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Lie algebras which have an ascending series with simple factors

Yoshimi KASHIWAGI

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Introduction

In this paper we investigate Lie algebras which have an ascending series whose factors are simple. Here simple Lie algebras are non-abelian simple. In [3] Levich has shown that if H is an ascendant subalgebra of a simple Lie algebra L, then $H=0$ or $H=L$. In particular H is a perfect characteristic ideal of L. In § 1 we shall show that, in a Lie algebra which has an ascending series whose factors are simple, every ascendant subalgebra is a perfect characteristic ideal. In § 2 we consider a special case and its application. In [6] it has been shown that, in the Lie algebra L of all endomorphisms of an infinite-dimensional vector space, every subideal is an ideal of L. We shall show in § 2 that every ascendant subalgebra of L is an ideal. In § 3 we shall show that $\acute{\mathrm{E}}(\lhd)\mathfrak{F} \leq \mathrm{L}\mathfrak{F}$. Using the results of §§ 1 and 3, we shall show in § 4 that in a Lie algebra which has an ascending series whose factors are finite-dimensional simple, every serial subalgebra is a perfect characteristic ideal. In § 5 we apply our results to prove that, in a semi-simple neoclassical algebra, serial subalgebras and local subideals are perfect characteristic ideals. In [7] it has been shown that every soluble Lie algebra, in which every ascendant subalgebra is an ideal, is either abelian or the split extension of an abelian Lie algebra by the 1-dimensional algebra of scalar multiplications and conversely. We shall finally show in § 6 that in the split extension of an abelian Lie algebra by the 1-dimensional algebra of scalar multiplications every serial subalgebra is an ideal.

Let H be a subalgebra of a Lie algebra L and let Σ be a totally ordered set. A series from H to L of type Σ is a family $\{\Lambda_\sigma, V_\sigma: \sigma \in \Sigma\}$ of subalgebras of L such that

(1) For all σ, $H \leq \Lambda_\sigma$ and $H \leq V_\sigma$,

(2) $L \setminus H = \cup_{\sigma\in\Sigma}(\Lambda_\sigma \setminus V_\sigma)$,

(3) $\Lambda_\tau \leq V_\sigma$ if $\tau < \sigma$,

(4) $V_\sigma \lhd \Lambda_\sigma$.

The quotient algebras Λ_σ/V_σ are the factors of the series. If Σ is well-ordered (resp. reversely well-ordered, finite), then the series is called an ascending series (resp. a descending series, a subideal) and we write $H \operatorname{ser} L$ (resp. $H \operatorname{desc} L$,

H si L) (cf. [1, p. 27]). We denote by $\mathfrak{A}$ (resp. E$\mathfrak{A}$, $\mathfrak{F}$) the class of abelian (resp. soluble, finite-dimensional) Lie algebras. For a class $\mathfrak{X}$ of Lie algebras, L$\mathfrak{X}$ is the class of Lie algebras in which every finite subset is contained in an $\mathfrak{X}$-subalgebra. É$(\lhd)\mathfrak{X}$ is the class of Lie algebras which have an ascending series of ideals whose factors belong to $\mathfrak{X}$. Any notation not explained here may be found in [1].

1.

We define a class $\mathfrak{X}_1$ of Lie algebras as follows: $L \in \mathfrak{X}_1$ if and only if either $L=0$ or L has an ascending series whose factors are simple. To investigate this class we need

LEMMA 1.1 ([3]). *Let L be a simple Lie algebra. If H is an ascendant subalgebra of L, then $H=0$ or L.*

We first state some simple properties of $\mathfrak{X}_1$.

LEMMA 1.2. (1) *If L belongs to $\mathfrak{X}_1$ and H is an ascendant subalgebra of L, then $H \in \mathfrak{X}_1$.*

(2) *$\mathfrak{X}_1$ is* Q*-closed.*

(3) $\mathfrak{A} \cap \mathfrak{X}_1 = (0)$.

PROOF. (1) Assume that $L \in \mathfrak{X}_1$ and $H \operatorname{asc} L$. If $L=0$, then $H=0 \in \mathfrak{X}_1$. Let σ be an ordinal and let $(L_\alpha)_{\alpha\le\sigma}$ be an ascending series of L such that $L_{\alpha+1}/L_\alpha$ is simple for any $\alpha<\sigma$. For $\alpha\le\sigma$ we put $H_\alpha = H \cap L_\alpha$. Let $\alpha<\sigma$. We have

$$H_{\alpha+1}/H_\alpha \cong ((H \cap L_{\alpha+1}) + L_\alpha)/L_\alpha.$$

Since $H \cap L_{\alpha+1} \operatorname{asc} L_{\alpha+1}$, we see that $((H \cap L_{\alpha+1})+L_\alpha)/L_\alpha \operatorname{asc} L_{\alpha+1}/L_\alpha$. Since $L_{\alpha+1}/L_\alpha$ is simple, it follows from Lemma 1.1 that $H_{\alpha+1}=H_\alpha$ or that $H_{\alpha+1}/H_\alpha$ is simple. It is immediate that $H \in \mathfrak{X}_1$.

(2) Assume that $L \in \mathfrak{X}_1$ and $I \lhd L$. If $L=0$, then $L/I=0 \in \mathfrak{X}_1$. Let $(L_\alpha)_{\alpha\le\sigma}$ be an ascending series of L such that $L_{\alpha+1}/L_\alpha$ is simple for any $\alpha<\sigma$. For $\alpha\le\sigma$ we put $\bar{L}_\alpha=(L_\alpha+I)/I$. Let $\alpha<\sigma$. Then we have

$$\bar{L}_{\alpha+1}/\bar{L}_\alpha \cong (L_{\alpha+1} + \mathrm{I})/(L_\alpha + I).$$

$(L_{\alpha+1}+I)/(L_\alpha+I)$ is a homomorphic image of the simple Lie algebra $L_{\alpha+1}/L_\alpha$. Hence $\bar{L}_{\alpha+1}=\bar{L}_\alpha$ or $\bar{L}_{\alpha+1}/\bar{L}_\alpha$ is simple, and therefore we have $L/I \in \mathfrak{X}_1$.

(3) Assume that $L \in \mathfrak{A} \cap \mathfrak{X}_1$. If $L \neq 0$, then L has a simple ascendant subalgebra and hence $L \notin \mathfrak{A}$. Therefore $L=0$.

LEMMA 1.3 ([1, Proposition 1.3.5]). *Let L be a Lie algebra. If H is a perfect ascendant subalgebra of L, then H is an ideal.*

LEMMA 1.4. *Let $L \in \mathfrak{X}_1$. If H is an ascendant subalgebra of L, then H is a perfect ideal of L.*

PROOF. By Lemma 1.2 (1) we have $H \in \mathfrak{X}_1$. Hence by Lemma 1.2 (2) $H/H^2 \in \mathrm{Q}\mathfrak{X}_1 \cap \mathfrak{A} = \mathfrak{X}_1 \cap \mathfrak{A}$. By Lemma 1.2 (3) $H = H^2$. The statement now follows from Lemma 1.3.

Now we show the following

THEOREM 1.5. *Let $L \in \mathfrak{X}_1$. If H is an ascendant or a descendant subalgebra of L, then H is a perfect characteristic ideal of L.*

PROOF. (1) Assume that $H \operatorname{asc} L$. By Lemma 1.4 H is a perfect ideal of L. Let δ be a derivation of L and form the split extension $K = L \dotplus \langle \delta \rangle$. Then H is a perfect subideal of K. By Lemma 1.3 $H \lhd K$. Hence $H\delta \subseteq H$.

(2) Let $(H_\alpha)_{\alpha \leq \sigma}$ be a descending series from H to L. We shall show by transfinite induction on α that $H_\alpha \lhd L$. If $\alpha = 0$, then the result is trivial. Let $\alpha > 0$ and assume that $H_\beta \lhd L$ for all $\beta < \alpha$. If α is a limit ordinal, then $H_\alpha = \bigcap_{\beta < \alpha} H_\beta \lhd L$. If α is not a limit ordinal, then $H_\alpha \lhd H_{\alpha-1} \lhd L$. By Lemma 1.4 we have $H_\alpha \lhd L$. Now the statement follows from (1).

COROLLARY 1.6. *Let L be a Lie algebra and let $(L_\alpha)_{\alpha \leq \sigma}$ be an ascending series of L such that $L_{\alpha+1}/L_\alpha$ is simple for any $\alpha < \sigma$. Then for any $\alpha \leq \sigma$, L_α is a perfect characteristic ideal of L.*

The next result is well known. For the proof see [1, Lemma 13.4.1].

LEMMA 1.7. *The sum of the minimal ideals of a Lie algebra is a direct sum of a subset of them.*

We shall give a characterization of $\mathfrak{X}_1$ by the following

PROPOSITION 1.8. *Let L be a Lie algebra. Suppose that L has an ascending series whose factors are generated by simple ascendant subalgebras. Then L belongs to $\mathfrak{X}_1$.*

PROOF. It follows easily from Lemma 1.3 that every simple ascendant subalgebra is a minimal ideal. By Lemma 1.7 each factor is a direct sum of simple ideals. Refining them, we have $L \in \mathfrak{X}_1$.

2.

In this section we consider special cases and applications of the results of § 1. First we need the next result. The proof can be found in [1, Lemma 13.4.3].

LEMMA 2.1. *Suppose that $L=\oplus_{\lambda\in\Lambda} L_\lambda$, where each L_λ is a simple ideal of L. Let I be an ideal of L. Then $I=\oplus_{\mu\in M} L_\mu$ for some subset M of Λ.*

We can generalize this lemma as follows:

PROPOSITION 2.2. *Let L be generated by a family $(L_\lambda)_{\lambda\in\Lambda}$ of simple ascendant subalgebras of L. If H is an ascendant or a descendant subalgebra of L, then H is a characteristic ideal of L and $H=\oplus_{\mu\in M} L_\mu$ for some subset M of Λ.*

PROOF. As in the proof of Proposition 1.8 we see that each L_λ is a simple ideal of L and $L=\oplus_{\nu\in N} L_\nu$ for some subset N of Λ. By Proposition 1.8 and Theorem 1.5 H is a characteristic ideal of L. Now Lemma 2.1 completes the proof.

PROPOSITION 2.3. *Let L be a Lie algebra and let $(L_\alpha)_{\alpha\leq\sigma}$ be an ascending series of L with simple factors. Then the following are equivalent*:

(1) *$C_{L/L_\alpha}(L_{\alpha+1}/L_\alpha)=0$ for all $\alpha<\sigma$.*
(2) *The only ascendant subalgebras of L are the L_α.*
(3) *The only descendant subalgebras of L are the L_α.*
(4) *The only subideals of L are the L_α.*
(5) *The only ideals of L are the L_α.*
(6) *The only characteristic ideals of L are the L_α.*

PROOF. By Corollary 1.6 each L_α is a characteristic ideal of L. Hence the statements (1), (3) and (6) make sense. Evidently we have the following implications:

$$(2) \Rightarrow (4),\ (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6).$$

By Theorem 1.5 we see that (6)⇒(2) and (6)⇒(3).

Now we show that (1)⇒(5). Let $H \trianglelefteq L$ and let α be the minimal ordinal with respect to $L_\alpha \nleq H$. Evidently α is neither 0 nor a limit ordinal. Hence $L_{\alpha-1}\leq H$. Since $H/L_{\alpha-1}\cap L_\alpha/L_{\alpha-1}$ is an ideal of the simple Lie algebra $L_\alpha/L_{\alpha-1}$, we see that $H/L_{\alpha-1}\cap L_\alpha/L_{\alpha-1}=0$ or $L_\alpha/L_{\alpha-1}\leq H/L_{\alpha-1}$. Since $L_\alpha\nleq H$, we have $H/L_{\alpha-1}\cap L_\alpha/L_{\alpha-1}=0$. Hence

$$[H/L_{\alpha-1}, L_\alpha/L_{\alpha-1}] \subseteq H/L_{\alpha-1} \cap L_\alpha/L_{\alpha-1} = 0.$$

Since $C_{L/L_{\alpha-1}}(L_\alpha/L_{\alpha-1})=0$, we have $H=L_{\alpha-1}$.

Next we show that (5)⇒(1). Suppose that $C_{L/L_\alpha}(L_{\alpha+1}/L_\alpha)\neq 0$ for some ordinal $\alpha<\sigma$. Since $L_{\alpha+1}/L_\alpha\lhd L/L_\alpha$, we have $C_{L/L_\alpha}(L_{\alpha+1}/L_\alpha)\lhd L/L_\alpha$. Hence we can find an ideal H of L such that $L_\alpha\lneqq H$ and $H/L_\alpha=C_{L/L_\alpha}(L_{\alpha+1}/L_\alpha)$. By (5) and $L_\alpha\lneqq H$ there exists an ordinal β such that $\beta\geq\alpha+1$ and $H=L_\beta$. Therefore we have

$$[L_{\alpha+1}, L_{\alpha+1}] \subseteq [L_\beta, L_{\alpha+1}] = [H, L_{\alpha+1}] \subseteq L_\alpha.$$

Hence $L_{\alpha+1}/L_\alpha$ is abelian. This is a contradiction.

This proposition is a generalization of [6, Lemma 7]. The following is there used: Let L be a Lie algebra having two subideals H, K such that K is simple. If $K \cap H=0$, then $[K, H]=0$ (cf. [5, Lemma 4.6]).

We generalize this in the following

PROPOSITION 2.4. *Let L be a Lie algebra with a subalgebra H and let K be a simple ascendant subalgebra of L such that $K \cap H=0$.*

(1) *If H is ascendant in L, then $[K, H]=0$.*

(2) *If H is descendant in L, then $[K, H]=0$.*

PROOF. By Lemma 1.3 we have $K \lhd L$. If $K=0$, then there is nothing to prove. We assume that $K \neq 0$.

(1) Let $(H_\alpha)_{\alpha\leq\sigma}$ be an ascending series from H to L and let α be the minimal ordinal with respect to $K \cap H_\alpha \neq 0$. Evidently α is neither 0 nor a limit ordinal. Hence $K \cap H_{\alpha-1}=0$. Since $K \cap H_\alpha$ is a non-zero ascendant subalgebra of K, we have $K \leq H_\alpha$ by Lemma 1.1. Since $H_{\alpha-1} \lhd H_\alpha$, we have

$$[K, H] \subseteq [K, H_{\alpha-1}] \subseteq K \cap H_{\alpha-1} = 0.$$

(2) Let $(H_\alpha)_{\alpha\leq\sigma}$ be a descending series from H to L and let α be the minimal ordinal with respect to $K \not\leq H_\alpha$. Clearly α is neigher 0 nor a limit ordinal. Hence $K \leq H_{\alpha-1}$. If $K \cap H_\alpha \neq 0$, then $K \cap H_\alpha$ is a non-zero descendant subalgebra of K. By Theorem 1.5 we have $K \leq H_\alpha$. Thus we have $K \cap H_\alpha=0$ and therefore

$$[K, H] \subseteq [K, H_\alpha] \subseteq K \cap H_\alpha = 0.$$

As an application of Propositions 2.3 and 2.4 we shall show that the Lie algebra of all endomorphisms of an infinite-dimensional vector space lies in the class $\mathfrak{M}'$ of Lie algebras in which every ascendant subalgebra is an ideal (cf. [7]).

Let c be an infinite cardinal with successor c^+. Let V be a vector space of dimension c. For any infinite cardinal $d \leq c^+$, $L(c, d)$ is the Lie algebra of all linear transformations α: $V \to V$ such that the image of α has dimension $<d$. Let $F=L(c, \aleph_0)$, T be the set of endomorphisms of trace zero (in the sense of [5, p. 306]), and S be the set of scalar multiplications. In [6] it has been shown that the ideals of $L=L(c, c^+)$ are precisely the following:

a) $L(c, d)$ for $\aleph_0 \leq d \leq c^+$,

b) $L(c, d)+S$ for $\aleph_0 \leq d \leq c$,

c) Any subspace X of L such that $T \leq X \leq F+S$,

d) S,

e) $\{0\}$.

Further, every subideal of L is an ideal.

Now we shall show the following

THEOREM 2.5. *Let* $L=L(c, c^+)$. *Then every ascendant subalgebra of* L *is an ideal.*

PROOF. Let $H \operatorname{asc} L$. If $H \cap T=0$, then by Proposition 2.4 $[H, T]=0$. (In [5] it has been shown that T is simple.) As in [6, p. 85] we have $H \leq S$. Hence $H \lhd L$. If $H \cap T \neq 0$, then by Lemma 1.1 $T \leq H$. By Proposition 2.3 and the argument of [6, p. 82] we see that every ascendant subalgebra of L which contains $F+S$ is of the form $L(c, d)+S$ with $\aleph_0 \leq d \leq c^+$. Hence $H+F+S= L(c, d)+S$ for some d with $\aleph_0 \leq d \leq c^+$. If $d=\aleph_0$, then $T \leq H \leq F+S$, which is the case c) of Theorem (A) in [6]. Let $d>\aleph_0$. Since $\dim (H+F+S)/(H+T+S) \leq \dim F/T=1$ and $H+T+S \operatorname{asc} H+F+S$, we have $H+T+S \lhd H+F+S$. Hence

$$(H + T + S)/(T + S) \lhd (L(c, d) + S)/(T + S).$$

By the argument of Lemma 12 in [6] $L(c, d)$ has no proper ideals of finite codimension. Hence $H+T+S=L(c, d)+S$. Since $T \leq H$, we have $H+S=L(c, d)+S$. Since

$$L(c, d)/H \cap L(c, d) \cong (L(c, d) + H)/H \subseteq (H + S)/H,$$

we have $\dim L(c, d)/H \cap L(c, d) \leq 1$. Hence $H \cap L(c, d)$ is an ascendant subalgebra of $L(c, d)$ of codimension ≤ 1 and so it is an ideal of $L(c, d)$ of codimension ≤ 1. Since $L(c, d)$ has no proper ideals of finite codimension, we have $H \cap L(c, d)=L(c, d)$. Hence $L(c, d) \leq H \leq L(c, d)+S$. Since $\dim S=1$, we have $H=L(c, d)$ or $L(c, d)+S$. Thus $H \lhd L$.

REMARK. It is easy to see that in a Lie algebra in which every subideal is an ideal, every descendant subalgebra is an ideal.

3.

In this section we shall give a sufficient condition for a Lie algebra to be locally finite.

Let L be a finitely generated Lie algebra. Let $\{a_1,\dots, a_n\}$ be a set of generators for L and let F be a free Lie algebra generated by $\{x_1,\dots, x_n\}$. Then there exists a homomorphism θ from F to L such that $\theta(x_i)=a_i$ $(1 \leq i \leq n)$. L is said to be finitely presented if there exist finitely many elements $y_1,\dots, y_m$ of F such that $\operatorname{Ker} \theta=\sum_{j=1}^{m} y_j^F$. (See [4, p. 31] for the well-definedness of the definition.) We denote by $\mathfrak{Fp}$ the class of finitely presented Lie algebras.

We shall show some properties of $\mathfrak{Fp}$.

LEMMA 3.1. (1) $\mathfrak{F} \leq \mathfrak{F}\mathfrak{p}$.

(2) *Let L be a finitely generated Lie algebra. If I is an ideal of L such that L/I is finitely presented, then I is finitely generated as an ideal.*

PROOF. (1) Let $L \in \mathfrak{F}$ and let $\{e_1, \dots, e_n\}$ be a basis for L. Then $[e_i, e_j] = \sum_{l=1}^{n} \alpha_{ijl} e_l$ for some $\alpha_{ijl} \in \mathfrak{k}$ $(1 \leq i, j, l \leq n)$, where $\mathfrak{k}$ is a base field. Let F be a free Lie algebra generated by $\{x_1, \dots, x_n\}$ and let θ be a homomorphism from F to L such that $\theta(x_i) = e_i$ $(1 \leq i \leq n)$. We put $y_{ij} = [x_i, x_j] - \sum_{l=1}^{n} \alpha_{ijl} x_l$ $(1 \leq i, j \leq n)$ and $I = \sum_{i,j} y_{ij}^F$, where y_{ij}^F is the ideal of F generated by y_{ij}. Clearly $I \leq \operatorname{Ker} \theta$. Hence we have a homomorphism

$$\psi: F/I \longrightarrow F/\operatorname{Ker} \theta.$$

Obviously we have an isomorphism $\bar{\theta}: F/\operatorname{Ker} \theta \to L$. Let $\phi = \bar{\theta} \circ \psi: F/I \to L$. Then $\phi(\bar{x}_i) = e_i$ $(1 \leq i \leq n)$, where $\bar{x}_i = x_i + I \in F/I$. Since $[\bar{x}_i, \bar{x}_j] = \sum_{l=1}^{n} \alpha_{ijl} \bar{x}_l$ in F/I, F/I is spanned by $\{\bar{x}_1, \dots, \bar{x}_n\}$. Since $\{\phi(\bar{x}_i)\} = \{e_i\}$ is a basis for L, $\{\bar{x}_1, \dots, \bar{x}_n\}$ is a basis for F/I. Hence ϕ is injective. Therefore ψ is injective. Thus we have $\operatorname{Ker} \theta = I = \sum_{i,j} y_{ij}^F$.

(2) Let $L = \langle a_1, \dots, a_n \rangle$ and let F be a free Lie algebra generated by $\{x_1, \dots, x_n\}$. We have a homomorphism $\theta: F \to L$ such that $\theta(x_i) = a_i$ $(1 \leq i \leq n)$. Let $R = \theta^{-1}(I)$. We have a homomorphism $\bar{\theta}: F \to L/I$ such that $\bar{\theta}(x_i) = \bar{a}_i \in L/I$ $(1 \leq i \leq n)$. Since L/I is finitely presented, there exist finitely many elements $y_1, \dots, y_m$ of F such that $\operatorname{Ker} \bar{\theta} = \sum_{j=1}^{m} y_j^F$. Clearly $\operatorname{Ker} \bar{\theta} = R$. Thus we have

$$I = \theta(R) = \theta(\textstyle\sum_{j=1}^{m} y_j^F) = \sum_{j=1}^{m} \theta(y_j)^L.$$

REMARK. Lemma 3.1 (2) is the Lie analogue of [4, Lemma 1.43 (i)]. The Lie analogue of [4, Lemma 1.43 (ii)] is also valid, i.e., $\mathfrak{F}\mathfrak{p}$ is E-closed.

Now we have the following

THEOREM 3.2. *Let $\mathfrak{X}$ be any {S, E}-closed subclass of $\mathfrak{F}\mathfrak{p}$. Then $\acute{\mathrm{E}}(\lhd)\mathfrak{X} \leq \mathrm{L}\mathfrak{X}$.*

PROOF. Let $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{X}$ and let $(L_\alpha)_{\alpha \leq \sigma}$ be an ascending series of ideals of L such that $L_{\alpha+1}/L_\alpha \in \mathfrak{X}$ for all $\alpha < \sigma$. Let H be a finitely generated subalgebra of L. Since $H/H \cap L_\sigma = 0$, there exists an ordinal α minimal with respect to $H/H \cap L_\alpha \in \mathfrak{X}$. Now suppose that α is a limit ordinal. Since H is finitely generated and $H/H \cap L_\alpha \in \mathfrak{X} \leq \mathfrak{F}\mathfrak{p}$, we can apply Lemma 3.1 (2) to see that there exist finitely many elements $x_1, \dots, x_n$ of $H \cap L_\alpha$ such that $H \cap L_\alpha = \sum_{i=1}^{n} x_i^H$. Since α is a limit ordinal, there exists an ordinal β such that $\beta < \alpha$ and $x_i \in L_\beta$ $(1 \leq i \leq n)$. Since $H \cap L_\beta \lhd H$, we have $H \cap L_\alpha = \sum_{i=1}^{n} x_i^H \leq H \cap L_\beta \leq H \cap L_\alpha$. Hence $H \cap L_\alpha = H \cap L_\beta$. Therefore $H/H \cap L_\beta = H/H \cap L_\alpha \in \mathfrak{X}$. This contradicts the minimality of α. Thus α is not a limit ordinal.

Next suppose that α is non-zero. We have

$$H \cap L_\alpha/H \cap L_{\alpha-1} \cong ((H \cap L_\alpha) + L_{\alpha-1})/L_{\alpha-1} \leq L_\alpha/L_{\alpha-1} \in \mathfrak{X}.$$

Hence $H \cap L_\alpha/H \cap L_{\alpha-1} \in \mathrm{s}\mathfrak{X} = \mathfrak{X}$. Therefore $H/H \cap L_{\alpha-1} \in \mathrm{E}\mathfrak{X} = \mathfrak{X}$, which contradicts the minimality of α. Thus α must be zero and hence $H \in \mathfrak{X}$.

COROLLARY 3.3. *Let* $\mathfrak{X}$ *be any* $\{\mathrm{s}, \mathrm{E}\}$*-closed subclass of* $\mathfrak{F}$. *Then* $\acute{\mathrm{E}}(\lhd)\mathfrak{X} \leq \mathrm{L}\mathfrak{X}$. *In particular* $\acute{\mathrm{E}}(\lhd)\mathfrak{F} \leq \mathrm{L}\mathfrak{F}$.

PROOF. By Lemma 3.1 (1) $\mathfrak{F} \leq \mathfrak{F}\mathfrak{p}$. Hence the statement follows from Theorem 3.2.

4.

In this section we shall find some classes of Lie algebras in which every serial subalgebra is a characteristic ideal. We begin by showing some elementary properties of serial subalgebras.

LEMMA 4.1. *Let* L *be a Lie algebra and let* H *be a serial subalgebra of* L.

(1) *If* $K \leq L$, *then* $H \cap K$ ser K.

(2) *If* $L \in \mathfrak{F}$, *then* H si L.

PROOF. (1) Let $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ be a series from H to L. Then clearly $\{\Lambda_\sigma \cap K, V_\sigma \cap K : \sigma \in \Sigma\}$ is a series from $H \cap K$ to K.

(2) Since $L \in \mathfrak{F}$, there exists a series from H to L of finite type.

For a locally finite Lie algebra we have some more properties.

LEMMA 4.2. *Let* $L \in \mathrm{L}\mathfrak{F}$ *and let* H *be a subalgebra of* L. *Then*

(1) H ser L *if and only if* $H \cap F$ si F *for every finite-dimensional subalgebra* F *of* L.

(2) *If* H ser L *and* θ *is a homomorphism of* L, *then* $\theta(H)$ ser $\theta(L)$.

PROOF. See the proof of [2, Lemma 2 and the conclusion of the proof of Theorem A].

REMARK. Since the join of two subideals of a finite-dimensional Lie algebra need not be a subideal, the second statement of [1, Proposition 13.2.4] is false.

LEMMA 4.3. *Let* $L \in \mathrm{L}\mathfrak{F}$. *If* H *is a finite-dimensional serial subalgebra of* L, *then* $H^\omega = \bigcap_{n=1}^\infty H^n \lhd L$.

PROOF. Let $x \in L$. Since $L \in \mathrm{L}\mathfrak{F}$ and $H \in \mathfrak{F}$, there exists a finite-dimensional subalgebra F of L such that $\langle x, H \rangle \leq F$. By Lemma 4.2 we see that $H = H \cap F$ si F. By [1, Lemma 1.3.2] $H^\omega \lhd F$. Hence $[x, H^\omega] \subseteq H^\omega$. Thus $H^\omega \lhd L$.

Now we define a class $\mathfrak{X}_2$ of Lie algebras as follows: $L \in \mathfrak{X}_2$ if and only if either $L=0$ or L has an ascending series whose factors are finite-dimensional and simple. By Corollaries 1.6 and 3.3 we see that $\mathfrak{X}_2 \leq \acute{\mathrm{E}}(\lhd)\mathfrak{F} \leq \mathrm{L}\mathfrak{F}$. Hence we can use Lemmas 4.2 and 4.3 for $\mathfrak{X}_2$-algebras. Now we have the following

THEOREM 4.4. *Let $L \in \mathfrak{X}_2$. If H is a serial subalgebra of L, then H is a perfect characteristic ideal of L.*

PROOF. If $L=0$, then there is nothing to prove. Let $(L_\alpha)_{\alpha \leq \sigma}$ be an ascending series of L such that $L_{\alpha+1}/L_\alpha$ is finite-dimensional simple for any $\alpha<\sigma$. We shall show the following by transfinite induction on α:

(*) If H ser L_α, then $H \lhd L$.

If $\alpha=0$, then the result is trivial. Let $\alpha>0$ and assume that (*) holds for any $\beta<\alpha$. Let H ser L_α. By Lemma 4.1 (1) we have $H \cap L_\beta$ ser L_β for all $\beta<\alpha$. It follows that $H \cap L_\beta \lhd L$ for all $\beta<\alpha$. If α is a limit ordinal, then we have

$$H = H \cap L_\alpha = \cup_{\beta<\alpha}(H \cap L_\beta) \lhd L.$$

If α is not a limit ordinal, then we have $(H+L_{\alpha-1})/L_{\alpha-1}$ ser $L_\alpha/L_{\alpha-1}$ by Lemma4.2 (2). Since $L_\alpha/L_{\alpha-1} \in \mathfrak{F}$, $(H+L_{\alpha-1})/L_{\alpha-1}$ si $L_\alpha/L_{\alpha-1}$ by Lemma 4.1 (2). Since $L_\alpha/L_{\alpha-1}$ is simple, we see that $H \leq L_{\alpha-1}$ or that $L_\alpha = H + L_{\alpha-1}$. In the former case $H = H \cap L_{\alpha-1} \lhd L$. So we consider the latter case. Since $H \cap L_{\alpha-1} \lhd L$ and $L_\alpha = H + L_{\alpha-1}$, we have

$$L_\alpha/H \cap L_{\alpha-1} = H/H \cap L_{\alpha-1} \dotplus L_{\alpha-1}/H \cap L_{\alpha-1}.$$

Hence $H/H \cap L_{\alpha-1} \cong L_\alpha/L_{\alpha-1}$. Therefore $H/H \cap L_{\alpha-1}$ is finite-dimensional and simple. By Lemma 4.2 (2) we have $H/H \cap L_{\alpha-1}$ ser $L/H \cap L_{\alpha-1}$. By Lemma 4.3

$$H/H \cap L_{\alpha-1} = (H/H \cap L_{\alpha-1})^\omega \lhd L/H \cap L_{\alpha-1}.$$

Thus we have $H \lhd L$. This completes the proof of (*). Now the statement follows from Theorem 1.5.

COROLLARY 4.5. *Let L be a Lie algebra over a field of characteristic zero. If L has an ascending series whose factors are finite-dimensional and semi-simple, then every serial subalgebra of L is a perfect characteristic ideal of L.*

PROOF. Since a finite-dimensional semi-simple Lie algebra over a field of characteristic zero is a direct sum of simple ideals, we have $L \in \mathfrak{X}_2$. Now Theorem 4.4 completes the proof.

5.

We consider a special case of $\mathfrak{X}_2$ in this section.

PROPOSITION 5.1. (1) *Let L be generated by a family* $(L_\lambda)_{\lambda\in\Lambda}$ *of finite-dimensional simple ascendant subalgebras. Then if H is a serial subalgebra of L, H is a perfect characteristic ideal of L and* $H=\oplus_{\mu\in M}L_\mu$ *for some subset M of* Λ.

(2) *Let L be a locally finite Lie algebra and let* $(L_\lambda)_{\lambda\in\Lambda}$ *be a family of finite-dimensional simple serial subalgebras of L such that* $L=\langle L_\lambda: \lambda\in\Lambda\rangle$. *If H is a serial subalgebra of L, then H is a perfect characteristic ideal of L and* $H=\oplus_{\mu\in M}L_\mu$ *for some subset M of* Λ.

PROOF. (1) As in the proof of Proposition 1.8 we see that $L_\lambda \lhd L$ for all $\lambda\in\Lambda$ and $L=\oplus_{\nu\in N}L_\nu$ for some $N\subseteq\Lambda$. Hence $L\in\mathfrak{X}_2$. By Theorem 4.4 we see that H is a perfect characteristic ideal of L. It follows from Lemma 2.1 that $H=\oplus_{\mu\in M}L_\mu$ for some subset M of Λ.

(2) By Lemma 4.3 we have $L_\lambda=L_\lambda^\omega \lhd L$ for all $\lambda\in\Lambda$. Argue as in (1).

Now we recall the definition of a semi-simple neoclassical algebra (cf. [1, Chap. 13]). Let $L\in \mathrm{L}\mathfrak{F}$. The sum $\sigma(L)$ of all locally soluble ideals of L is the largest locally soluble ideal of L. L is called semi-simple if $\sigma(L)=0$. Let L be a Lie algebra and let H be a subalgebra of L. H is said to be a local subideal of L, denoted by H lsi L, if H si $\langle H, X\rangle$ for any finite subset X of L. Over a field of characteristic zero L is said to be a neoclassical algebra if L is generated by finite-dimensional local subideals of L. The class of neoclassical algebras is denoted by $\mathfrak{H}$. Generally we have $\mathfrak{H}\leq \mathrm{L}\mathfrak{F}$.

LEMMA 5.2. *Let L be a locally finite Lie algebra. If H is a local subideal of L, then H is a serial subalgebra of L.*

PROOF. Let F be a finite-dimensional subalgebra of L. Since H lsi L, we have H si $\langle H, F\rangle$. Hence $H\cap F$ si F. By Lemma 4.2 (1) we have H ser L.

We shall prove that in a semi-simple neoclassical algebra every serial subalgebra is a perfect characteristic ideal.

THEOREM 5.3. *Let L be a semi-simple neoclassical algebra over a field of characteristic zero. If H is a serial subalgebra or a local subideal of L, then H is a perfect characteristic ideal of L and is a direct sum of some finite-dimensional simple ideals of L.*

PROOF. By [1, Theorem 13.4.2] L is a direct sum of finite-dimensional simple ideals of L. The statement follows from Proposition 5.1 (1) and Lemma 5.2.

COROLLARY 5.4. *Let L be a neoclassical algebra over a field of characteristic zero. If H is a serial subalgebra of L, then* $H+\sigma(L)\lhd L$. *In particular if H is a local subideal of L, then* $H+\sigma(L)\lhd L$.

PROOF. By Lemma 4.2 (2) we have

$$(H + \sigma(L))/\sigma(L) \text{ ser } L/\sigma(L).$$

By [1, Theorem 13.3.9] $L/\sigma(L)$ is a semi-simple $\mathfrak{H}$-algebra. By Theorem 5.3 we have

$$(H + \sigma(L))/\sigma(L) \lhd L/\sigma(L).$$

Thus $H+\sigma(L) \lhd L$. If $H \operatorname{lsi} L$, then by Lemma 5.2 we have $H \operatorname{ser} L$. Hence $H+\sigma(L) \lhd L$.

Corresponding to Proposition 2.3 we have

PROPOSITION 5.5. *Let L be a Lie algebra and let $(L_\alpha)_{\alpha\leq\sigma}$ be an ascending series of L such that $L_{\alpha+1}/L_\alpha$ is finite-dimensional and simple for any $\alpha<\sigma$. Then the following are equivalent*:

(1) $C_{L/L_\alpha}(L_{\alpha+1}/L_\alpha)=0$ *for all* $\alpha<\sigma$.
(2) *The only serial subalgebras of L are the L_α.*
(3) *The only local subideals of L are the L_α.*
(4) *The only ascendant subalgebras of L are the L_α.*
(5) *The only descendant subalgebras of L are the L_α.*
(6) *The only subideals of L are the L_α.*
(7) *The only ideals of L are the L_α.*
(8) *The only characteristic ideals of L are the L_α.*

PROOF. By Proposition 2.3 we have the equivalence of (1), (4), (5), (6), (7) and (8). By Lemma 5.2 we see that (2)⇒(3). Clearly (3)⇒(6). By Theorem 4.4 we see that (8)⇒(2).

6.

It is trivial that $\mathfrak{X}_2 \leq \mathfrak{X}_1$ and $\mathfrak{X}_2 \cap \mathfrak{F} = \mathfrak{X}_1 \cap \mathfrak{F}$. Since there exists an infinite-dimensional simple Lie algebra ([1, Theorem 10.3.1] or [5, Lemma 4.1]), we have $\mathfrak{X}_2 \lneqq \mathfrak{X}_1$. We have a characterization of an $\mathfrak{X}_1 \cap \mathfrak{F}$-algebra.

PROPOSITION 6.1. *Let L be a Lie algebra.*

(1) *L belongs to $\mathfrak{X}_1 \cap \mathfrak{F}$ if and only if L is finite-dimensional and every 2-step subideal of L is perfect.*

(2) *Over a field of characteristic zero L belongs to $\mathfrak{X}_1 \cap \mathfrak{F}$ if and only if L is finite-dimensional semi-simple.*

PROOF. (1) The necessity follows from Theorem 4.4. Let L be a finite-dimensional Lie algebra in which every 2-step subideal is perfect. Let I be a minimal ideal of L. If $J \lhd I$, then $J=J^2$. By Lemma 1.3 $J \lhd L$. Hence $J=0$

or I. Therefore I is simple. Induction on dim L completes the proof.

(2) First we shall show that if $L \in \mathfrak{X}_1$, then the only LE$\mathfrak{A}$-ideal of L is the zero ideal. Let I be an LE$\mathfrak{A}$-ideal of L and let $(L_\alpha)_{\alpha \leq \sigma}$ be an ascending series of L such that $L_{\alpha+1}/L_\alpha$ is simple for any $\alpha < \sigma$. We show by transfinite induction on α that $I \cap L_\alpha = 0$. If $\alpha = 0$, then there is nothing to prove. Let $\alpha > 0$ and assume that $I \cap L_\beta = 0$ for all $\beta < \alpha$. If α is a limit ordinal, then $I \cap L_\alpha = \bigcup_{\beta < \alpha}(I \cap L_\beta) = 0$. If α is not a limit ordinal, then $((I \cap L_\alpha) + L_{\alpha-1})/L_{\alpha-1}$ is an LE$\mathfrak{A}$-ideal of $L_\alpha/L_{\alpha-1}$. Since $L_\alpha/L_{\alpha-1}$ is simple, we see that $L_\alpha/L_{\alpha-1} = ((I \cap L_\alpha) + L_{\alpha-1})/L_{\alpha-1}$ or $I \cap L_\alpha \leq L_{\alpha-1}$. By [1, Lemma 8.5.4] $L_\alpha/L_{\alpha-1}$ cannot be locally soluble. Hence $I \cap L_\alpha \leq L_{\alpha-1}$. Therefore $I \cap L_\alpha = I \cap L_{\alpha-1} = 0$. Thus if $L \in \mathfrak{X}_1 \cap \mathfrak{F}$, then L is semi-simple. Since a finite-dimensional semi-simple Lie algebra over a field of characteristic zero is a direct sum of simple ideals, we see that if L is finite-dimensional semi-simple over a field of characteristic zero, then $L \in \mathfrak{X}_1 \cap \mathfrak{F}$.

By the proof of Proposition 6.1 (2) we have the following

PROPOSITION 6.2. $\mathfrak{X}_1 \cap$ LE$\mathfrak{A} = (0)$.

Now we define classes $\mathfrak{X}_3$ and $\mathfrak{X}_4$ of Lie algebras as follows: $L \in \mathfrak{X}_3$ if and only if H ser L implies $H \lhd L$. $L \in \mathfrak{X}_4$ if and only if H ser L implies H ch L. In [7] the class $\mathfrak{M}'$ is investigated. Clearly $\mathfrak{X}_4 \leq \mathfrak{X}_3 \leq \mathfrak{M}'$. By [7, Theorem 5.2] E$\mathfrak{A} \cap \mathfrak{M}' =$ E$\mathfrak{A} \cap \mathfrak{T}$, where $\mathfrak{T}$ is the class of Lie algebras in which every subideal is an ideal. We shall show that E$\mathfrak{A} \cap \mathfrak{X}_3 =$ E$\mathfrak{A} \cap \mathfrak{T}$.

LEMMA 6.3. *Let $L = A \dotplus \langle z \rangle$, where A is an abelian ideal of L and* $\mathrm{ad}_A z = \mathrm{id}_A$. *If H is a serial subalgebra of L, then H is an ideal.*

PROOF. Let $a_1, \ldots, a_n \in A$. We put $F = \langle a_1 \rangle + \cdots + \langle a_n \rangle + \langle z \rangle$. Clearly $F \leq L$. Hence $L \in$ L$\mathfrak{F}$. Now suppose that H is a serial subalgebra not contained in A. Then there exists $x = a + z$ in H with $a \in A$. Put $E = \langle a \rangle + \langle z \rangle$. Since H ser L and $L \in$ L$\mathfrak{F}$, $H \cap E$ si E by Lemma 4.2 (1). Since $\dim E \leq 2$, we have $H \cap E \lhd E$. Hence $a = [a + z, z] \in H \cap E$. Therefore $z \in H$. Let $b \in A$ and put $G = \langle b \rangle + \langle z \rangle$. We have $H \cap G \lhd G$ and hence $b = [b, z] \in H \cap G$. Therefore $H = L$. Thus we see that if H ser L, then $H \leq A$ or $H = L$. In each case we have $H \lhd L$.

THEOREM 6.4. E$\mathfrak{A} \cap \mathfrak{X}_3 =$ E$\mathfrak{A} \cap \mathfrak{M}' =$ E$\mathfrak{A} \cap \mathfrak{T}$.

PROOF. Clearly $\mathfrak{X}_3 \leq \mathfrak{M}' \leq \mathfrak{T}$. Let $L \in$ E$\mathfrak{A} \cap \mathfrak{T}$. By [7, Theorem 5.2] $L \in \mathfrak{A}$ or L is as in Lemma 6.3. Thus we have $L \in \mathfrak{X}_3$.

As for $\mathfrak{X}_4$ we have the following

THEOREM 6.5. *$L \in$* E$\mathfrak{A} \cap \mathfrak{X}_4$ *if and only if $L \in \mathfrak{F}_1$ or $L \in \mathfrak{F}_2 \backslash \mathfrak{A}$.*

PROOF. Clearly $\mathfrak{F}_1 \leq$ E$\mathfrak{A} \cap \mathfrak{X}_4$. Let $L \in \mathfrak{F}_2 \backslash \mathfrak{A}$. Since L is complete, we have $\mathrm{Der}(L)=\mathrm{Inn}(L)$. Let $H \operatorname{ser} L$. Since $\dim L=2$, we have $H \lhd L$. Hence $H\,\mathrm{Der}(L) \subseteq H$. Thus $L \in \mathfrak{X}_4$. Conversely let $L \in \mathrm{E}\mathfrak{A} \cap \mathfrak{X}_4$. By [7, Theorem 5.2] $L \in \mathfrak{A}$ or $L=A \dotplus \langle z \rangle$, where A is an $\mathfrak{A}$-ideal of L and $\mathrm{ad}_A z=\mathrm{id}_A$. Clearly $\mathfrak{X}_4 \cap \mathfrak{A} \leq \mathfrak{F}_1$. Let f be an endomorphism of A. Then f induces a derivation $\bar{f}$ of L such that $z\bar{f}=0$. Since $L \in \mathfrak{X}_4$, we have $A \in \mathfrak{F}_1$. Thus we conclude that $L \in \mathfrak{F}_1$ or $L \in \mathfrak{F}_2 \backslash \mathfrak{A}$.

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Department of Mathematics,
Faculty of Science,
Hiroshima University