \text{(by (3.6))}$$

$$= -(-1)^{l(s)}\sum_{i=0}^{n(s)}\binom{n(s)}{i}p^{r-t-2+i}\sigma'(t-1)A(t)^i\sigma(t)^{n(t)-(p-1)(i-1)} \qquad \text{(by (3.7))}$$

$$= -(-1)^{l(t)}\sum_{i=1}^{n(s)}\binom{n(s)}{i}p^{r-t-2+i}\sigma'(t-1)\sigma(t)^{n(t)-(p-1)(i-1)} \qquad \text{(by (3.5))}$$

$$= -\sum_{i=1}^{n(s)}\binom{n(s)}{i}(-1)^i p^{r-t-2+l(t)}\sigma'(t) = p^{r-t-2+l(t)}\sigma'(t) \qquad \text{(by (3.6))};$$

and hence that $p^{r-s-2+l(s)}\sigma'(s)=p^{r-2+l}\sigma'(0)$ in $\tilde{K}(L_0^n(p^r))$.

Now, the last equality, Lemmas 3.3 (ii) and 2.8 imply the lemma. q.e.d.

PROOF OF THEOREM 3.1. Let $f(n,r;s)$ be the non-negative integer with

$$\sharp Jr\sigma(s) = p^{f(n,r;s)} \quad \text{in} \quad \tilde{J}(L_0^n(p^r))$$

by (2.5). Then, by Lemma 3.3 (i) and by noticing the similar result to Lemma 2.7, we see that the theorem follows from the equality

(3.8) $f(n,r;s) = \max\{t-s+[l/p^t]p^{t-s}: s \leqq t < r \text{ and } p^t \leqq l\}$

$$\text{for} \quad n = (p-1)l.$$

For $s=0$, this is (2.11). Let $s \geqq 1$. If $r \leqq s$ or $l=0$, then the both sides of (3.8) are 0, since $\sigma(s)=0$ in $\tilde{K}(L_0^n(p^r))$ $(r \leqq s)$, $L_0^0(p^r)=*$ and $\max \emptyset = 0$. Furthermore (3.8) also holds if $r>s$ and $l=mp^{r-1}>0$, by Lemma 3.4. For the case $r>s \geqq 1$ and $mp^{r-1}<l<(m+1)p^{r-1}$, we can prove the equality

$$f(n,r;s) = \max\{f(n-p+1,r;s), f(n,r-1;s)\} \quad (n=(p-1)l)$$

by the same proof as Lemma 2.9, and hence (3.8) inductively by the same way as the proof of (2.11). Thus, we have obtained (3.8) and Theorem 3.1. q.e.d.

References

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