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Existence of non-tangential limits of solutions of non-linear Laplace equation

Yoshihiro MIZUTA

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Our aim in this note is to study the boundary behavior of (weak) solutions of the non-linear Laplace equation

$$(1)\qquad -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(|\mathrm{grad}\, u|^{p-2}\frac{\partial u}{\partial x_i}\right)=0 \qquad \text{on} \quad \Omega,$$

where Ω is a domain in the n-dimensional Euclidean space R^n.

We say that $\xi\in\partial\Omega$ satisfies the interior cone condition if there is an open truncated cone Γ in Ω with vertex at ξ. Let F be the set of all $\xi\in\partial\Omega$ satisfying the interior cone condition. We can show that F is an F_σ-set*).

A function u on Ω is said to have a non-tangential limit at $\xi\in F$ if for any open truncated cone $\Gamma\subset\Omega$ with vertex at ξ,

$$\lim_{x\to\xi,\, x\in\Gamma'} u(x)$$

exists and is finite whenever Γ' is a cone with vertex at ξ whose closure $\bar{\Gamma}'$ is included in $\Gamma\cup\{\xi\}$.

In this note let $1<p<\infty$ and let $\rho(x)$ denote the distance of x from $R^n-\Omega$.

THEOREM. *Let $1<p\leqq n$ and let u be a function satisfying the following properties:*

i) *u is continuous on Ω;*

ii) *u is p-precise**) on any relatively compact open subset of Ω;*

iii) *u satisfies* (1) *in the weak sense* (*cf.* [4]);

iv) $\int_\Omega |\mathrm{grad}\, u(x)|^p\rho(x)^\alpha dx<\infty$ *for* $\alpha<p$.

Then there exists a set $E\subset\partial\Omega$ such that $B_{1-\alpha/p,p}(E)=0$ and u has a non-tangential limit at each point of $F-E$.

Here $B_{1-\alpha/p,p}$ denotes the Bessel capacity of index $(1-\alpha/p,\ p)$ (see [1]). In case $p=2$, our theorem is shown in [3; Theorem 2′].

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*) This fact was pointed out by Professor Makoto Sakai.

**) For the definition of p-precise functions, see Ziemer [5].

PROOF. Set

$$E' = \left\{\xi \in \partial\Omega; \int_{B(\xi,1)\cap\Omega} |\xi - y|^{1-\alpha/p-n}[|\mathrm{grad}\, u(y)|\rho(y)^{\alpha/p}]dy = \infty\right\},$$

where, in general, $B(\xi, r)$ denotes the open ball with center at ξ and radius r. Let $\xi \in F - E'$ and let Γ, Γ', Γ'' be cones with vertexes at ξ and $\bar{\Gamma}' - \{\xi\} \subset \Gamma'' \subset \bar{\Gamma}'' - \{\xi\} \subset \Gamma \subset \Omega$. Then, since there exists $c > 0$ such that $c|y-\xi| \leqq \rho(y) \leqq |y-\xi|$ for all $y \in \Gamma'$,

$$\int_{\Gamma'} |\xi - y|^{1-n}|\mathrm{grad}\, u(y)|dy < \infty.$$

Hence as in the proof of [3; Lemma 4] we can find a line ℓ such that $\ell \cap \Gamma' \neq \emptyset$ and $\lim_{x\to\xi, x\in\ell} u(x)$ exists and is finite. Denote the limit by a.

On the other hand, we can find c', $0 < c' < 1$, such that $B(x, c'|x-\xi|) \subset \Gamma''$ whenever $x \in \Gamma'$. For $x \in \Gamma'$, we set $r = |x-\xi|$, $\Gamma(r) = \{y \in \Gamma''; |y-\xi| < (1+c')r\}$, and denote by $|\Gamma(r)|$ the n-dimensional measure of $\Gamma(r)$. By [4; Theorems 1 and 2], we have

$$\begin{aligned}
&\left|u(x) - \frac{1}{|\Gamma(r)|}\int_{\Gamma(r)} u(y)dy\right| \\
&\leqq C_1(c'r)^{-n/p}\left\{\int_{B(x,c'r)}\left|u(z) - \frac{1}{|\Gamma(r)|}\int_{\Gamma(r)} u(y)dy\right|^p dz\right\}^{1/p} \\
&\leqq C_2 r^{-n/p}\left[\int_{\Gamma(r)}\left\{\frac{1}{|\Gamma(r)|}\int_{\Gamma(r)} |u(z) - u(y)|dy\right\}^p dz\right]^{1/p} \\
&\leqq C_2 r^{-n(1+1/p)}\left[\int_{\Gamma(r)}\left\{\int_{\Gamma(r)}\left(\int_0^1 |z-y|\,|\mathrm{grad}\, u(y+t(z-y))|dt\right)dy\right\}^p dz\right]^{1/p}.
\end{aligned}$$

By the change of variables and Hölder's inequality, we have

$$\begin{aligned}
&\int_{\Gamma(r)}\left\{\int_0^{1/2} |z - y|\,|\mathrm{grad}\, u(y + t(z-y))|dt\right\}dy \\
&= \int_0^{1/2} (1-t)^{-n-1}\left\{\int_{\Gamma(r)} |z - y|\,|\mathrm{grad}\, u(y)|dy\right\}dt \\
&\leqq C_3 r^{1+n-n/p}\left\{\int_{\Gamma(r)} |\mathrm{grad}\, u(y)|^p dy\right\}^{1/p},
\end{aligned}$$

and by using Minkowski's inequality, we obtain

$$\begin{aligned}
&\left[\int_{\Gamma(r)}\left\{\int_{\Gamma(r)}\left(\int_{1/2}^1 |z-y|\,|\mathrm{grad}\, u(y+t(z-y))|dt\right)dy\right\}^p dz\right]^{1/p} \\
&\leqq C_4 r\int_{\Gamma(r)}\left\{\int_{1/2}^1\left(\int_{\Gamma(r)} |\mathrm{grad}\, u(y+t(z-y))|^p dz\right)^{1/p} dt\right\}dy
\end{aligned}$$

$$\leqq C_5 r^{1+n}\left(\int_{\Gamma(r)} |\mathrm{grad}\ u(z)|^p dz\right)^{1/p}.$$

Hence

$$\left|u(x) - \frac{1}{|\Gamma(r)|}\int_{\Gamma(r)} u(y)dy\right| \leqq C_6\left\{r^{p-n}\int_{\Gamma(r)} |\mathrm{grad}\ u(y)|^p dy\right\}^{1/p}$$

$$\leqq C_7\left\{r^{p-\alpha-n}\int_{\Gamma(r)} |\mathrm{grad}\ u(y)|^p \rho(y)^\alpha dy\right\}^{1/p}.$$

Here $C_1 \sim C_7$ are positive constants independent of $x \in \Gamma'$. Thus, denoting by $x^* \in \ell$ the point with $|x^* - \xi| = r$, we have established

$$|u(x) - u(x^*)| \leqq 2C_7\left\{r^{p-\alpha-n}\int_{\Gamma(r)} |\mathrm{grad}\ u(y)|^p \rho(y)^\alpha dy\right\}^{1/p}.$$

Define a function f by

$$f(y) = \begin{cases} |\mathrm{grad}\ u(y)|\rho(y)^{\alpha/p}, & \text{if } y \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f \in L^p(R^n)$ by our assumption iv). If we set

$$E'' = \left\{\xi \in \partial\Omega;\ \limsup_{t \downarrow 0} t^{p-\alpha-n}\int_{B(\xi,t)} f(y)^p dy > 0\right\},$$

then $B_{1-\alpha/p,p}(E'')=0$ on account of [2; Theorem 1] (see also [3; Lemma 6]). If $\xi \in F-(E' \cup E'')$, then

$$\lim_{x \to \xi, x \in \Gamma'} |u(x) - u(x^*)| = 0,$$

which implies that $\lim_{x \to \xi, x \in \Gamma'} u(x) = \lim_{x^* \to \xi, x^* \in \ell} u(x^*) = a$. Our theorem is now proved with $E = E' \cup E''$.

REMARK 1. The same conclusion as in the theorem holds for any u satisfying i), ii), iv) and

iii)′ $|u(x) - a| \leqq C\left\{r^{-n}\int_{B(x,r)} |u(y) - a|^p dy\right\}^{1/p}$

for all numbers a and r with $B(x, r) \subset \Gamma'$, where C is a positive constant independent of a, r and x.

Therefore, in view of [4; Theorems 1 and 2], we may replace the equation (1) by a more general equation of the form

(1)′ $$\mathrm{div}\ \mathbf{A}(x, \mathrm{grad}\ u) = 0,$$

where $\mathbf{A}(x, \eta)$ is an R^n-valued (measurable) function on $\Omega \times R^n$ such that $|\mathbf{A}(x, \eta)| \leqq a|\eta|^{p-1}$ ($a>0$: const.) and $(\mathbf{A}(x, \eta), \eta) \geqq |\eta|^p$ for all $x \in \Omega$ and $\eta \in R^n$.

REMARK 2. In case $p>n$, for any function u on Ω satisfying i), ii) and iv) in the theorem, the same conclusion as in the theorem holds.

In fact, with the same notation as in the proof, we can show

$$|u(x) - \frac{1}{|\Gamma(r)|}\int_{\Gamma(r)} u(y)dy| \leqq C\left\{r^{p-\alpha-n}\int_{\Gamma(r)} |\mathrm{grad}\, u(y)|^p \rho(y)^\alpha dy\right\}^{1/p},$$

which gives

$$|u(x) - u(x^*)| \leqq 2C\left\{r^{p-\alpha-n}\int_{\Gamma(r)} |\mathrm{grad}\, u(y)|^p \rho(y)^\alpha dy\right\}^{1/p}.$$

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Department of Mathematics,
Faculty of Integrated Arts and Sciences,
Hiroshima University