Hiroshima Math. J.
11 (1981), 173–180

A classification of certain symmetric Lie algebras

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(Received September 12, 1980)

Introduction

Harish-Chandra's profound investigations tell us many important facts concerning semisimple Lie groups. The following is one of them; a connected real semisimple Lie group G has discrete series if and only if G has a compact Cartan subgroup. If a connected noncompact real simple Lie group has a compact Cartan subgroup, its Lie algebra is of inner type. One can find the classification of real simple Lie algebras of inner type in Murakami [6]. For affine symmetric spaces, in [3] Flensted-Jensen has proved a similar result to Harish-Chandra's. That is to say; if an affine symmetric space G/H of a connected noncompact real semisimple Lie group G has a compact Cartan subspace, then the regular representation of G on $L^2(G/H)$ contains closed invariant subspaces. Recently, in [5] Matsumoto has given a sufficient condition for that;

(*) Some representations belonging to the discrete series of a connected semisimple Lie group G appear as closed invariant subspaces in the regular representation of G on $L^2(G/H)$. Here $L^2(G/H)$ is the space of all square integrable functions on an affine symmetric space G/H with respect to the invariant measure.

In this article, we shall sort symmetric real simple Lie algebras which satisfy the condition in [5]. For this, we shall describe the condition in terms of the root theory. Let $(\mathfrak{g}, \sigma)$ be a pair consisting of a real simple Lie algebra and its involutive automorphism (the so-called symmetric Lie algebra). In our case $\mathfrak{g}$ may be determined by a Dynkin diagram and a simple root, and σ will be characterized by a Satake diagram. A classification of all symmetric real simple Lie algebras has been shown by Berger, so that we will search the list in [2]. Also, Flensted-Jensen has obtained a sufficient condition for (*) ([3], Theorem 7.14). In view of the classification, his condition seems different from Matsumoto's.

Throughout the paper, we assume that Lie algebras are defined over the field of real numbers, and we denote the dual space of a real or complex vector space V by V^*. In addition, we denote by V_c the complexification of a real vector space V.

1. Preliminaries

This section is devoted to recalling some basic terminology. Let $\mathfrak{g}$ be a

Lie algebra and σ be an involutive automorphism of $\mathfrak{g}$. Then a pair $(\mathfrak{g}, \sigma)$ is called a *symmetric Lie algebra*. Let $\mathrm{Aut}(\mathfrak{g})$ be the group of all automorphisms of $\mathfrak{g}$ and $\mathrm{Int}(\mathfrak{g})$ be the connected component of the identity element of $\mathrm{Aut}(\mathfrak{g})$. Two symmetric Lie algebras $(\mathfrak{g}, \sigma)$ and $(\mathfrak{g}, \sigma')$, of the same Lie algebra $\mathfrak{g}$, defined by the involutive automorphisms σ and σ', are said to be *isomorphic*, if there exists an automorphism ϕ in $\mathrm{Int}(\mathfrak{g})$ such that $\sigma' = \phi\sigma\phi^{-1}$ ([2]).

Let $(\mathfrak{g}, \sigma)$ be a symmetric Lie algebra. Let $\mathfrak{h}$ denote the 1-eigensubspace of σ in $\mathfrak{g}$ and $\mathfrak{q}$ denote the (-1)-eigensubspace of σ in $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ (direct sum) and $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. We shall refer to this decomposition of $\mathfrak{g}$ as the decomposition with σ. We assume that $\mathfrak{g}$ is noncompact semisimple and we identify the Lie algebra of $\mathrm{Int}(\mathfrak{g})$ with $\mathfrak{g}$. Let H be the analytic subgroup of $\mathrm{Int}(\mathfrak{g})$ corresponding to $\mathfrak{h}$. Then there exists a Cartan involution θ of $\mathfrak{g}$ commutative with σ, and every Cartan involution of $\mathfrak{g}$ commutative with σ is conjugate under H; it can be written as $h\theta h^{-1}$ with an h in H ([4], Lemmas 3 and 4).

By $(\mathfrak{g}, \sigma, \theta)$ we denote a triple consisting of a noncompact semisimple Lie algebra $\mathfrak{g}$, an involutive automorphism σ of $\mathfrak{g}$ and a Cartan involution θ of $\mathfrak{g}$ commutative with σ. Such a triple is called a *symmetric Lie algebra*, too. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition with θ and $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ be the decomposition with σ. Then we have $\mathfrak{g} = \mathfrak{k} \cap \mathfrak{h} + \mathfrak{k} \cap \mathfrak{q} + \mathfrak{p} \cap \mathfrak{h} + \mathfrak{p} \cap \mathfrak{q}$ (direct sum) because θ commutes with σ. Put $\mathfrak{h}^0 = \mathfrak{h} \cap \mathfrak{k} + i(\mathfrak{h} \cap \mathfrak{p})$ and $\mathfrak{q}^0 = i(\mathfrak{k} \cap \mathfrak{q}) + \mathfrak{p} \cap \mathfrak{q}$. We define a real subalgebra $\mathfrak{g}^0$ of $\mathfrak{g}_c$ by $\mathfrak{g}^0 = \mathfrak{h}^0 + \mathfrak{q}^0$. As usual, we extend θ and σ on $\mathfrak{g}_c$ as complex automorphisms. Then it is easy to see that; $\mathfrak{g}^0$ is θ- and σ-stable, σ induces a Cartan involution of $\mathfrak{g}^0$, and $\mathfrak{g}^0 = \mathfrak{h}^0 + \mathfrak{q}^0$ denotes the Cartan decomposition with σ. Thus $(\mathfrak{g}^0, \theta, \sigma)$ makes a symmetric Lie algebra (the so-called *dual* symmetric Lie algebra).

Here we prepare some background material. Let $\mathfrak{g}$ be a *noncompact simple Lie algebra of inner type* and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition. We assume that the center $\mathfrak{z}$ of $\mathfrak{k}$ is nonzero. Then $\dim \mathfrak{z} = 1$, and there exists an element Z in $\mathfrak{z}$ such that $\mathrm{ad}\, Z$ gives a complex structure on $\mathfrak{p}$. Such a Lie algebra is said to be *of Hermitian type*. By $\mathfrak{p}^+$ we denote the i or $(-i)$-eigensubspace of $\mathrm{ad}\, Z$ in $\mathfrak{p}_c$. Fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ contained in $\mathfrak{k}$ (i.e. a compact Cartan subalgebra). Nonzero roots of the pair $(\mathfrak{g}_c, \mathfrak{t}_c)$ are called $\mathfrak{t}$-*roots* (of $\mathfrak{g}$). A $\mathfrak{t}$-root is said to be *compact* (resp. *noncompact*) if its root subspace is contained in $\mathfrak{k}_c$ resp. $\mathfrak{p}_c$). A positive system of $\mathfrak{t}$-roots is called *compatible* if $\mathfrak{g}^+$ contains $\mathfrak{p}^+$, where $\mathfrak{g}^+$ denotes the sum of all root subspaces corresponding to the positive $\mathfrak{t}$-roots. Also, a fundamental system of $\mathfrak{t}$-roots is said to be *compatible* if it is contained in a compatible positive system. It is well-known that the adjoint representation of $\mathfrak{k}_c$ on $\mathfrak{p}^+$ is irreducible. Therefore a compatible fundamental system of $\mathfrak{t}$-roots contains a unique noncompact (simple) root. In the equation, which expresses the maximal root as a positive integral linear combination of simple roots, the coefficient of the noncompact simple root equals 1 ([6], Théorème 1).

In the remainder of this section, we consider the condition in [5], under the assumption that the Lie algebra is *simple*. Let $(\mathfrak{g}, \sigma, \theta)$ be a symmetric noncompact simple Lie algebra. Let $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ denote the decomposition with σ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ denote the Cartan decomposition with θ. In the next place, we restate the condition in [5].

MC_1: $\mathfrak{g}$ is of Hermitian type ([5], A III).

MC_2: Let $\mathfrak{t}_\mathfrak{q}$ be a maximal abelian subspace in $\mathfrak{k}\cap\mathfrak{q}$. Then $\mathfrak{t}_\mathfrak{q}$ is a maximal abelian subspace in $\mathfrak{q}$ consisting of semisimple elements of $\mathfrak{g}$ ([5], A I).

MC_3: $\mathfrak{t}_\mathfrak{q}$ contains the center of $\mathfrak{k}$ ([5], A III).

MC_4: The centralizer of $\mathfrak{t}_\mathfrak{q}$ in $\mathfrak{g}$ is contained in $\mathfrak{k}$ ([5], A II).

Remark; [MC] refers to the above four statements.

LEMMA 1. *Let $(\mathfrak{g}, \sigma, \theta)$ be a symmetric noncompact simple Lie algebra satisfying [MC]. Let $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ denote the decomposition with σ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ denote the Cartan decomposition with θ. Then there exists a compact Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ contained in $\mathfrak{k}$ such that $\mathfrak{t}$ is σ-stable and $\mathfrak{t}\cap\mathfrak{q}$ is a maximal abelian subspace in $\mathfrak{q}$.*

PROOF. Note that $\mathfrak{k}=\mathfrak{k}\cap\mathfrak{h}+\mathfrak{k}\cap\mathfrak{q}$ (direct sum). Let $\mathfrak{t}_\mathfrak{q}$ be a maximal abelian subspace in $\mathfrak{k}\cap\mathfrak{q}$ and $\mathfrak{m}$ be the centralizer of $\mathfrak{t}_\mathfrak{q}$ in $\mathfrak{k}\cap\mathfrak{h}$. Put $\mathfrak{b}=$a Cartan subalgebra of $\mathfrak{m}$ and $\mathfrak{t}=\mathfrak{b}+\mathfrak{t}_\mathfrak{q}$. Obviously $\mathfrak{t}$ is σ-stable. It is easy to see that $[\mathfrak{k}\cap\mathfrak{h}, \mathfrak{k}\cap\mathfrak{q}]$ is contained in $\mathfrak{k}\cap\mathfrak{q}$ and $[\mathfrak{k}\cap\mathfrak{q}, \mathfrak{k}\cap\mathfrak{q}]$ is contained in $\mathfrak{k}\cap\mathfrak{h}$. Therefore a usual argument implies that $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$. Because $\mathfrak{g}$ is of Hermitian type, we have a required Cartan subalgebra of $\mathfrak{g}$.

Such a Cartan subalgebra in Lemma 1 is called *compatible*. Let a symmetric noncompact simple Lie algebra $(\mathfrak{g}, \sigma, \theta)$ satisfy [MC]. Let $(\mathfrak{g}^0, \theta, \sigma)$ be the dual symmetric Lie algebra and $\mathfrak{t}$ be a compatible Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{q}$, $\mathfrak{k}$, and $\mathfrak{p}$ denote as in Lemma 1. Set $\mathfrak{a}=\mathfrak{t}\cap\mathfrak{h}+i(\mathfrak{t}\cap\mathfrak{q})$. Then $\mathfrak{a}$ is a σ-stable Cartan subalgebra of $\mathfrak{g}^0$ and its vector part is maximal. [Let $\mathfrak{g}^0=\mathfrak{h}^0+\mathfrak{q}^0$ denote the Cartan decomposition with σ. MC_2 means that $i(\mathfrak{t}\cap\mathfrak{q})$ is a maximal abelian subspace in $\mathfrak{q}^0$.] Let η be the conjugation of $\mathfrak{g}_c$ relative to $\mathfrak{g}^0$. Note that $\mathfrak{t}_c$ $(=\mathfrak{a}_c)$ is η-stable. For α in $\mathfrak{t}_c^*$, we define $\eta\alpha$ in $\mathfrak{t}_c^*$ by conj. $\alpha(\eta\cdot)$; conj. denotes the complex conjugate. Let (Δ, η) be a pair consisting of $\mathfrak{t}$-roots and the involution on $\mathfrak{t}_c^*$ defined as above. Then the pair (Δ, η) becomes a normal σ-system of roots ([1], 1.5). Let $\{H_j; j=1,..., \dim\mathfrak{t}\}$ be a basis of $i\mathfrak{t}$ such that $\{H_j; j=1,..., \dim\mathfrak{t}\cap\mathfrak{q}\}$ makes a basis of $i(\mathfrak{t}\cap\mathfrak{q})$. By MC_3, we may assume that iH_1 belongs to the center of $\mathfrak{k}$ and gives a complex structure on $\mathfrak{p}$. Let Δ^+ denote the positive system of $\mathfrak{t}$-roots with respect to the lexicographic order on $i\mathfrak{t}^*$ corresponding to H_j's.

LEMMA 2. *Retain above notations. Then Δ^+ is compatible and σ-positive; if a positive root α is not equal to $-\eta\alpha$, then $\eta\alpha$ is positive, too.*

PROOF. ad H_1 is an involution on $\mathfrak{p}_c$, hence Δ^+ is compatible. Every $\mathfrak{t}$-root takes real values on $i\mathfrak{t}$. The conjugation η is $-$*identity* on $i(\mathfrak{t} \cap \mathfrak{h})$ and *identity* on $i(\mathfrak{t} \cap \mathfrak{q})$. These imply the rest of the lemma ([1], 2.8).

In view of Lemma 2, a compatible fundamental system of roots is called *σ-fundamental* if the corresponding positive system becomes σ-positive (with respect to the dual symmetric Lie algebra). Under [MC], a compatible σ-fundamental system will enable us to describe two real forms, symmetric Lie algebra and its dual.

2. The principle of the classification

In this section, we shall state the principle of the classification of symmetric noncompact simple Lie algebras which satisfy [MC]. We start with the following.

DEFINITION. Let (S, β) be a pair consisting of the Satake diagram of a σ-fundamental system of a normally extendable irreducible σ-system of roots ([1], 2.3) and a simple root in S (we shall abuse the notation by treating as a σ-fundamental system). (S, β) is called an *mc-pair* if;

mc_1: In the equation, which expresses the maximal root as a positive integral linear combination of simple roots, the coefficient of β equals 1, and

mc_2: In S, β is denoted by a white circle and connected to no circle (simple root) by a curved arrow.

LEMMA 3. *Let a symmetric noncompact simple Lie algebra $(\mathfrak{g}, \sigma, \theta)$ satisfy [MC]. Let $\mathfrak{t}$ be a compatible Cartan subalgebra of $\mathfrak{g}$. Let (S, β) denote the pair consisting of the Satake diagram of a compatible σ-fundamental system of $\mathfrak{t}$-roots and the unique noncompact simple root. Then (S, β) is an mc-pair.*

PROOF. Clearly MC_1 implies mc_1. Let $(\mathfrak{g}^0, \theta, \sigma)$ denote the dual symmetric Lie algebra of $(\mathfrak{g}, \sigma, \theta)$, and η be the conjugation of $\mathfrak{g}_c$ relative to $\mathfrak{g}^0$. Then η commutes with θ. Hence, if a $\mathfrak{t}$-root α is compact (resp. noncompact), $\eta\alpha$ is compact (resp. noncompact). Let $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ denote the decomposition with σ. Then MC_2 and MC_4 imply that no noncompact root vanishes on $\mathfrak{t} \cap \mathfrak{q}$. This means that β is denoted by a white circle in S. Since S contains a unique noncompact simple root, β cannot be connected to any circles by a curved arrow.

For a symmetric noncompact simple Lie algebra satisfying [MC], the *mc*-pair in Lemma 3 is said to be *associated*. Let (S, β) be an *mc*-pair and D be the Dynkin diagram which has the same shape as S. The Dynkin diagram D determines a simple compact Lie algebra $\mathfrak{u}$. Let $\mathfrak{t}$ denote a Cartan subalgebra of $\mathfrak{u}$ and H_β denote the unique element in $i\mathfrak{t}$ such that $\beta H_\beta = 1$ and $\alpha H_\beta = 0$ for all simple roots α's in S which are different from β, here we identify S with a fundamental system of roots of the pair $(\mathfrak{u}_c, \mathfrak{t}_c)$. Set $\theta = \exp \pi i \operatorname{ad} H_\beta$ in $\mathrm{Int}(\mathfrak{u}_c)$ and $\tau =$ the conjugation of $\mathfrak{u}_c$ relative to $\mathfrak{u}$. Let η be a conjugation of $\mathfrak{u}_c$ characterized by the Satake diagram S. We may assume that $\eta\mathfrak{t} = \mathfrak{t}$. Then mc_2 implies $\eta H_\beta = H_\beta$. Hence η and θ are commutative with each other. Let $\mathfrak{g}$ denote the real form of $\mathfrak{u}_c$ corresponding to the conjugation $\theta\tau$. Théoèrme 1 in [6] implies that $\mathfrak{g}$ is a noncompact simple Lie algebra of Hermitian type and $\mathfrak{t}$ is a compact Cartan subalgebra of $\mathfrak{g}$. Put $\sigma = \eta\tau$. Since $\theta\tau$ is commutative with σ, $\mathfrak{g}$ is σ-stable. Thus we obtain a symmetric noncompact simple Lie algebra $(\mathfrak{g}, \sigma, \theta)$, which satisfies [$MC$]. The above considerations can be summarized as follows.

PROPOSITION 1. *Let (S, β) be an mc-pair. Then there exists a symmetric noncompact simple Lie algebra such that; it satisfies [MC], and the associated mc-pair agrees with (S, β).*

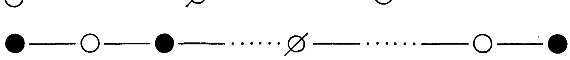
Let two symmetric Lie algebras $(\mathfrak{g}, \sigma, \theta)$ and $(\mathfrak{g}, \sigma', \theta')$ satisfy [MC]. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ and $\mathfrak{g} = \mathfrak{h}' + \mathfrak{q}'$ denote the decomposition with σ and σ' respectively. We assume that they are isomorphic. Let ϕ be an element of $\mathrm{Int}(\mathfrak{g})$ such that $\sigma' = \phi\sigma\phi^{-1}$. Then $\phi\mathfrak{h} = \mathfrak{h}'$, $\phi\mathfrak{q} = \mathfrak{q}'$, and $\phi\theta\phi^{-1}$ is a Cartan involution commutative with σ'. We may assume that $\theta' = \phi\theta\phi^{-1}$ (see the section 1). Let $\mathfrak{t}$ be a compatible Cartan subalgebra of $(\mathfrak{g}, \sigma, \theta)$. $\phi\mathfrak{t}$ becomes a compatible Cartan subalgebra of $(\mathfrak{g}, \sigma', \theta')$. Therefore, the following is an immediate consequence of Corollary 2.15 in [1].

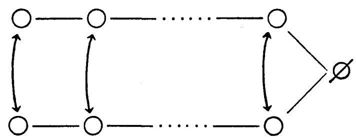
PROPOSITION 2. *Two symmetric noncompact simple Lie algebras which satisfy [MC] are isomorphic, if and only if the associated mc-pairs equal each other.*

3. The assortment

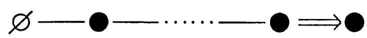
By inspection of the list of Satake diagrams in [1], one obtains the following *mc*-pairs. Here $\varnothing$ denotes the unique noncompact simple root.

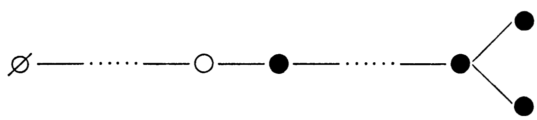
1)

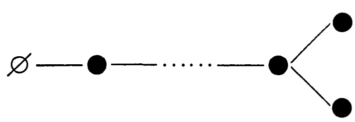
2) 

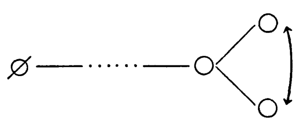
3) 

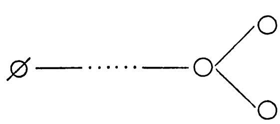
4) 

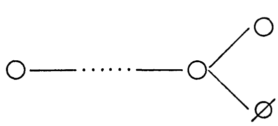
5) 

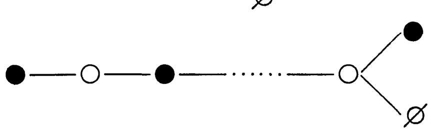
6) 

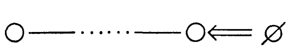
7) 

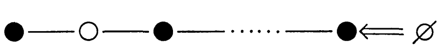
8) 

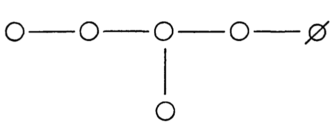
9) 

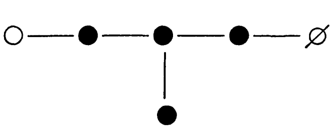
10) 

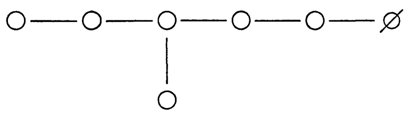
11) 

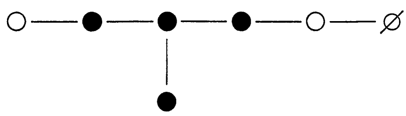
12) 

13) 

14) 

15) 

16) 

17) 

It is interesting to see what an above mc-pair corresponds to in Berger's list ([2], chapitre IV, tableau II). Let $(\mathfrak{g}, \sigma, \theta)$ be a symmetric noncompact simple Lie algebra satisfying $[MC]$. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ denote the Cartan decomposition with θ and $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ denote the decomposition with σ. By $\mathfrak{s}$ we denote the Satake diagram removing $\varnothing$ from the mc-pair associated with $(\mathfrak{g}, \sigma, \theta)$. Put $\mathfrak{k}^1=[\mathfrak{k}, \mathfrak{k}]$. Then the Satake diagram $\mathfrak{s}$ is associated with the compact symmetric Lie algebra $(\mathfrak{k}^1, \sigma)$. Hence one knows the type of $\mathfrak{k}^1 \cap \mathfrak{h}$. MC_3 implies that $\mathfrak{k} \cap \mathfrak{h}=\mathfrak{k}^1 \cap \mathfrak{h}$. The classification in [2] is based on the type of $\mathfrak{k} \cap \mathfrak{h}$; Berger denotes $\mathfrak{k} \cap \mathfrak{h}$ by $\mathfrak{g}_{11}$. By this procedure, one can sort the correspondings to the above mc-pairs 1),..., 17). In the following we give the list (for notations, see [2], chapitre III).

1) $SU^i(n)/SO^i(n)$

2) $SU^{2i}(2n)/Sp^i(n)$

3) $SU^n(2n)/SL(n, \boldsymbol{C}) + \boldsymbol{R}$

4), 5), 6), 7), 8), 9)
$SO^2(n)/SO^1(h) + SO^1(n-h)$ $\quad [SO^1(1) = 0,\ SO^1(2) = \boldsymbol{R}]$

10) $SO^*(2n)/SO(n, \boldsymbol{C})$

11) $SO^*(4n)/SU^*(2n) + \boldsymbol{R}$

12) $Sp(n, \boldsymbol{R})/SL(n, \boldsymbol{R}) + \boldsymbol{R}$

13) $Sp(2n, \boldsymbol{R})/Sp(n, \boldsymbol{C})$

14) $E_6^3/Sp^2(4)$

15) E_6^3/F_4^2

16) $E_7^3/SU^*(8)$

17) $E_7^3/E_6^4 + \boldsymbol{R}$

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