HIROSHIMA MATH. J.
10 (1980), 337–349

Estimates on the support of solutions of parabolic variational inequalities in bounded cylindrical domains

Naoki YAMADA

(Received December 12, 1979)

1. Introduction

Consider a solution u of the following Cauchy problem of the parabolic variational inequality:

$$\frac{\partial u}{\partial t} - \Delta u \leqq f, \qquad u \leqq 0,$$

$$u\left(\frac{\partial u}{\partial t} - \Delta u - f\right) = 0 \qquad \text{in} \quad R^n \times]0, T[,$$

$$u(x, 0) = u_0(x) \qquad \text{in} \quad R^n.$$

The support $S(t)$ of the function $x \to u(x, t)$ has been studied by Bensoussan and Lions [2], Brezis and Friedman [4] and Evans and Knerr [6]. They proved that

$$S(t) \subset S(0) + B(c(t|\log t|)^{1/2})$$

for sufficiently small $t > 0$, where $+$ denotes the vector sum, $B(\rho) = \{x \mid |x| \leqq \rho\}$, and c is a positive constant.

Some results which conclude

$$u(x, t) = 0 \qquad \text{for} \quad t > (\text{some constant}), \ x \in R^n,$$

or

$$u(x, t) = 0 \qquad \text{for} \quad |x| > (\text{some constant}), \ t > 0$$

are stated in the book by Bensoussan and Lions [3, Chapter 3, § 2.16].

In this paper we shall consider a solution u of the parabolic variational inequality with Dirichlet boundary condition

$$\frac{\partial u}{\partial t} - \Delta u \leqq f, \qquad u \leqq 0,$$

$$u\left(\frac{\partial u}{\partial t} - \Delta u - f\right) = 0 \qquad \text{in} \quad \Omega \times]0, T[,$$

$$u(x, t) = \phi(x, t) \qquad \text{on} \quad \Gamma \times]0, T[,$$

$$u(x, 0) = u_0(x) \qquad \text{on} \quad \Omega,$$

where Ω is a bounded domain in R^n with smooth boundary Γ.

We are interested to estimate the size of the set

$$S = \{(x, t) \in \Omega \times]0, T[\mid u(x, t) = 0\}$$

by the data f, ϕ and u_0. A comparison theorem which will be stated in section 2 is the main tool to prove our results.

Main results of this paper are stated in section 3. Some examples are also given.

For elliptic variational inequalities, analogous estimates on the support of the solution has been obtained by Bensoussan, Brezis and Friedman [1], Nagai [8] and the author [9], [10] under various boundary conditions. It is shown that, by a formal argument, our results for parabolic variational inequalities imply the corresponding ones for elliptic variational inequalities.

In section 4 we treat the one phase Stefan problem of one space dimension. It is known that the Stefan problem can be transformed into a parabolic variational inequality. However, we cannot apply our result in section 3, directly, to this variational inequality to estimate the free boundary of the Stefan problem, because, by the physical meaning, the data have definite signs.

Nevertheless, by using the same idea as in section 3, we can construct a comparison function to the solution of the variational inequality and give an estimate on the free boundary.

2. The problem and a comparison theorem

Let Ω be a bounded domain in R^n with smooth boundary Γ, and let T be a positive number.

We consider a solution u of the following parabolic variational inequality:

$$\frac{\partial u}{\partial t} - \Delta u \leqq f, \qquad u \leqq 0,$$

$$u\left(\frac{\partial u}{\partial t} - \Delta u - f\right) = 0 \qquad \text{in} \quad \Omega \times]0, T[, \tag{2.1}$$

$$u(x, t) = \phi(x, t) \qquad \text{on} \quad \Gamma \times]0, T[,$$

$$u(x, 0) = u_0(x) \qquad \text{on} \quad \Omega.$$

Let Φ be a solution of the initial-boundary value problem

$$\frac{\partial \Phi}{\partial t} - \Delta \Phi = 0 \qquad \text{in} \quad \Omega \times]0, T[,$$

(2.2)
$$\Phi(x, 0) = -u_0(x) \qquad \text{on} \quad \Omega,$$
$$\Phi(x, t) = -\phi(x, t) \qquad \text{on} \quad \Gamma \times]0, T[.$$

Then $\tilde{u} = u + \Phi$ satisfies the following variational inequality:

$$\frac{\partial \tilde{u}}{\partial t} - \Delta \tilde{u} \leqq f, \qquad \tilde{u} \leqq \Phi,$$

(2.3)
$$(\tilde{u} - \Phi)\left(\frac{\partial \tilde{u}}{\partial t} - \Delta \tilde{u} - f\right) = 0 \qquad \text{in} \quad \Omega \times]0, T[,$$
$$\tilde{u}(x, t) = 0 \qquad \text{on} \quad \Gamma \times]0, T[,$$
$$\tilde{u}(x, 0) = 0 \qquad \text{on} \quad \Omega.$$

Assuming that

$$f \in L^p(\Omega \times]0, T[)$$

and

$$\Phi \in W^{2,p}(\Omega \times]0, T[)$$

for some $p \geqq 2$, Charrier and Troianiello [5, Remark 4, p. 120] proved that there exists a unique solution $\tilde{u}$ of (2.3) and, moreover, the solution is continuous on $\bar{\Omega} \times [0, T]$ if p is large enough.

In the following of this paper, we shall always assume these conditions. Therefore, there always exists a unique continuous solution u of (2.1) as well as a solution $\tilde{u}$ of (2.3).

We are interested to estimate how the size of the set

$$S = \{(x, t) \in \Omega \times]0, T[\mid u(x, t) = 0\}$$

depends on the data f, ϕ and u_0.

The next theorem is a main tool to estimate the solution u of (2.1). However, since we already proved an analogous comparison theorem for elliptic variational inequalities in [9, Theorem 2.1, p. 8], the proof is omitted.

THEOREM 2.1. *Let u be a solution of* (2.1). *Suppose that $w \in L^2(0, T; H^2(\Omega))$ satisfies the differential inequalities*

$$\frac{\partial w}{\partial t} - \Delta w \leqq f, \quad w \leqq 0 \qquad \text{in} \quad \Omega \times]0, T[,$$
$$w(x, t) \leqq \phi(x, t) \qquad \text{on} \quad \Gamma \times]0, T[,$$
$$w(x, 0) \leqq u_0(x) \qquad \text{on} \quad \Omega.$$

Then we have $u \geqq w$ *a.e. in* $\Omega \times]0, T[$.

COROLLARY. *Let u be a solution of* (2.1). *Suppose that* $\hat{w} \in L^2(0, T; H^2(\Omega))$ *satisfies the differential inequalities*

$$f \leqq \frac{\partial \hat{w}}{\partial t} - \Delta \hat{w}, \quad \hat{w} \leqq 0 \qquad \text{in} \quad \Omega \times]0, T[,$$

$$\phi(x, t) \leqq \hat{w}(x, t) \qquad \text{on} \quad \Gamma \times]0, T[,$$

$$u_0(x) \leqq \hat{w}(x, 0) \qquad \text{on} \quad \Omega.$$

Then we have $u \leqq \hat{w}$ *a.e. in* $\Omega \times]0, T[$.

3. Main results

First, we shall give an estimate of the set S from above.

THEOREM 3.1. *If there exist positive constants* $\hat{\gamma}$ *and* $\hat{\delta}$ *such that*

$$f(x, t) \leqq \hat{\gamma} \qquad \text{in} \quad \Omega \times]0, T[,$$

$$u_0(x) \leqq -\hat{\delta} \qquad \text{on} \quad \Omega,$$

$$\phi(x, t) \leqq \hat{\gamma}(t - (\hat{\delta}/\hat{\gamma})) \qquad \text{for all} \quad x \in \Gamma, \ 0 < t < \hat{\delta}/\hat{\gamma},$$

then we have, for a solution u of (2.1),

$$u(x, t) < 0 \qquad \text{for} \quad (x, t) \in \Omega \times]0, \hat{\delta}/\hat{\gamma}[.$$

In other words, the lower part of the cylinder does not intersect the set S.

PROOF. If we choose a comparison function $\hat{w}$ as

$$\hat{w}(x, t) = \hat{\gamma}(t - (\hat{\delta}/\hat{\gamma})),$$

we have

$$\frac{\partial \hat{w}}{\partial t} - \Delta \hat{w} = \hat{\gamma} \geqq f, \quad \hat{w} \leqq 0 \qquad \text{in} \quad \Omega \times]0, \hat{\delta}/\hat{\gamma}[,$$

$$\hat{w}(x, 0) = -\hat{\delta} \geqq u_0(x) \qquad \text{on} \quad \Omega.$$

For $x \in \Gamma$ and $0 < t < \hat{\delta}/\hat{\gamma}$, we have, from the assumption,

$$\hat{w}(x, t) \geqq \phi(x, t).$$

Applying the corollary to Theorem 2.1 in $\Omega \times]0, \hat{\delta}/\hat{\gamma}[$, we have

$$u(x, t) \leqq \hat{w}(x, t) < 0 \qquad \text{in} \quad \Omega \times]0, \hat{\delta}/\hat{\gamma}[.$$

Now we are in a position to state the main theorem of this paper.

THEOREM 3.2. *Suppose that there exist positive constants γ and δ such that*

$$f(x, t) \geqq \gamma \qquad in \quad \Omega \times]0, T[,$$

$$u_0(x) \geqq -\delta \qquad on \quad \Omega.$$

Assume also that there exists $t_0 \in]\delta/\gamma, T[$ such that:

(3.1) *There exists a continuously differentiable, nondecreasing function c defined on $[0, t_0]$ and satisfying*

$$0 \leqq c(t) \leqq \frac{1}{2n}\left(\gamma - \frac{\delta}{t_0}\right) \qquad for\ all \quad t \in [0, t_0].$$

(3.2) *There exists a nonnegative constant ε depending on t_0 such that*

$$\phi(x, t) \geqq -\varepsilon c(t) + \frac{\delta}{t_0}(t - t_0) \qquad for\ all \quad t \in [0, t_0],\ x \in \Gamma.$$

If $x_0 \in \Omega$ satisfies

$$\operatorname{dist}(x_0, \Gamma) \geqq \varepsilon^{1/2}, \tag{3.3}$$

then we have, for a solution u of (2.1),

$$u(x_0, t_0) = 0.$$

PROOF. Let $(x_0, t_0) \in \Omega \times]0, T[$ satisfy the assumptions of the theorem. Choose a comparison function w as

$$w(x, t) = -c(t)|x - x_0|^2 + \frac{\delta}{t_0}(t - t_0)$$

for $x \in \Omega$, $0 < t < t_0$. Then we have, from (3.1),

$$\frac{\partial w}{\partial t} - \Delta w = -c'(t)|x - x_0|^2 + \frac{\delta}{t_0} + 2nc(t)$$

$$\leqq \frac{\delta}{t_0} + \left(\gamma - \frac{\delta}{t_0}\right) = \gamma.$$

By the definition of w, it is obvious that $w(x, t) \leqq 0$.

Again from (3.1), we have

$$w(x, 0) = -c(0)|x - x_0|^2 - \delta \leqq -\delta.$$

Let $x \in \Gamma$, $0 < t < t_0$. We have, from (3.2) and (3.3),

$$\begin{aligned} w(x, t) &= -c(t)|x - x_0|^2 + \frac{\delta}{t_0}(t - t_0) \\ &\leqq -c(t)(\operatorname{dist}(x_0, \Gamma))^2 + \frac{\delta}{t_0}(t - t_0) \\ &\leqq -\varepsilon c(t) + \frac{\delta}{t_0}(t - t_0) \leqq \phi(x, t). \end{aligned}$$

Hence, applying Theorem 2.1 in $\Omega \times]0, t_0[$, we obtain

$$w(x, t) \leqq u(x, t) \quad \text{in} \quad \Omega \times]0, t_0[.$$

By the continuity of u and w, we have

$$u(x_0, t_0) = 0.$$

Before stating some versions of this theorem, we give two examples of the function $c(t)$.

EXAMPLE 3.1. If there exists $t_0 > \delta/\gamma$ such that

$$\phi(x, t) \geqq (\delta/t_0)(t - t_0)$$

for $x \in \Gamma$, $0 < t < t_0$, then $u(x, t_0) = 0$ for any $x \in \Omega$.

Indeed, we may in this case choose $c(t) \equiv 0$ and $\varepsilon = 0$ in Theorem 3.1.

EXAMPLE 3.2. In the case that there exist $t_0 > \delta/\gamma$ and $\varepsilon_0 > 0$ such that for some $p \geqq 1$,

$$\phi(x, t) \geqq -\varepsilon_0 t^p + \frac{\delta}{t_0}(t - t_0) \quad \text{for} \quad x \in \Gamma,\ 0 < t < t_0, \tag{3.4}$$

we have $u(x_0, t_0) = 0$ if $x_0 \in \Omega$ satisfies the inequality

$$\operatorname{dist}(x_0, \Gamma) \geqq \left(\frac{2nt_0^{p+1}\varepsilon_0}{\gamma t_0 - \delta}\right)^{1/2}. \tag{3.5}$$

Indeed, it is sufficient to choose

$$c(t) = \frac{\gamma t_0 - \delta}{2nt_0^{p+1}} t^p \quad \text{and} \quad \varepsilon = \frac{2nt_0^{p+1}\varepsilon_0}{\gamma t_0 - \delta}.$$

REMARK 3.1. If $\phi(x, t)$ does not depend on t, then the compatibility condition yields that $\phi(x) \geqq -\delta$ for $x \in \Gamma$. This is the case in (3.4) that $\varepsilon_0 = \delta/t_0$ and $p = 1$. Formally speaking, by letting $t_0 \to \infty$ in (3.5) with $p = 1$, we have

$$\operatorname{dist}(x_0, \Gamma) \geqq \left(\frac{2n\delta}{\gamma}\right)^{1/2}. \tag{3.6}$$

(3.6) is the same estimate which Bensoussan, Brezis and Friedman [1, Theorem 3.1, p. 307] imposed on the point x_0 in order to conclude $u(x_0)=0$ for the solution u of the elliptic variational inequality

$$\begin{aligned} &- \Delta u + \alpha u \leqq f, \qquad u \leqq 0, \\ &u(- \Delta u + \alpha u - f) = 0 \qquad \text{in} \quad \Omega, \\ &u = \phi \qquad\qquad\qquad\qquad \text{on} \quad \Gamma, \end{aligned}$$

where α is a positive constant.

In the following part of this section, we state some generalizations of Theorem 3.2.

THEOREM 3.3. *Suppose that there exist positive constants γ and δ such that*

$$\begin{aligned} &f(x, t) \geqq \gamma \qquad \textit{in} \quad \Omega \times]0, T[, \\ &u_0(x) \geqq - \delta \qquad \textit{on} \quad \Omega. \end{aligned}$$

Assume also that there exists $t_0 \in]\delta/\gamma, T[$ satisfying the following conditions:

(3.7) *There exists a constant α such that $\delta/t_0 \leqq \alpha \leqq \gamma$.*

(3.8) *There exists a continuously differentiable, nondecreasing function c_1 defined on $[0, t_0]$ and satisfying*

$$0 \leqq c_1(t) \leqq \frac{1}{2n}(\gamma - \alpha).$$

(3.9) *There exists a twice continuously differentiable function c_2 defined on $[0, \operatorname{diam} \Omega]$ such that*

$$c_2'(\rho) \leqq 0, \quad c_2''(\rho) \leqq 0$$

and

$$\delta/t_0 \leqq c_2(\rho) \leqq \alpha.$$

(3.10) *There exists a nonnegative constant ε such that*

$$\phi(x, t) \geqq - \varepsilon c_1(t) + c_2(\operatorname{diam} \Omega)(t - t_0) \qquad \textit{for} \quad x \in \Gamma, \ 0 < t < t_0.$$

Then we have $u(x_0, t_0)=0$ if $x_0 \in \Omega$ satisfies

(3.11) $$\operatorname{dist}(x_0, \Gamma) \geqq \varepsilon^{1/2}.$$

PROOF. Let $(x_0, t_0) \in \Omega \times]0, T[$ satisfy the assumptions. We choose a comparison function w as

$$w(x, t) = -c_1(t)|x - x_0|^2 + c_2(|x - x_0|)(t - t_0)$$

in $\Omega \times]0, t_0[$, and compare w with the solution u in $\Omega \times]0, t_0[$.

It is obvious that $w \leqq 0$. From (3.7), (3.8) and (3.9), we have

$$\frac{\partial w}{\partial t} - \Delta w = -c_1'(t)|x - x_0|^2 + c_2(|x - x_0|) + 2nc_1(t)$$

$$- c_2''(|x - x_0|)(t - t_0) - \frac{n-1}{|x - x_0|} c_2'(|x - x_0|)(t - t_0)$$

$$\leqq \alpha + (\gamma - \alpha) = \gamma.$$

By (3.8) and (3.9), we have

$$w(x, 0) = -c_1(0)|x - x_0|^2 - c_2(|x - x_0|)t_0$$

$$\leqq -\delta.$$

Let $x \in \Gamma$. Applying (3.9), (3.11) and (3.10), we have

$$w(x, t) = -c_1(t)|x - x_0|^2 + c_2(|x - x_0|)(t - t_0)$$

$$\leqq -c_1(t)\varepsilon + c_2(\operatorname{diam} \Omega)(t - t_0)$$

$$\leqq \phi(x, t).$$

Therefore, we can apply Theorem 2.1 to conclude

$$u(x, t) \geqq w(x, t) \quad \text{in} \quad \Omega \times]0, t_0[,$$

and by the continuity of u and w, we obtain the assertion.

REMARK 3.2. Theorem 3.2 is the special case of Theorem 3.3 where $c_2(\rho) \equiv \alpha = \delta/t_0$.

In [9] and [10], we proved some estimates of the set S when x_0 might be on the boundary. In this direction, we have the following theorem.

THEOREM 3.4. *Suppose that there exist positive constants γ and δ such that*

$$f(x, t) \geqq \gamma \quad \textit{in} \quad \Omega \times]0, T[,$$

$$u_0(x) \geqq -\delta \quad \textit{on} \quad \Omega.$$

Assume also that there exists $(x_0, t_0) \in \bar{\Omega} \times]0, T[$ such that $t_0 > \delta/\gamma$ and satisfying the following conditions:

(3.12) *There exist a constant α and functions $c_1(t)$, $c_2(\rho)$ which satisfies* (3.7), (3.8) *and* (3.9) *in Theorem* 3.2, *respectively.*

(3.13) *There exist nonnegative constants* ε *and* r *such that*

$$\phi(x, t) \geqq -\varepsilon c_1(t) + c_2(\operatorname{diam} \Omega)(t - t_0)$$

$$\text{on} \quad \{x \in \Gamma \mid |x - x_0| \geqq \operatorname{dist}(x_0, \Gamma) + r\} \times]0, t_0[,$$

$$\phi(x, t) = 0$$

$$\text{on} \quad \{x \in \Gamma \mid |x - x_0| < \operatorname{dist}(x_0, \Gamma) + r\} \times]0, t_0[.$$

If

$$s = r + \operatorname{dist}(x_0, \Gamma) - \varepsilon^{1/2}$$

is nonnegative, then we have

$$u(x, t_0) = 0$$

for all $x \in \Omega$ *such that* $|x - x_0| \leqq s$.

PROOF. Define the comparison function w by

$$w(x, t) = \begin{cases} -c_1(t)(|x - x_0| - s)^2 + c_2(|x - x_0|)(t - t_0) \\ \qquad\qquad\qquad \text{if} \quad |x - x_0| \geqq s, \ 0 < t < t_0, \\ c_2(s)(t - t_0) \qquad \text{if} \quad |x - x_0| < s, \ 0 < t < t_0, \end{cases}$$

and compare it with u in $\Omega \times]0, t_0[$.

First, it is obvious by the definition that $w \leqq 0$ in $\Omega \times]0, t_0[$.

In the set $\{x \in \Omega \mid |x - x_0| \geqq s\} \times]0, t_0[$, we have from (3.12)

$$\begin{aligned} \frac{\partial w}{\partial t} - \Delta w = & -c_1'(t)(|x - x_0| - s)^2 + c_2(|x - x_0|) + 2c_1(t) \\ & + 2(n-1)c_1(t)\frac{|x - x_0| - s}{|x - x_0|} \\ & - (t - t_0)\left\{c_2''(|x - x_0|) + \frac{n-1}{|x - x_0|}c_2'(|x - x_0|)\right\} \\ \leqq & \ \alpha + (\gamma - \alpha) = \gamma. \end{aligned}$$

On the other hand, in the region $\{x \in \Omega \mid |x - x_0| < s\} \times]0, t_0[$, we have

$$\frac{\partial w}{\partial t} - \Delta w = c_2(s) \leqq \alpha \leqq \gamma.$$

Hence we obtain

$$\frac{\partial w}{\partial t} - \Delta w \leqq f \qquad \text{in} \quad \Omega \times]0, t_0[.$$

It is easy to see that

$$w(x, 0) \leqq -\delta \qquad \text{on} \quad \Omega$$

by the definition of w and (3.12).

Finally, let $x \in \Gamma$ and $0 < t < t_0$. If $|x - x_0| < s$, then we have from (3.13)

$$w(x, t) = c_2(s)(t - t_0) \leqq 0 = \phi(x, t) .$$

If $s \leqq |x - x_0| < r + \operatorname{dist}(x_0, \Gamma)$, then we have

$$\begin{aligned} w(x, t) &= -c_1(t)(|x - x_0| - s)^2 + c_2(|x - x_0|)(t - t_0) \\ &\leqq 0 = \phi(x, t) . \end{aligned}$$

In the case that $r + \operatorname{dist}(x_0, \Gamma) \leqq |x - x_0|$, we have using (3.13)

$$\begin{aligned} w(x, t) &= -c_1(t)(|x - x_0| - s)^2 + c_2(|x - x_0|)(t - t_0) \\ &\leqq -c_1(t)(r + \operatorname{dist}(x_0, \Gamma) - s)^2 + c_2(|x - x_0|)(t - t_0) \\ &= -c_1(t)\varepsilon + c_2(|x - x_0|)(t - t_0) \\ &\leqq \phi(x, t) . \end{aligned}$$

Combining these estimates, we obtain

$$w(x, t) \leqq \phi(x, t) \qquad \text{on} \quad \Gamma \times]0, t_0[.$$

Hence we can apply Theorem 2.1 to u and w, and the assertion is obtained.

4. Estimates on the free boundary arising in a Stefan problem

Let us consider the following one phase Stefan problem of one space dimension

$$(4.1) \qquad \begin{aligned} &\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0 && 0 < x < y(t), \quad 0 < t \leqq T, \\ &v(0, t) = g(t) && 0 < t \leqq T, \\ &v(y(t), t) = 0 && 0 \leqq t \leqq T, \\ &v(x, 0) = v_0(x) && 0 < x < \ell, \\ &\dot{y}(t) = -v_x(y(t), t) && 0 < t < T, \\ &y(0) = \ell, \end{aligned}$$

where $T>0$, $\ell>0$ are given constants, $g(t)$ and $v_0(x)$ are given functions, and the unknowns are $v(x, t)$ and $y(t)$.

Under suitable assumptions, it is known that the solution of the Stefan problem (4.1) exists and the free boundary $x=y(t)$ is monotone increasing in t (see, for instance, [11, Chap. 4, § 3, pp. 66–74]).

It is also known that the Stefan problem (4.1) can be represented as a variational inequality. Let $L>y(T)$ be a constant and define

$$Q = [0, L] \times [0, T],$$

$$D = \{(x, t) \mid 0 \leqq x \leqq y(t), 0 \leqq t \leqq T\}.$$

Extend the solution v to Q as

$$\tilde{v}(x, t) = \begin{cases} v(x, t) & \text{if } (x, t) \in D, \\ 0 & \text{if } (x, t) \notin D, \end{cases}$$

and define the new unknown $\theta(x, t)$ by

$$\theta(x, t) = \int_0^t \tilde{v}(x, s) ds. \tag{4.2}$$

It is shown in Lions [7, Chap. II, § 7, pp. 70–74] that $\theta(x, t)$ satisfies the following parabolic variational inequality:

$$\begin{aligned} &\frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x^2} - \theta_0(x) \geqq 0, \qquad \theta \geqq 0, \\ &\theta\Big(\frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x^2} - \theta_0(x)\Big) = 0 \qquad \text{in } Q, \\ &\theta(x, 0) = 0, \qquad \theta(L, t) = 0, \\ &\theta(0, t) = \int_0^t g(s) ds. \end{aligned} \tag{4.3}$$

Here we have set

$$\theta_0(x) = \begin{cases} v_0(x) & \text{if } 0 < x < \ell, \\ -1 & \text{if } \ell \leqq x \leqq L. \end{cases}$$

We are interested to estimate the size of the set

$$\begin{aligned} S &= \{(x, t) \in Q \mid \theta(x, t) = 0\} \\ &= \{(x, t) \in Q \mid \tilde{v}(x, t) = 0\} = Q \setminus D \end{aligned}$$

as has been studied in section 3.

By the physical meaning of the Stefan problem, it is required that $g(t) \geqq 0$ and $v_0(x) \geqq 0$.

The variational inequality (4.3) does not satisfy the assumptions $f(x, t) \geqq \gamma$ and $u_0(x) \geqq -\delta$ for some positive constants γ and δ in Theorems 3.2–3.4. But we can construct a comparison function to the solution of (4.3) and have the following estimate of the set S.

THEOREM 4.1. *Suppose that there exist positive constants μ and ν such that*

$$g(t) \leqq \mu \qquad 0 < t < T,$$

$$v_0(x) \leqq \nu \qquad 0 < x < \ell.$$

If a point $(x_0, t_0) \in Q$ satisfies the inequalities

$$\ell^2 \geqq 2\mu t_0, \tag{4.4}$$

$$x_0 \geqq \ell + \sqrt{2t_0(\nu + 1)}, \tag{4.5}$$

then we have

$$\tilde{v}(x, t) = 0 \qquad \textit{for} \quad (x, t) \in [x_0, L] \times [0, t_0].$$

PROOF. Let (x_0, t_0) satisfy the assumptions. Define a comparison function $w(x, t)$ by

$$w(x, t) = \begin{cases} \dfrac{t}{2t_0} |x - x_0|^2 & 0 < x < x_0, t < t_0, \\ 0 & x_0 \leqq x, t < t_0, \end{cases}$$

and compare it with θ in $]0, L[\times]0, t_0[$.

By easy calculation and (4.5), we have

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} \geqq \begin{cases} \dfrac{|\ell - x_0|^2}{2t_0} - \dfrac{t}{t_0} & \text{if} \quad x \leqq \ell, \\ -\dfrac{t}{t_0} & \text{if} \quad \ell < x < x_0, \\ 0 & \text{if} \quad x_0 \leqq x, \end{cases}$$

$$\geqq \theta_0(x),$$

$$w(x, 0) = 0 = \theta(x, 0),$$

$$w(L, t) = 0 = \theta(L, t).$$

The assumption (4.4) leads us to

$$w(0, t)=\frac{tx_0^2}{2t_0} \geqq \mu t \geqq \int_0^t g(s)ds.$$

Therefore we have

$$w(x, t) \geqq \theta(x, t) \qquad \text{in} \quad]0, L[\times]0, t_0[.$$

The continuity of the solution $\theta(x, t)$ and the definition of $w(x, t)$ imply that

$$\theta(x, t)=0 \qquad \text{in} \quad [x_0, L]\times[0, t_0].$$

Hence, by the definition (4.2) of $\theta(x, t)$, we obtain

$$\tilde{v}(x, t)=0 \qquad \text{in} \quad [x_0, L]\times[0, t_0].$$

References

[1] A. Bensoussan, H. Brezis and A. Friedman, Estimates on the free boundary for quasi variational inequalities, Comm. Partial Differential Equations, **2** (1977), 297–321.

[2] A. Bensoussan and J. L. Lions, On the support of the solution of some variational inequalities of evolution, J. Math. Soc. Japan, **28** (1976), 1–17.

[3] A. Bensoussan et J. L. Linons, Applications des inéquations variationnelles en contrôle stochastique, Dunod, Paris, 1978.

[4] H. Brezis and A. Friedman, Estimates on the support of solutions of parabolic variational inequalities, Illinois J. Math., **20** (1976), 82–97.

[5] P. Charrier and G. M. Troianiello, On strong solutions to parabolic unilateral problems with obstacle dependent on time, J. Math. Anal. Appl., **65** (1978), 110–125.

[6] L. C. Evans and B. F. Knerr, Instantaneous shrinking of the support of nonnegative solutions to certain nonlinear parabolic equations and variational inequalities, Illinois J. Math., **23** (1979), 153–166.

[7] J. L. Lions, Sur quelques questions d'analyse, de mécanique et de contrôle optimal, Univ. Montréal, Montréal, 1976.

[8] T. Nagai, Estimates for the coincidence sets of solutions of elliptic variational inequalities, Hiroshima Math. J., **9** (1979), 335–346.

[9] N. Yamada, Estimates on the support of solutions of elliptic variational inequalities in bounded domains, Hiroshima Math. J., **9** (1979), 7–16.

[10] ———, Estimates on the support of solutions of elliptic variational inequalities in bounded domains II, Funkcial. Ekvac., **22** (1979), 339–349.

[11] M. Yamaguchi and T. Nogi, The Stefan problem, Sangyo-tosho, Tokyo, 1977 (in Japanese).

Department of Mathematics,
Faculty of Science,
Kobe University