HIROSHIMA MATH. J.
11 (1981), 53–79

Discrete series for an affine symmetric space

Shuichi MATSUMOTO

(Received July 28, 1980)

§ 1. Introduction

We introduce the four dimensional linear space $\boldsymbol{R}^4$ with the bilinear form

$$[x, y] = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$$

defined on it. Let H^3 (resp. H_I^3) be the set of all lines passing through the origin of $\boldsymbol{R}^4$ and lying inside (resp. outside) the cone whose equation is $[x, x] = x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0$, that is, all lines whose points satisfy the inequality $[x, x] < 0$ (resp. $[x, x] > 0$). Then naturally they may be interpreted as open submanifolds of the three dimensional projective space $P^3(\boldsymbol{R})$, and moreover they are homogeneous spaces:

$$H^3 = SO(3, 1)/S(O(3) \times O(1)) \quad \text{and} \quad H_I^3 = SO(3, 1)/S(O(1) \times O(2, 1)).$$

H^3 and H_I^3 are called the Lobachevskian space and the imaginary Lobachevskian space respectively. As is well known, in each $SO(3, 1)$-invariant riemannian structure on H^3 (such a structure exists) the space H^3 is a riemannian symmetric space. However, the imaginary Lobachevskian space H_I^3 has not an $SO(3, 1)$-invariant riemannian structure. Let us now go on to discuss "affine symmetric structure" on the space H_I^3.

For this purpose we consider the involutive automorphism σ of $SO(3, 1)$ defined by $\sigma: g \to J({}^tg)^{-1}J$, where $J = \text{diag.}(-1, 1, 1, -1)$. Then a simple calculation shows that the isotropy subgroup $S(O(1) \times O(2, 1))$ is exactly the set of all fixed points of σ.

On the other hand a manifold M with an affine connection is called an affine symmetric space if each $p \in M$ is an isolated fixed point of an involutive affine transformation s_p of M, which is called the symmetry at p. It is well known that the group of affine transformations $A(M)$ of M is a Lie group (see, [12]). Let $G = A(M)$ and let H be the isotropy subgroup at $p \in M$. Then M can be identified with G/H and s_p induces an involutive automorphism $\sigma: g \to s_p \circ g \circ s_p$ of G such that $(H_\sigma)_0 \subset H \subset H_\sigma$, where H_σ denotes the subgroup of G consisting of fixed points of σ and $(H_\sigma)_0$ is the identity component of H_σ.

Conversely, let G be a Lie group with an involutive automorphism σ and let H be a closed subgroup such that $(H_\sigma)_0 \subset H \subset H_\sigma$. Then the coset space G/H carries a canonical affine connection. Furthermore the manifold G/H is an affine symmetric space with symmetry derived from σ in an obvious manner (see, [20]).

Thus the imaginary Lobachevskian space can be regarded as an affine symmetric space.

In harmonic analysis on homogeneous spaces, riemannian symmetric spaces have been extensively studied. But when "riemannian" is replaced by "affine", systematic studies have been done only for some special cases. For the imaginary Lobachevskian space the work of I. M. Gelfand, M. I. Graef and N. Ya. Vilenkin [5] makes the Plancherel theorem quite explicit. It is very interesting for us to prove the analogue of the Plancherel theorem for a fairly general class of affine symmetric spaces.

From now on, we shall restrict ourselves to an affine symmetric space G/H such that (1) G is a connected non compact semisimple Lie group with finite center, and (2) $H=H_\sigma$. We note that such an affine symmetric space G/H_σ has a G-invariant measure.

Now for a semisimple Lie group G, which may be identified with the affine symmetric space $G\times G/\{(g, g): g\in G\}$, the Plancherel formula has been proved by Harish-Chandra. The first and basic step is the identification of the discrete part of $L^2(G)$. Similarly, when we approach to the Plancherel theorem for an affine symmetric space G/H_σ, we should start with the discrete part of $L^2(G/H_\sigma)$.

By the discrete series for G/H_σ, we shall mean the set of all equivalence classes of the representations of G on minimal closed invariant subspaces of $L^2(G/H_\sigma)$. In this paper we shall prove (Theorem 3 in §7) that if the four assumptions AI$\sim$AIV (see, §2) are satisfied, then some representations of the holomorphic discrete series of G occur in the discrete series for G/H_σ.

The paper is arranged as follows. In Section 2 we introduce the four assumptions AI$\sim$AIV under which we shall discuss the discrete series. Further, for such a space G/H_σ, we fix a Cartan subalgebra $\mathfrak{t}$ of the Lie algebra $\mathfrak{g}$ of G and we define a set L of integral forms on $\mathfrak{t}$. At the end of Section 2, to each element $\Lambda\in L$ we associate in a natural way an irreducible unitary representation $\mathscr{H}_\Lambda$, which is an element of holomorphic discrete series of G. In Section 3 we define a distinguished function f_Λ ($\Lambda\in L$) on H/G_σ. In section 4 we consider an integration formula on G/H_σ. In Section 5 we obtain the next result: if $\Lambda\in L$ is sufficiently regular, then f_Λ is in $L^2(G/H_\sigma)$. In Section 6 we assume that f_Λ ($\Lambda\in L$) is in $L^2(G/H_\sigma)$. Let H_Λ be the closed invariant subspace of $L^2(G/H_\sigma)$ generated by f_Λ. Then we assert that H_Λ is irreducible. In Section 7 we obtain the final result: if $\Lambda\in L$ is sufficiently regular, then $\mathscr{H}_\Lambda$ is a representation of the discrete series for G/H_σ. We shall obtain this by showing that $\mathscr{H}_\Lambda\cong H_\Lambda$.

Throughout the paper let $\boldsymbol{Z}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ be the sets of integers, real numbers and complex numbers respectively. Set $i=(-1)^{1/2}$. For any z in $\boldsymbol{C}$, the complex conjugate will be referred to as $\bar{z}$ or $\{z\}^{-}$. For a real vector space, we use the superscript ^{C} in referring to its complexification. We denote the dual space of a vector space V by V^*.

It is a pleasant duty to express my gratitude to Professor K. Okamoto for his guidance and encouragement.

§ 2. Preliminary arguments

Let G be a connected noncompact semisimple Lie group with finite center. We assume, for convenience, that G has a simply connected complex form G^C. Let $\mathfrak{g}$ be the Lie algebra of G. Let σ be a fixed involutive automorphism of G $(\sigma \neq I)$. We extend σ to an automorphism of G^C and the differential of it will then be denoted by the same letter σ.

Put $H_\sigma = \{g \in G: \sigma g = g\}$, $\mathfrak{h} = \{X \in \mathfrak{g}: \sigma X = X\}$, $\mathfrak{q} = \{X \in \mathfrak{g}: \sigma X = -X\}$. Then $\mathfrak{h}$ is the Lie algebra of the closed subgroup H_σ and $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ (direct sum). Let θ be a fixed Cartan involution of $\mathfrak{g}$ commuting with σ (for the existence, see [14, I, p. 153]) and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition. Then $\theta(\mathfrak{h}) = \mathfrak{h}$, so $\mathfrak{h}$ is reductive. Furthermore since H_σ has only a finite number of connected components ([14, I, p. 171]), the space G/H_σ has an invariant measure dx. We denote by $L^2(G/H_\sigma)$ the Hilbert space of square integrable functions on G/H_σ with respect to dx. Let π be the left regular representation of G on $L^2(G/H_\sigma)$.

DEFINITION 1. By the *discrete series* for G/H_σ, we shall mean the set of all equivalence classes of the representations of G on minimal closed invariant subspaces of $(\pi, L^2(G/H_\sigma))$.

DEFINITION 2. A *Cartan subalgebra* of G/H_σ is an abelian subspace $\mathfrak{a}_\mathfrak{q}$ of $\mathfrak{q}$ satisfying the following conditions:

(1) $\mathfrak{a}_\mathfrak{q}$ is maximal subject to the condition that $[X, Y] = 0$ for X, Y in $\mathfrak{a}_\mathfrak{q}$, and

(2) for each $H \in \mathfrak{a}_\mathfrak{q}$, the endomorphism ad H of $\mathfrak{g}^C$ is semisimple.

In broad outline the main results concerning the Cartan subalgebra may be listed as follows (see, [22]):

(1) There exists at least one Cartan subalgebra of G/H_σ.

(2) Each Cartan subalgebra of G/H_σ is H_σ-conjugate to θ-stable one.

(3) There are only a finite number of H_σ-conjugacy classes of Cartan subalgebras.

(4) Select a maximal set $\mathfrak{a}_{\mathfrak{q},i}$ $(1 \leqq i \leqq r)$ of Cartan subalgebras no two of which are H_σ-conjugate. Then $\bigcup_i \mathrm{Ad}(H_\sigma)\mathfrak{a}_{\mathfrak{q},i}$ is dense in $\mathfrak{q}$.

DEFINITION 3. A Cartan subalgebra $\mathfrak{a}_{\mathfrak{q}}$ of G/H_σ is said to be *compact* if for each $H \in \mathfrak{a}_{\mathfrak{q}}$ the eigenvalues of ad H are all pure imaginary.

A compact Cartan subalgebra is always H_σ-conjugate to one which is contained in $\mathfrak{q} \cap \mathfrak{k}$ (see the statement (2) listed above). On the other hand, when we denote by K the analytic subgroup of G corresponding to $\mathfrak{k}$, $(K, K \cap H_\sigma)$ is a riemannian symmetric pair. Hence all maximal abelian subalgebras of $\mathfrak{q} \cap \mathfrak{k}$ are $K \cap H_\sigma$-conjugate ([9, Ch. V, Lemma 6.3]). Therefore all compact Cartan subalgebras of G/H_σ are H_σ-conjugate.

Now we describe the four assumptions AI$\sim$AIV for the space G/H_σ.

AI: G/H_σ has a compact Cartan subalgebra.

We fix a compact Cartan subalgebra $\mathfrak{t}_{\mathfrak{q}}$ of G/H_σ such that $\mathfrak{t}_{\mathfrak{q}} \subset \mathfrak{q} \cap \mathfrak{k}$.

AII: $Z_G(\mathfrak{t}_{\mathfrak{q}})$ (=the centralizer of $\mathfrak{t}_{\mathfrak{q}}$ in G) is compact.

As was mentioned above, any two compact Cartan subalgebras are conjugate under Ad(H_σ). Hence the assumption AII is independent of the choice of $\mathfrak{t}_{\mathfrak{q}}$. Furthermore from AII we can conclude that $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}_{\mathfrak{q}}) \subset \mathfrak{k}$, where $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}_{\mathfrak{q}})$ is the centralizer of $\mathfrak{t}_{\mathfrak{q}}$ in $\mathfrak{g}$. We fix a maximal abelian subalgebra $\mathfrak{t}_{\mathfrak{h}}$ of $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{t}_{\mathfrak{q}})$ (=the centralizer of $\mathfrak{t}_{\mathfrak{q}}$ in $\mathfrak{h}$), and put $\mathfrak{t} = \mathfrak{t}_{\mathfrak{q}} + \mathfrak{t}_{\mathfrak{h}}$. Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ in the usual sense, and $\mathfrak{t} \subset \mathfrak{k}$. Let $\mathfrak{c}$ be the center of $\mathfrak{k}$ and let $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]$. Then $\mathfrak{k} = \mathfrak{c} + \mathfrak{k}'$ (direct sum) and $\mathfrak{c} \subset \mathfrak{t}$. Since $\sigma(\mathfrak{c}) = \mathfrak{c}$ and $\sigma(\mathfrak{k}') = \mathfrak{k}'$ it follows that $\mathfrak{t}_{\mathfrak{q}} = \mathfrak{c}_{\mathfrak{q}} + \mathfrak{t}'_{\mathfrak{q}}$ and $\mathfrak{t}_{\mathfrak{h}} = \mathfrak{c}_{\mathfrak{h}} + \mathfrak{t}'_{\mathfrak{h}}$ where $\mathfrak{c}_{\mathfrak{q}} = \mathfrak{c} \cap \mathfrak{q}$, $\mathfrak{c}_{\mathfrak{h}} = \mathfrak{c} \cap \mathfrak{h}$ and $\mathfrak{t}'_{\mathfrak{q}} = \mathfrak{t}_{\mathfrak{q}} \cap \mathfrak{k}'$, $\mathfrak{t}'_{\mathfrak{h}} = \mathfrak{t}_{\mathfrak{h}} \cap \mathfrak{k}'$.

AIII: $\mathfrak{c}_{\mathfrak{q}} \neq 0$.

Let Φ be the set of non zero roots of $(\mathfrak{g}^C, \mathfrak{t}^C)$. $\mathfrak{g}_\alpha$ be the root space corresponding to $\alpha \in \Phi$. Then $\mathfrak{g}_\alpha \subset \mathfrak{k}^C$ or $\mathfrak{g}_\alpha \subset \mathfrak{p}^C$, and we call α compact or non compact accordingly. Let Φ_k and Φ_n be the sets of compact and non compact roots, respectively.

AIV: If β is a non compact root then it is not identically zero on $\mathfrak{c}_{\mathfrak{q}}$.

REMARK. *If G is simple,* AIV *holds automatically under the assumptions* AI, AII *and* AIII. This may be proved as follows. If G is simple, then dim $\mathfrak{c} \leqq 1$. Therefore it follows from AIII that dim $\mathfrak{c} = 1$ and $\mathfrak{c}_{\mathfrak{q}} = \mathfrak{c}$. Now let Q be the set of all roots in Φ which are not identically zero on $\mathfrak{c}$. Then Q is contained in Φ_n and, since $\mathfrak{g}$ has center $\{0\}$, Q is not empty. Put $\mathfrak{p}_Q = \sum_{\alpha \in Q} \mathfrak{g}_\alpha$ and let $(\mathfrak{p}_Q)^\perp$ be the orthogonal complement of $\mathfrak{p}_Q$ in $\mathfrak{p}^C$ (under the Hermitian form $B(X, \tilde{\theta}X)$, where B is the Killing form of $\mathfrak{g}^C$ and $\tilde{\theta}$ is the conjugate linear mapping of $\mathfrak{g}^C$ such that $\tilde{\theta} | \mathfrak{g} = \theta$). Let $\mathfrak{k}_Q$ be the centralizer of $(\mathfrak{p}_Q)^\perp$ in $\mathfrak{k}^C$, and let $\mathfrak{g}_Q = \mathfrak{k}_Q + \mathfrak{p}_Q$. Then we shall prove that $[\mathfrak{p}_Q, (\mathfrak{p}_Q)^C] = 0$. Let $X \in \mathfrak{p}_Q$, $Y \in (\mathfrak{p}_Q)^\perp$ and $Z \in \mathfrak{k}^C$. Then since $[\tilde{\theta}Z, X] \in \mathfrak{p}_Q$,

$$B([X, Y], \theta Z) = B(Y, [\theta Z, X]) = 0.$$

But $[X, Y] \in \mathfrak{k}^C$ and so $[X, Y]=0$. We next prove that $\mathfrak{g}_Q$ is an ideal in $\mathfrak{g}^C$. Since $\mathfrak{p}_Q$ is invariant under $\mathrm{ad}\,\mathfrak{k}^C$, we have that $[\mathfrak{k}^C, (\mathfrak{p}_Q)^\perp] \subset (\mathfrak{p}_Q)^\perp$ and $[\mathfrak{k}^C, \mathfrak{k}_Q] \subset \mathfrak{k}_Q$. This implies that $[\mathfrak{k}^C, \mathfrak{g}_Q] \subset \mathfrak{g}_Q$. Moreover $[(\mathfrak{p}_Q)^\perp, \mathfrak{g}_Q]=0$. On the other hand, $\mathfrak{g}^C = \mathfrak{k}^C + (\mathfrak{p}_Q)^\perp + \mathfrak{p}_Q$. Therefore we have only to show that $[\mathfrak{p}_Q, \mathfrak{g}_Q] \subset \mathfrak{g}_Q$. But $\mathfrak{g}_Q = \mathfrak{k}_Q + \mathfrak{p}_Q$ and $[\mathfrak{p}_Q, \mathfrak{k}^C] \subset \mathfrak{p}_Q$, so it is enough to prove that $[\mathfrak{p}_Q, \mathfrak{p}_Q] \subset \mathfrak{k}_Q$. Let $X, Y \in \mathfrak{p}_Q$ and let $Z \in (\mathfrak{p}_Q)^\perp$. Then $[X, Y]$ lies in $\mathfrak{k}^C$ and

$$[[X, Y], Z] = [[X, Z], Y] + [X, [Y, Z]] = 0$$

since $[\mathfrak{p}_Q, (\mathfrak{p}_Q)^\perp]=0$. This implies that $[X, Y] \in \mathfrak{k}_Q$ and therefore $\mathfrak{g}_Q$ is an ideal in $\mathfrak{g}^C$ as asserted. So it follows from the simplicity of G that $\mathfrak{g}_Q = \mathfrak{g}^C$. This implies that $Q = \Phi_n$. Therefore, since $\mathfrak{c} = \mathfrak{c}_{\mathfrak{q}}$, each non compact root does not vanish on $\mathfrak{c}_{\mathfrak{q}}$.

EXAMPLE. From among the affine symmetric spaces in the M. Berger's list [1, p. 157], we choose the spaces which satisfy the our assumptions AI$\sim$AIV. (We restrict ourselves to the case that G is a simple classical group.) They are as follows: $SU(p, q)/SO(p, q)$, $SU(n, n)/SL(n, \boldsymbol{C}) + \boldsymbol{R}$, $SO^*(2n)/SO(n, \boldsymbol{C})$, $SO(2, q)/SO(1, q-k) \times SO(1, k)$, $Sp(n, \boldsymbol{R})/SL(n, \boldsymbol{R}) + \boldsymbol{R}$, $SU(2p, 2q)/Sp(p, q)$, $SO^*(4n)/SU^*(2n) + \boldsymbol{R}$, $Sp(2n, \boldsymbol{R})/Sp(n, \boldsymbol{C})$. (This result was pointed out to me by H. Doi. See [2].)

From now on, in all our discussions we shall always assume the assumptions AI$\sim$AIV.

We fix a basis for the real vector space $i\mathfrak{t}$, the first r_1 members and the second r_2 members of which span $i\mathfrak{c}_{\mathfrak{q}}$ and $i\mathfrak{t}'_{\mathfrak{q}}$ respectively ($r_1 = \dim \mathfrak{c}_{\mathfrak{q}}$, $r_2 = \dim \mathfrak{t}'_{\mathfrak{q}}$). Taking the lexicographic order relative to this basis, we obtain an ordering of Φ such that if $\alpha \in \Phi$, $\alpha > 0$ and $\alpha | \mathfrak{t}_{\mathfrak{q}} \neq 0$, then $-\alpha \circ \sigma > 0$. Set $P = \{\alpha \in \Phi : \alpha > 0\}$, $P_k = P \cap \Phi_k$ and $P_n = P \cap \Phi_n$. Put $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$. Then $\mathfrak{u}$ is a compact real form of $\mathfrak{g}^C$. We denote by $\tilde{\theta}$ and η the conjugations of $\mathfrak{g}^C$ with respect to $\mathfrak{u}$ and $\mathfrak{g}$ respectively. We extend θ, $\tilde{\theta}$ and η to automorphisms of G^C.

LEMMA 1. *For each root $\alpha \in \Phi$ we can choose an element $X_\alpha \in \mathfrak{g}_\alpha$ such that*

(1) $\alpha(H_\alpha) = 2$ *where* $H_\alpha = [X_\alpha, X_{-\alpha}]$,

(2) $\eta(X_\alpha) = \varepsilon_\alpha X_{-\alpha}$ *where* $\varepsilon_\alpha = -1$ *or* 1 *according as* α *is compact or not,*

(3) $\tilde{\theta} X_\alpha = -X_{-\alpha}$, *and*

(4) *if* α *is not identically zero on* $\mathfrak{t}_{\mathfrak{q}}$ *then* $\sigma X_\alpha = -X_{\alpha\circ\sigma}$.

PROOF. For the various roots $\alpha \in \Phi$ we can choose the elements $X'_\alpha \in \mathfrak{g}_\alpha$ such that

(a) $\alpha(H_\alpha) = 2$ where $H_\alpha = [X'_\alpha, X'_{-\alpha}]$, and

(b) $X'_\alpha - X'_{-\alpha}$ and $i(X'_\alpha + X'_{-\alpha})$ both lie in $\mathfrak{u}$ ([9, p. 219]).

It follows from (b) that $\tilde{\theta}X'_\alpha = -X'_{-\alpha}$. Since $\sigma\mathfrak{g}_\alpha = \mathfrak{g}_{\alpha\circ\sigma}$ there exist complex numbers c_α $(\alpha \in \Phi)$ such that $\sigma X'_\alpha = c_\alpha X'_{\alpha\circ\sigma}$. We claim that $|c_\alpha| = 1$. We denote by B the Killing form of $\mathfrak{g}^C$. Then

$$B(H_\alpha, H_\alpha) = B(H_\alpha, [X'_\alpha, X'_{-\alpha}]) = 2B(X'_\alpha, X'_{-\alpha}).$$

On the other hand $\sigma H_\alpha = \sigma[X'_\alpha, X'_{-\alpha}] \in \boldsymbol{C}[X'_{\alpha\circ\sigma}, X'_{-\alpha\circ\sigma}] = \boldsymbol{C}H'_{\alpha\circ\sigma}$. Moreover $(\alpha\circ\sigma)(\sigma H_\alpha) = \alpha(H_\alpha) = 2$, therefore $\sigma H_\alpha = H_{\alpha\circ\sigma}$. Hence

$$B(X'_\alpha, X'_{-\alpha}) = 2^{-1}B(H_\alpha, H_\alpha) = 2^{-1}B(H_{\alpha\circ\sigma}, H_{\alpha\circ\sigma}) = B(X'_{\alpha\circ\sigma}, X'_{-\alpha\circ\sigma}).$$

It is well known that $(X, Y) = -B(X, \tilde{\theta}Y)$ $(X, Y \in \mathfrak{g}^C)$ is a positive definite inner product in $\mathfrak{g}^C$. Put $\|X\| = (X, X)^{1/2}$ $(X \in \mathfrak{g}^C)$. Since $\tilde{\theta}X'_\alpha = -X'_{-\alpha}$ (for all $\alpha \in \Phi$) and $\|\sigma X\| = \|X\|$ $(X \in \mathfrak{g}^C)$,

$$\|\sigma X'_\alpha\|^2 = B(X'_\alpha, X'_{-\alpha}) = B(X'_{\alpha\circ\sigma}, X'_{-\alpha\circ\sigma}) = \|X'_{\alpha\circ\sigma}\|^2.$$

This implies that $|c_\alpha| = 1$.

Now $\sigma X'_{-\alpha} = -\sigma\tilde{\theta}X'_\alpha = -\tilde{\theta}\sigma X'_\alpha = -\tilde{\theta}(c_\alpha X'_{\alpha\circ\sigma}) = \bar{c}_\alpha X'_{-\alpha\circ\sigma}$, hence $c_{-\alpha} = \bar{c}_\alpha$. Moreover $\sigma X'_{-\alpha\circ\sigma} = -\sigma\tilde{\theta}X'_{\alpha\circ\sigma} = -\tilde{\theta}(c_\alpha^{-1}X'_\alpha) = c_\alpha X'_{-\alpha}$, hence $c_{-\alpha\circ\sigma} = c_\alpha$. We know that if α is positive and not identically zero on $\mathfrak{t}_\mathfrak{q}$ then $-\alpha\circ\sigma$ is positive. Therefore, for each root α which does not vanish on $\mathfrak{t}_\mathfrak{q}$, we can take a number a_α such that

$$c_\alpha = -a_\alpha^2, \quad \bar{a}_{-\alpha} = a_\alpha = a_{-\alpha\circ\sigma}.$$

Set $X_\alpha = \bar{a}_\alpha X'_\alpha$ or X'_α according as $\alpha | \mathfrak{t}_\mathfrak{q} \neq 0$ or $=0$, respectively. If $\alpha | \mathfrak{t}_\mathfrak{q} \neq 0$, then

$$\sigma X_\alpha = \bar{a}_\alpha \sigma X'_\alpha = \bar{a}_\alpha c_\alpha X'_{\alpha\circ\sigma} = -a_\alpha X'_{\alpha\circ\sigma} = -\bar{a}_{\alpha\circ\sigma} X'_{\alpha\circ\sigma} = -X_{\alpha\circ\sigma}$$

and $[X_\alpha, X_{-\alpha}] = [\bar{a}_\alpha X'_\alpha, \bar{a}_{-\alpha} X'_{-\alpha}] = [X'_\alpha, X'_{-\alpha}] = H_\alpha$, moreover $\tilde{\theta}X_\alpha = a_\alpha \tilde{\theta}X'_\alpha = -\bar{a}_{-\alpha}X'_{-\alpha} = -X_{-\alpha}$. Hence the conditions (1), (3) and (4) hold. Since $\eta = \tilde{\theta}\circ\theta$, (2) is immediate. The proof is now complete.

Choose and fix the elements X_α $(\alpha \in \Phi)$ as in Lemma 1.

LEMMA 2. *Any non compact positive root is totally positive.*

PROOF. Let β be a non compact positive root. Then $\beta | \mathfrak{c}_\mathfrak{q} \neq 0$. Let $\alpha_1, \ldots, \alpha_k$ be all the positive compact roots of $\mathfrak{g}^C$, and suppose that $\gamma = \beta + n_1\alpha_1 + \cdots + n_k\alpha_k$ $(n_j \in \boldsymbol{Z})$ is a root. Then since α_j are all identically zero on $\mathfrak{c}_\mathfrak{q}$, $\gamma - \beta$ vanishes on $\mathfrak{c}_\mathfrak{q}$. Hence it follows from our definition of the order on Φ that γ is positive. This shows that β is totally positive (see, p. 759 of [6]).

LEMMA 3.

(1) *Let β and γ be non compact positive roots. Then $[X_\beta, X_\gamma] = 0$, that is, $\beta + \gamma \notin \Phi$.*

(2) *Put* $\mathfrak{p}_+ = \sum_{\gamma \in P_n} \boldsymbol{C} X_\gamma$ *and* $\mathfrak{p}_- = \sum_{\gamma \in P_n} \boldsymbol{C} X_{-\gamma}$. *Then* $[\mathfrak{k}^C, \mathfrak{p}_+] \subset \mathfrak{p}_+$ *and* $[\mathfrak{k}^C, \mathfrak{p}_-] \subset \mathfrak{p}_-$.

(3) *Let* β *be a non compact positive root. Then* $s\beta$ $(s \in W_k)$ *is also non compact positive, where* W_k *denotes the Weyl group of* $(\mathfrak{k}^C, \mathfrak{t}^C)$.

For a proof, see [6, §4].

Let Σ be the set of all non zero roots of $\mathfrak{g}^C$ with respect to $\mathfrak{t}_{\mathfrak{q}}^C$. Then Σ is exactly the set of restrictions to $\mathfrak{t}_{\mathfrak{q}}^C$ of the elements of Φ which do not vanish on $\mathfrak{t}_{\mathfrak{q}}^C$. Fix an ordering of Σ which satisfies the condition:

$$\text{If} \quad \alpha \in P \quad \text{and} \quad \alpha \,|\, \mathfrak{t}_{\mathfrak{q}}^C \neq 0, \quad \text{then} \quad \alpha \,|\, \mathfrak{t}_{\mathfrak{q}}^C > 0 \quad \text{in} \quad \Sigma.$$

Set $\Sigma_+ = \{\lambda \in \Sigma : \lambda > 0\}$. For each $\lambda \in \Sigma$, set $\mathfrak{g}_\lambda = \{X \in \mathfrak{g}^C : [H, X] = \lambda(H)X$, for all $H \in \mathfrak{t}_{\mathfrak{q}}^C\}$ and take the element $H'_\lambda \in \mathfrak{t}_{\mathfrak{q}}^C$ such that $B(H'_\lambda, \cdot) = \lambda$. Put $H_\lambda = 2\{\lambda(H'_\lambda)\}^{-1} H'_\lambda$.

LEMMA 4. *If* $\lambda \in \Sigma$, *then* $\mathfrak{g}_\lambda$ *is contained either in* $\mathfrak{k}^C$ *or in* $\mathfrak{p}^C$.

PROOF. For each $\lambda \in \Sigma$, put $\Sigma(\lambda) = \{\alpha \in \Phi : \alpha \,|\, \mathfrak{t}_{\mathfrak{q}}^C = \lambda\}$. Then $\mathfrak{g}_\lambda = \sum_{\alpha \in \Sigma(\lambda)} \mathfrak{g}_\alpha$. On the other hand a root $\alpha \in \Phi$ is identically zero on $\mathfrak{c}_{\mathfrak{q}}$ or not according as $\alpha \in \Phi_k$ or $\alpha \in \Phi_n$. Therefore $\mathfrak{g}_\lambda \subset \mathfrak{k}^C$ or $\mathfrak{g}_\lambda \subset \mathfrak{p}^C$ according as $\lambda \,|\, \mathfrak{c}_{\mathfrak{q}} = 0$ or $\neq 0$. This implies our assertion.

Let $\{\beta_1, \ldots, \beta_l\}$ be the set of all simple roots in P. We may assume that $\beta_j \in P_n$ $(1 \leqq j \leqq t)$, $\beta_j \in P_k$ $(t < j \leqq l)$ and that $\beta_j \,|\, \mathfrak{t}_{\mathfrak{q}}^C \neq 0$ $(1 \leqq j \leqq q)$, $\beta_j \,|\, \mathfrak{t}_{\mathfrak{q}}^C = 0$ $(q < j \leqq l)$, where $1 \leqq t \leqq q \leqq l$. There exists a permutation $i \to i'$ of order 2 of the set $\{1, \ldots, q\}$ such that

$$-\beta_i \circ \sigma = \beta_{i'} + \sum_{j=q+1}^{l} n_j^i \beta_j \qquad (n_j^i \in \boldsymbol{Z},\ n_j^i \geqq 0)$$

(see, [28, p. 23]). It is obvious that the sets $\{1, \ldots, t\}$ and $\{t+1, \ldots, q\}$ are stable under the assignment $i \to i'$. So we may assume that

$$i' = \begin{cases} i & \text{for} \quad 1 \leqq i \leqq t_1, \\ i + t_2 & \text{for} \quad t_1 < i \leqq t_1 + t_2, \\ i - t_2 & \text{for} \quad t_1 + t_2 < i \leqq t, \end{cases}$$

$$i' = \begin{cases} i & \text{for} \quad t + 1 \leqq i \leqq t + s_1, \\ i + s_2 & \text{for} \quad t + s_1 < i \leqq t + s_1 + s_2, \\ i - s_2 & \text{for} \quad t + s_1 + s_2 < i \leqq q, \end{cases}$$

where $t = t_1 + 2t_2$, $q - t = s_1 + 2s_2$. Set $\mu_j = \beta_j \,|\, \mathfrak{t}_{\mathfrak{q}}^C$ $(1 \leqq j \leqq t_1 + t_2)$, $\mu_j = \beta_{j+t_2} \,|\, \mathfrak{t}_{\mathfrak{q}}^C$ $(t_1 + t_2 < j \leqq t_1 + t_2 + s_1 + s_2)$. Let $p = t_1 + t_2$ and $r = t_1 + t_2 + s_1 + s_2$. Then it follows from the definition of μ_j that the set $\{H_{\mu_1}, \ldots, H_{\mu_r}\}$ is linearly independent and spans $\mathfrak{t}_{\mathfrak{q}}^{C}$. It is obvious that every element in Σ_+ can be written as an integral linear combination of $\{\mu_1,\dots,\mu_r\}$ where the integers are all non negative. From the $0-1$ property of coefficients of the non compact simple roots ([6, p. 761]) we obtain the following lemma.

LEMMA 5. *Suppose* $\lambda=n_1\mu_1+\cdots+n_r\mu_r$ *(n_j are all non negative integers) is a root in* Σ_+. *Then* $n_1+\cdots+n_p=0$ *or* 1 *according as* $\mathfrak{g}_\lambda\subset\mathfrak{k}^C$ *or* $\mathfrak{g}_\lambda\subset\mathfrak{p}^C$.

Now set $\Gamma_{\mathfrak{t}}=\{H\in\mathfrak{t}\colon \exp H=1\}$ and set $\Gamma_{\mathfrak{t}_{\mathfrak{q}}}=\{H\in\mathfrak{t}_{\mathfrak{q}}\colon \exp H\in H_\sigma\}$. Let U be the analytic subgroup of G^C corresponding to $\mathfrak{u}$. Then since G^C is simply connected, U is simply connected (note that U is a maximal compact subgroup of G^C). Therefore Theorem 4.6.7 of [27] says that $\Gamma_{\mathfrak{t}}$ is the lattice generated by $\{2\pi iH_{\beta_1},\dots,2\pi iH_{\beta_l}\}$. We define the roots λ_j $(j=1,\dots,r)$ in Σ by

$$\lambda_j=\begin{cases}\mu_j & \text{if}\quad 2\mu_j\notin\Sigma\,,\\ 2\mu_j & \text{if}\quad 2\mu_j\in\Sigma\,.\end{cases}$$

LEMMA 6. $\Gamma_{\mathfrak{t}_{\mathfrak{q}}}$ *is the lattice generated by* $\{\pi iH_{\lambda_1},\dots,\pi iH_{\lambda_r}\}$.

PROOF. Let $H_U=\{g\in U\colon \sigma g=g\}$. Then H_U is connected [9, p. 272] and so U/H_U is simply connected. Therefore we conclude from [14, II, p. 77] that the lattice $\Gamma_{\mathfrak{t}_{\mathfrak{q}}}$ is generated by the vectors πiH_λ $(\lambda\in\Sigma)$. Hence it is enough to prove that H_λ is in $\sum_{j=1}^{r}\boldsymbol{Z}H_{\lambda_j}$ for each λ in Σ. If λ and 2λ are both in Σ then, obviously, $H_\lambda=2H_{2\lambda}$. Therefore we need only to show that H_λ is in $\sum_{j=1}^{r}\boldsymbol{Z}H_{\lambda_j}$ for each $\lambda\in\Sigma_*$. Here $\Sigma_*=\{\lambda\in\Sigma\colon 2\lambda\notin\Sigma\}$. Let W_Σ be the Weyl group of $(\mathfrak{g}^C, \mathfrak{t}_{\mathfrak{q}}^C)$. Then one can show by standard arguments that

(a) W_Σ acts transitively on the Weyl chambers of $\mathfrak{t}_{\mathfrak{q}}^C$,

(b) W_Σ is generated by the s_{λ_j} $(j=1,\dots,r)$, where s_{λ_j} is the Weyl reflection with respect to λ_j, and

(c) if λ and $c\lambda$ $(c\in\boldsymbol{C})$ are in Σ_* then $c=\pm1$.

Fix λ in Σ^*. By (c) there exists an element H_0 in $i\mathfrak{t}_{\mathfrak{q}}$ so that (1) $\lambda(H_0)=0$, (2) if $\mu\neq\pm\lambda$ $(\mu\in\Sigma_*)$ then $\mu(H_0)\neq0$. Let B be a sufficiently small open ball $(H_0\in\mathrm{B})$ in $i\mathfrak{t}_{\mathfrak{q}}$ such that if $\mu\neq\pm\lambda$ $(\mu\in\Sigma_*)$ then the real numbers $\mu(H)$ and $\mu(H_0)$ have the same sign for each H in B. Let Q be the Weyl chamber containing $\mathrm{B}\cap\{H\in i\mathfrak{t}_{\mathfrak{q}}\colon \lambda(H)>0\}$. Let $\mathrm{Q}^+=\{H\in i\mathfrak{t}_{\mathfrak{q}}\colon \mu(H)>0$ for all μ in $\Sigma_+\}$. Then by (a) one can choose an element s in W_Σ so that $s\mathrm{Q}^+=\mathrm{Q}$. We assert that $\lambda=s\lambda_j$ or $-s\lambda_j$ for some j $(1\leqq j\leqq r)$. For otherwise suppose $\lambda\neq\pm s\lambda_j$ for all j. Then since $s\lambda_j(H)=\lambda_j(s^{-1}H)>0$ for any H in $\mathrm{B}\cap\{H\in i\mathfrak{t}_{\mathfrak{q}}\colon \lambda(H)>0\}$, $s\lambda_j(H_0)>0$ $(j=1,\dots,r)$. But this means that $s^{-1}H_0\in\mathrm{Q}^+$, and so $H_0\in s\mathrm{Q}^+=\mathrm{Q}$. This is a contradiction, and therefore our assertion is true. Thus $H_\lambda=sH_{\lambda_j}$ or $-sH_{\lambda_j}$ for some j. This combined with (b) says that H_λ is in $\sum\boldsymbol{Z}H_{\lambda_j}$. Our proof is now complete.

We say that $\Lambda\in(\mathfrak{t}^C)^*$ is an integral form on $\mathfrak{t}^C$ if $\Lambda(H)\in2\pi i\boldsymbol{Z}$ for all $H\in\Gamma_{\mathfrak{t}}$.

Similarly $\Lambda \in (\mathfrak{t}_{\mathfrak{q}}^{C})^*$ is called an integral form on $\mathfrak{t}_{\mathfrak{q}}^{C}$ if $\Lambda(H) \in 2\pi i\boldsymbol{Z}$ for all $H \in \Gamma_{\mathfrak{t}_{\mathfrak{q}}}$. If we extend an integral form Λ on $\mathfrak{t}_{\mathfrak{q}}^{C}$ to all of $\mathfrak{t}^{C}$ by rendering it trivial on $\mathfrak{t}_{\mathfrak{h}}^{C}$, we get an integral form on $\mathfrak{t}^{C}$. Thus we can regard integral forms on $\mathfrak{t}_{\mathfrak{q}}^{C}$ as those on $\mathfrak{t}^{C}$.

Let L be the set of all integral forms Λ on $\mathfrak{t}_{\mathfrak{q}}^{C}$ such that

(1) $\Lambda(H_\alpha) \geqq 0$ for all α in P_k, and
(2) $(\Lambda+\rho)(H_\gamma)<0$ for all γ in P_n, where $2\rho=\sum_{\alpha\in P}\alpha$.

Then it follows from Lemma 5 that L is an infinite set.

Put $\mathfrak{n}=\sum_{\alpha\in P}\boldsymbol{C}X_\alpha$ and $\bar{\mathfrak{n}}=\sum_{\alpha\in P}\boldsymbol{C}X_{-\alpha}$. We denote by T, T^C, N, $\bar{N}$ the real analytic subgroups of G^C corresponding to $\mathfrak{t}$, $\mathfrak{t}^C$, $\mathfrak{n}$, $\bar{\mathfrak{n}}$ respectively. Then $GT^C\bar{N}$ is open in G^C (see, [7, p. 3]). For any Λ in L we can define the character ξ_Λ on T^C so that $\xi_\Lambda(\exp H)=e^{\Lambda(H)}$ $(H \in \mathfrak{t}^C)$. Let Γ_Λ be the set of all holomorphic functions ψ on $GT^C\bar{N}$ such that $\psi(wt\bar{n})=\xi_\Lambda(t^{-1})\psi(w)$ $(w \in GT^C\bar{N},\ t \in T^C,\ \bar{n} \in \bar{N})$. For each ψ in Γ_Λ let $\|\psi\|^2=\int_G |\psi(g)|^2 dg$, where dg is an invariant measure on G. Let $\mathscr{H}_\Lambda$ be the subspace of Γ_Λ of functions of finite norm. Then $\mathscr{H}_\Lambda$ is a Hilbert space and we can define the action of G on it by $U_\Lambda(g)\psi(x)=\psi(g^{-1}x)$. The work of Harish-Chandra [8] tells us that if Λ is in L then $(U_\Lambda, \mathscr{H}_\Lambda)$ is an irreducible unitary representation of G.

§ 3. Construction of f_Λ

We define $\Lambda_j \in (\mathfrak{t}_{\mathfrak{q}}^{C})^*$ by $\Lambda_j(H_{\lambda_k})=2\delta_{jk}$ $(1\leqq j, k\leqq r)$. Then $\Lambda_1,\dots,\Lambda_r$ are integral forms on $\mathfrak{t}_{\mathfrak{q}}^{C}$. Clearly $\Lambda_j(H_\lambda)\geqq 0$ for all $\lambda \in \Sigma_+$ $(1\leqq j\leqq r)$.

Now let Λ be an element in L. Define a linear form Λ_0 on $\mathfrak{t}_{\mathfrak{q}}^{C}$ by the conditions $\Lambda_0(H_{\lambda_j})=0$ $(1\leqq j\leqq p)$ and $\Lambda_0(H_{\lambda_j})=\Lambda(H_{\lambda_j})$ $(p<j\leqq r)$. Then Λ_0 is an integral form on $\mathfrak{t}_{\mathfrak{q}}^{C}$, and furthermore $\Lambda_0(H_\lambda)\geqq 0$ for all $\lambda \in \Sigma_+$. Put $\Lambda_-=\Lambda-\Lambda_0$, then $\Lambda_-(H_{\lambda_j})=\Lambda(H_{\lambda_j})\in 2\boldsymbol{Z}$ $(1\leqq j\leqq p)$ and $\Lambda_-(H_{\lambda_j})=0$ $(p<j\leqq r)$. So if we put $m_j=2^{-1}\Lambda(H_{\lambda_j})$ $(1\leqq j\leqq p)$, then m_j are integers and $\Lambda_-=m_1\Lambda_1+\cdots+m_p\Lambda_p$. On the other hand $\rho(H_\alpha)>0$ for $\alpha\in P$, hence $\Lambda(H_\alpha)<-\rho(H_\alpha)<0$ for $\alpha \in P_n$ and therefore m_j $(1\leqq j\leqq p)$ are negative.

LEMMA 7. *Let X be a connected simply connected Lie group. Then the fixed point set of any involutive automorphism of X is connected.*

For a proof, see [13, p. 293].

Now let $G^0=\{g\in G^C\colon \sigma\circ\tilde{\theta}(g)=g\}$. Then Lemma 7 implies that G^0 is a connected closed subgroup of G^C. Let $\mathfrak{g}^0=\{X\in\mathfrak{g}^C\colon \sigma\circ\tilde{\theta}(X)=X\}$. Then $\mathfrak{g}^0$ is the Lie algebra of G^0 and $\mathfrak{g}^0=(\mathfrak{h}\cap\mathfrak{k})+i(\mathfrak{h}\cap\mathfrak{p})+i(\mathfrak{q}\cap\mathfrak{k})+(\mathfrak{q}\cap\mathfrak{p})$. Therefore G^0 is a real form of G^C. Moreover the restriction $\sigma^0=\sigma\,|\,\mathfrak{g}^0$ of σ to $\mathfrak{g}^0$ is a Cartan involution of $\mathfrak{g}^0$. Put $\mathfrak{h}^0=(\mathfrak{h}\cap\mathfrak{k})+i(\mathfrak{h}\cap\mathfrak{p})$ and put $\mathfrak{q}^0=i(\mathfrak{q}\cap\mathfrak{k})+(\mathfrak{q}\cap\mathfrak{p})$. Then $\mathfrak{g}^0=\mathfrak{h}^0+\mathfrak{q}^0$ is the Cartan decomposition of $\mathfrak{g}^0$ corresponding to σ^0 and $i\mathfrak{t}_{\mathfrak{q}}$ is a maximal abelian subspace of $\mathfrak{q}^0$. Moreover $\mathfrak{t}_{\mathfrak{h}}$ is a maximal abelian subalgebra of $\mathfrak{z}_{\mathfrak{h}^0}(i\mathfrak{t}_{\mathfrak{q}})$. Let H^0 be the analytic subgroup of G^C corresponding to $\mathfrak{h}^0$.

For each dominant integral form μ on $\mathfrak{t}^C$ (i.e. integral form on $\mathfrak{t}^C$ such that $\mu(H_\alpha)\geqq 0$ for all α in P), we consider the irreducible holomorphic representation τ_μ of G^C on the finite dimensional vector space V_μ with the highest weight μ. Since U is compact, we can regard V_μ as a Hilbert space in such a way that τ_μ becomes unitary on U. (An inner product is assumed to be linear in the first variable and conjugate linear in the second.) Set $V_{\mu,H^0}=\{\psi\in V_\mu\colon \tau_\mu(h)\psi=\psi$ for all h in $H^0\}$ and let ϕ_μ be the unit vector in V_μ belonging to the weight μ.

LEMMA 8. *Fix a dominant integral form μ on $\mathfrak{t}^C$ which satisfies the conditions*

(1) $\mu\,|\,\mathfrak{t}_{\mathfrak{h}}^C=0$, *and*

(2) $\mu(H_\lambda)/2$ *is a non negative integer for each λ in Σ_+.*

Then $V_{\mu,H^0}\neq 0$ *and* $\dim V_{\mu,H^0}=1$. *Moreover if ψ is a non zero vector in V_{μ,H^0}, then $(\psi,\phi_\mu)\neq 0$.*

PROOF. Theorem 3.3.1.1 of [28, p. 210] says that $V_{\mu,H^0}\neq 0$. Now put $\mathfrak{n}_-^0=(\sum_{\lambda\in\Sigma_+}\mathfrak{g}_{-\lambda})\cap\mathfrak{g}^0$ then $\mathfrak{g}^0=\mathfrak{n}_-^0+(i\mathfrak{t}_{\mathfrak{q}})+\mathfrak{h}^0$ is an Iwasawa decomposition of $\mathfrak{g}^0$ and therefore $U(\mathfrak{g}^0)^C=U(\mathfrak{n}_-^0)^C U(i\mathfrak{t}_{\mathfrak{q}})^C U(\mathfrak{h}^0)^C$, where $U(\cdot)$ denotes the corresponding universal enveloping algebra. Observe that since τ_μ is unitary on U, the adjoint of the operator $\tau_\mu(X)$ is $-\tau_\mu(\tilde{\theta}(X))$ $(X\in\mathfrak{g}^C)$. Let ψ be a non zero vector in V_{μ,H^0}. We define the function F on $U(\mathfrak{g}^0)^C$ by

$$F(u) = (\tau_\mu(u)\psi,\ \phi_\mu)\,.$$

Since $\tilde{\theta}(\mathfrak{n}_-^0)\subset\mathfrak{n}\,(=\sum_{\alpha\in P}\boldsymbol{C}X_\alpha)$ and ϕ_μ belongs to the highest weight, $F(U(\mathfrak{g}^0)^C)\subset\boldsymbol{C}(\psi,\phi_\mu)$. Therefore if $(\psi,\phi_\mu)=0$ then $F=0$. But τ_μ is irreducible and so $\tau_\mu(U(\mathfrak{g}^0)^C)\psi=V_\mu$, this implies that $F\neq 0$. Hence $(\psi,\phi_\mu)\neq 0$. Now we consider the linear mapping: $V_{\mu,H^0}\ni\psi\to(\psi,\phi_\mu)\in\boldsymbol{C}$. Then it follows from the above argument that this mapping is injective and therefore $\dim V_{\mu,H^0}=1$. Hence the lemma follows.

Recall that Λ_j $(0\leqq j\leqq p)$ are all integral forms on $\mathfrak{t}_{\mathfrak{q}}^C$ and that $\Lambda_j(H_\lambda)\geqq 0$ $(0\leqq j\leqq p,\ \lambda\in\Sigma_+)$. Since $\Gamma_{\mathfrak{t}}$ and $\Gamma_{\mathfrak{t}_{\mathfrak{q}}}$ are generated by $\{2\pi iH_\beta\colon\beta\in P\}$ and $\{\pi iH_\lambda\colon\lambda\in\Sigma_+\}$ respectively, the forms Λ_j $(0\leqq j\leqq p)$ satisfy the conditions in Lemma 8. For simplicity we shall write τ_j, V_j, ϕ_j instead of τ_{Λ_j}, V_{Λ_j}, ϕ_{Λ_j} respectively. We can choose elements ψ_j in V_j such that (1) $\tau_j(h)\psi_j=\psi_j$ for all h in H^0, and (2) $(\psi_j,\phi_j)=1$ (Lemma 8).

LEMMA 9. *$\tau_j(h)\psi_j=\psi_j$ for all h in H_σ $(0\leqq j\leqq p)$.*

PROOF. Let $H_\sigma^C=\{g\in G^C: \sigma g=g\}$. Then H_σ^C is a connected complex group (Lemma 7). Clearly, the subgroups H_σ and H^0 are both real forms of H_σ^C. On the other hand $(\tau_j\,|\,H_\sigma^C, V_j)$ is a holomorphic representation of H_σ^C and $\tau_j(H^0)\psi_j=\psi_j$. Hence $\tau_j(H_\sigma^C)\psi_j=\psi_j$ and therefore $\tau_j(H_\sigma)\psi_j=\psi_j$.

LEMMA 10. $(\phi_j, \tau_j(x)\psi_j)\neq 0$ *for all* x *in* G $(1\leqq j\leqq p)$.

PROOF. Fix an index j $(1\leqq j\leqq p)$. It is known that the mapping: $K\times(\mathfrak{q}\cap\mathfrak{p})\times(\mathfrak{h}\cap\mathfrak{p})\ni(k, Y, Z)\to k\cdot\exp Y\cdot\exp Z\in G$ is a diffeomorphism ([14, I, p. 161]). Therefore it is enough to prove $(\phi_j, \tau_j(k\cdot\exp Y)\psi_j)\neq 0$ for a pair (k, Y) in $K\times(\mathfrak{q}\cap\mathfrak{p})$. Let K' be the analytic subgroup of G corresponding to $\mathfrak{k}'$. Then we can write $k=k'\cdot\exp H_0$ where $k'\in K'$ and $H_0\in\mathfrak{c}$. On the other hand Lemma 5 says that the set $\{H_{\lambda_{p+1}},\dots, H_{\lambda_r}\}$ spans $(\mathfrak{t}'_\mathfrak{q})^C$ and so, from the definition of Λ_j, we have $\Lambda_j(H)=0$ for all H in $\mathfrak{t}'^C$ (note: $\mathfrak{t}'=\mathfrak{t}'_\mathfrak{q}+\mathfrak{t}'_\mathfrak{h}$). Since $\tau_j(X_\alpha)\phi_j=0$ for all α in P_k, it follows from Lemma 1 of [6] that $\tau_j(X_{-\alpha})\phi_j=0$ for α in P_k. Hence $\tau_j(X)\phi_j=0$ for X in $\mathfrak{k}'^C$, and so $\tau_j(k')\phi_j=\phi_j$. Therefore

$$(\phi_j, \tau_j(k\cdot\exp Y)\psi_j) = e^{-\Lambda_j(H_0)}(\phi_j, \tau_j(\exp Y)\psi_j).$$

So we have only to prove that $(\phi_j, \tau_j(\exp Y)\psi_j)\neq 0$. If N_-^0 is the analytic subgroup of G corresponding to $\mathfrak{n}_-^0$, then $G^0=N_-^0\exp(i\mathfrak{t}_\mathfrak{q})\cdot H^0$ is an Iwasawa decomposition of G^0. Since $\exp Y$ is in G^0, we can find elements $n_-^0\in N_-^0$, $H\in\mathfrak{t}_\mathfrak{q}$ and $h^0\in H^0$ such that $\exp Y=n_-^0\exp(iH)h^0$. Noting that $\tilde{\theta}(\mathfrak{n}_-^0)\subset\mathfrak{u}$ and that $(\phi_j, \psi_j)=1$, we get $(\phi_j, \tau_j(\exp Y)\psi_j)=e^{i\Lambda_j(H)}\neq 0$. Our proof is now complete.

We recall the relation $\Lambda-\Lambda_0=\Lambda_-=\sum_{j=1}^p m_j\Lambda_j$ (m_j are all negative integers). We define the function f_Λ on G/H_σ by

$$f_\Lambda(x) = (\phi_0, \tau_0(x)\psi_0)\prod_{j=1}^p(\phi_j, \tau_j(x)\psi_j)^{m_j} \qquad (x\in G).$$

Then it is a well defined C^∞ function (Lemmas 9, 10). Let π be the representation of G on $C^\infty(G/H_\sigma)$ given by $\pi(g)f(x)=f(g^{-1}x)$. Here $C^\infty(G/H_\sigma)$ denotes the space of C^∞ complex valued functions on G/H_σ.

LEMMA 11. $\pi(H)f_\Lambda = \Lambda(H)f_\Lambda \quad (H\in\mathfrak{t}^C), \qquad \pi(X_\alpha)f_\Lambda = 0 \quad (\alpha\in P).$

PROOF. For any H in $\mathfrak{t}$, $(\phi_j, \tau_j(\exp(-H)x)\psi_j)=e^{\Lambda_j(H)}(\phi_j, \tau_j(x)\psi_j)$ $(0\leqq j\leqq p$, $x\in G)$. Since $\Lambda=\Lambda_0+\Sigma m_j\Lambda_j$, it follows from the above that $f_\Lambda(\exp(-H)x)=e^{\Lambda(H)}f_\Lambda(x)$, and so $\pi(H)f_\Lambda=\Lambda(H)f_\Lambda$. To prove the second assertion we define T^C, $\bar{N}$, ξ_Λ, Γ_Λ as in Section 2. Let

$$F_\Lambda(w) = (\tau_0(w^{-1})\psi_0, \phi_0)\prod_{j=1}^p(\tau_j(w^{-1})\psi_j, \phi_j)^{m_j} \qquad (w\in GT^C\bar{N}).$$

Then Lemma 10 implies that the function F_Λ is a well defined holomorphic function on $GT^C\bar{N}$. A computation shows that $F_\Lambda(wt\bar{n})=\xi_\Lambda(t)^{-1}F_\Lambda(w)$ $(w\in$

$GT^C\bar{N}$, $t \in T^C$, $\bar{n} \in \bar{N}$), that is, $F_A \in \Gamma_A$. Moreover $f_A(x) = \{F_A(x^{-1})\}^-$ $(x \in G)$. For each α in P, we can write $X_\alpha = Y_\alpha + iZ_\alpha$ (Y_α, $Z_\alpha \in \mathfrak{g}$). Therefore

$$\begin{aligned}\pi(X_\alpha)f_A(x) &= \pi(Y_\alpha)f_A(x) + i\pi(Z_\alpha)f_A(x)\\ &= D_t\{F_A(x^{-1}\exp(tY_\alpha))\}^-|_{t=0} + iD_t\{F_A(x^{-1}\exp(tZ_\alpha))\}^-|_{t=0}\\ &= D_t\{F_A(x^{-1}\exp(tY_\alpha)) - iF_A(x^{-1}\exp(tZ_\alpha))\}^-|_{t=0}\\ &= D_t\{F_A(x^{-1}\exp t(Y_\alpha - iZ_\alpha))\}^-|_{t=0}\\ &= 0 \qquad (\text{note that } Y_\alpha - iZ_\alpha = \eta X_\alpha \in \bar{\mathfrak{n}}).\end{aligned}$$

§4. An integration formula

In this paragraph we give an integration formula given by M. F. Jensen [11, Theorem 2.6].

Let $\mathfrak{b}$ be a maximal abelian subspace of $\mathfrak{q} \cap \mathfrak{p}$. For each β in $\mathfrak{b}^*$, let $\mathfrak{g}^\beta = \{X \in \mathfrak{g}: [H, X] = \beta(H)X$ for all H in $\mathfrak{b}\}$. Put

$$\Delta_{\mathfrak{b}} = \{\beta \in \mathfrak{b}^*: \beta \neq 0, \mathfrak{g}^\beta \neq 0\}.$$

Since $\sigma\circ\theta \,|\, \mathfrak{b} = I$, $\mathfrak{g}^\beta$ is $\sigma\circ\theta$-stable and so

$$\mathfrak{g}^\beta = \mathfrak{g}^\beta \cap \{(\mathfrak{h} \cap \mathfrak{k}) + (\mathfrak{q} \cap \mathfrak{p})\} + \mathfrak{g}^\beta \cap \{(\mathfrak{h} \cap \mathfrak{p}) + (\mathfrak{q} \cap \mathfrak{k})\}.$$

For each β in $\Delta_{\mathfrak{b}}$, we put

$$p_\beta = \dim\{\mathfrak{g}^\beta \cap ((\mathfrak{h} \cap \mathfrak{k}) + (\mathfrak{q} \cap \mathfrak{p}))\} \text{ and } q_\beta = \dim\{\mathfrak{g}^\beta \cap ((\mathfrak{h} \cap \mathfrak{p}) + (\mathfrak{q} \cap \mathfrak{k}))\}.$$

Let $\mathfrak{b}' = \{H \in \mathfrak{b}: \beta(H) \neq 0$ for each β in $\Delta_{\mathfrak{b}}$ so that $p_\beta > 0\}$, and we fix a connected component $\mathfrak{b}^+$ of $\mathfrak{b}'$. Put

$$D(\exp H) = \{\textstyle\prod_{\beta \in \Delta_{\mathfrak{b}}} |\sinh \beta(H)|^{p_\beta} |\cosh \beta(H)|^{q_\beta}\}^{1/2} \qquad (H \in \mathfrak{b}).$$

Then the invariant measure dH on $\mathfrak{b}$ can be normalized in such a way that for all compactly supported continuous functions f,

$$\int_{G/H_\sigma} f(x)dx = \int_K \int_{\mathfrak{b}^+} f(k \exp H) D(\exp H) dk dH$$

where dk is the normalized Haar measure on K.

§5. Computation of $\|f_A\|^2$

We shall define the sequences $\mathfrak{g}^C = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots$ of subalgebras of $\mathfrak{g}^C$ and $P_n = Q_1 \supset Q_2 \supset \cdots$ of subsets of P_n. The inductive definition is as follows. Put $\mathfrak{g}_1 = \mathfrak{g}^C$ and $Q_1 = P_n$. For $j \geq 1$ if $\mathfrak{g}_j \subset \mathfrak{k}^C$, then $\mathfrak{g}_{j+1} = \mathfrak{g}_j$ and $Q_{j+1} = \phi$. Otherwise let γ_j be the lowest root in Q_j. Then $\mathfrak{g}_{j+1}$ is the centralizer of $\boldsymbol{C}H_{\gamma_j}+\boldsymbol{C}\sigma H_{\gamma_j}+\boldsymbol{C}X_{\gamma_j}+\boldsymbol{C}X_{-\gamma_j}+\boldsymbol{C}\sigma X_{\gamma_j}+\boldsymbol{C}\sigma X_{-\gamma_j}$ in $\mathfrak{g}_j$ and Q_{j+1} is the set of all $\gamma\in Q_j$ such that (1) $\gamma\neq\gamma_j$ and $\gamma\neq-\gamma_j\circ\sigma$, (2) $\gamma-\gamma_j\notin\Phi$ and $\gamma+(\gamma_j\circ\sigma)\notin\Phi$. Using Lemma 3(1), we can prove by induction on j that

$$\mathfrak{g}_j=\mathfrak{g}_j\cap\mathfrak{k}^C+\textstyle\sum_{\gamma\in Q_j}(\boldsymbol{C}X_\gamma+\boldsymbol{C}X_{-\gamma})\qquad(j\geqq 1).$$

It is obvious that $\dim\mathfrak{g}_{j+1}<\dim\mathfrak{g}_j$ unless $\mathfrak{g}_j\subset\mathfrak{k}^C$. Let s be the least integer such that $\mathfrak{g}_{s+1}\subset\mathfrak{k}^C$. We define the elements H^{γ_j}, X^{γ_j} and $X^{-\gamma_j}$ ($j=1,\dots,s$) as follows:

$$H^{\gamma_j}=H_{\gamma_j},\qquad X^{\gamma_j}=X_{\gamma_j},\qquad X^{-\gamma_j}=X_{-\gamma_j}\qquad\text{if}\quad\gamma_j\,|\,\mathfrak{t}_{\mathfrak{h}}=0,$$

$$H^{\gamma_j}=H_{\gamma_j}-\sigma H_{\gamma_j},\quad X^{\gamma_j}=X_{\gamma_j}-\sigma X_{-\gamma_j},\quad X^{-\gamma_j}=X_{-\gamma_j}-\sigma X_{\gamma_j}\quad\text{otherwise.}$$

It is easy to check that $H^{\gamma_j}\in\mathfrak{t}_{\mathfrak{q}}^C$.

LEMMA 12. *$\gamma_i\pm\gamma_j\notin\Phi$, $\gamma_i\pm\gamma_j\circ\sigma\notin\Phi$ $(1\leqq i<j\leqq s)$, and $\mathfrak{b}=\sum_{i=1}^{s}\boldsymbol{R}(X^{\gamma_i}+X^{-\gamma_i})$ is an abelian subspace of $\mathfrak{q}\cap\mathfrak{p}$.*

PROOF. If $i<j$, then $\mathfrak{g}_{i+1}\supset\mathfrak{g}_j$ and therefore $[X_{\pm\gamma_i}, X_{\gamma_j}]=[\sigma X_{\pm\gamma_i}, X_{\gamma_j}]=0$. This implies $\gamma_i\pm\gamma_j\notin\Phi$, $\gamma_i\pm\gamma_j\circ\sigma\notin\Phi$ and that $\mathfrak{b}$ is abelian. We know that $\eta X_\gamma=X_{-\gamma}$ for any γ in Φ_n (Lemma 1). Hence $\eta(X^{\gamma_i}+X^{-\gamma_i})=X^{\gamma_i}+X^{-\gamma_i}$, and so $X^{\gamma_i}+X^{-\gamma_i}\in\mathfrak{p}$ $(1\leqq i\leqq s)$. Moreover $\sigma X_\gamma=-X_{\gamma\circ\sigma}$ for any γ in Φ_n (see, Lemma 1), therefore $\sigma(X^{\gamma_i}+X^{-\gamma_i})=-(X^{\gamma_i}+X^{-\gamma_i})$, that is, $X^{\gamma_i}+X^{-\gamma_i}\in\mathfrak{q}^C$ $(1\leqq i\leqq s)$. Hence $\mathfrak{b}\subset\mathfrak{q}\cap\mathfrak{p}$.

LEMMA 13. *If $\alpha\in\Phi$, then $\alpha+\alpha\circ\sigma\notin\Phi$.*

PROOF. Let α, β be two elements in Φ such that $\alpha+\beta\in\Phi$. We define the number $N_{\alpha,\beta}$ by $[X_\alpha, X_\beta]=N_{\alpha,\beta}X_{\alpha+\beta}$. Applying $\tilde{\theta}$ to this identity, we have $N_{-\alpha,-\beta}=-\bar{N}_{\alpha,\beta}$. Obviously $N_{\alpha,\beta}=-N_{\beta,\alpha}$. Now fix an element α in Φ. If $\alpha\,|\,\mathfrak{t}_{\mathfrak{q}}^C=0$, then $\alpha+\alpha\circ\sigma=2\alpha\notin\Phi$. This being so, assume that $\alpha\,|\,\mathfrak{t}_{\mathfrak{q}}^C\neq 0$. If $\alpha+\alpha\circ\sigma\in\Phi$, then plainly $(\alpha+\alpha\circ\sigma)\,|\,\mathfrak{t}_{\mathfrak{q}}^C=0$, hence $X_{\alpha+\alpha\circ\sigma}\in\mathfrak{h}^C$. From Lemma 1, $\tilde{\theta}\circ\sigma X_\alpha=-\tilde{\theta}X_{\alpha\circ\sigma}=X_{-\alpha\circ\sigma}$ and $\tilde{\theta}\circ\sigma X_{-\alpha}=-\tilde{\theta}X_{-\alpha\circ\sigma}=X_{\alpha\circ\sigma}$. Therefore

$$[X_\alpha, \tilde{\theta}\circ\sigma X_{-\alpha}]=[X_\alpha, X_{\alpha\circ\sigma}]=N_{\alpha,\alpha\circ\sigma}X_{\alpha+\alpha\circ\sigma}.$$

Applying $\tilde{\theta}\circ\sigma$ to this identity, we have

$$[\tilde{\theta}\circ\sigma X_\alpha, X_{-\alpha}]=\bar{N}_{\alpha,\alpha\circ\sigma}\tilde{\theta}\circ\sigma X_{\alpha+\alpha\circ\sigma}=\bar{N}_{\alpha,\alpha\circ\sigma}\tilde{\theta}X_{\alpha+\alpha\circ\sigma}=N_{-\alpha,-\alpha\circ\sigma}X_{-\alpha-\alpha\circ\sigma}.$$

On the other hand

$$[\tilde{\theta}\circ\sigma X_\alpha, X_{-\alpha}]=[X_{-\alpha\circ\sigma}, X_{-\alpha}]=N_{-\alpha\circ\sigma,-\alpha}X_{-\alpha\circ\sigma-\alpha}=-N_{-\alpha,-\alpha\circ\sigma}X_{-\alpha-\alpha\circ\sigma}.$$

Hence $N_{-\alpha,-\alpha\circ\sigma}=-N_{-\alpha,-\alpha\circ\sigma}$ and so $N_{-\alpha,-\alpha\circ\sigma}=0$. This is a contradiction. Hence the lemma follows.

LEMMA 14. *Let β, δ be non compact positive roots such that $\beta-\delta\in\Phi$ and $\delta\,|\,\mathfrak{t}_{\mathfrak{h}}^{C}\neq 0$. Then $(\beta-\delta)-\delta\circ\sigma\notin\Phi$.*

PROOF. Let us suppose $(\beta-\delta)-\delta\circ\sigma\in\Phi$. Then $[X_{-\delta\circ\sigma}, [X_\beta, X_{-\delta}]]\neq 0$. On the other hand $[X_{-\delta\circ\sigma}, [X_\beta, X_{-\delta}]]=[[X_{-\delta}, X_{-\delta\circ\sigma}], X_\beta]+[[X_{-\delta\circ\sigma}, X_\beta], X_{-\delta}]$. But $\delta+\delta\circ\sigma$ cannot be a root nor zero (Lemma 13), and so $[X_{-\delta}, X_{-\delta\circ\sigma}]=0$. Furthermore $[X_{-\delta\circ\sigma}, X_\beta]=0$ (Lemma 3). Thus $[X_{-\delta\circ\sigma}, [X_\beta, X_{-\delta}]]=0$. This is a contradiction.

LEMMA 15. *For each i $(1\leqq i\leqq s)$, $\boldsymbol{C}(X^{\gamma_i}+X^{-\gamma_i})+\mathfrak{g}_{i+1}\cap(\mathfrak{q}^C\cap\mathfrak{p}^C)$ is the set of all elements in $\mathfrak{g}_i\cap(\mathfrak{q}^C\cap\mathfrak{p}^C)$ which commute with $X^{\gamma_i}+X^{-\gamma_i}$.*

PROOF. Let $Q_{i,1}=\{\gamma\in Q_i\colon \gamma\neq\gamma_i \text{ and } \gamma\neq-\gamma_i\circ\sigma\}$. We have two cases: (1) $\gamma_i\,|\,\mathfrak{t}_{\mathfrak{h}}^{C}=0$, (2) $\gamma_i\,|\,\mathfrak{t}_{\mathfrak{h}}^{C}\neq 0$.

In the first case $X^{\gamma_i}+X^{-\gamma_i}=X_{\gamma_i}+X_{-\gamma_i}$. If $X\in\mathfrak{g}_i\cap(\mathfrak{q}^C\cap\mathfrak{p}^C)$, we can write $X=c'_{\gamma_i}X_{\gamma_i}+c'_{-\gamma_i}X_{-\gamma_i}+\sum_{\gamma\in Q_{i,1}}(c'_\gamma X_\gamma+c'_{-\gamma}X_{-\gamma})$ where $c'_{\pm\gamma_i}$, $c'_{\pm\gamma}$ are complex numbers. Since $\sigma X=-X$ and $\sigma X_{\pm\gamma_i}=-X_{\mp\gamma_i}$, we can write

$$X = c_{\gamma_i}(X_{\gamma_i} + X_{-\gamma_i}) + \sum_{\gamma\in Q_{i,1}} c_\gamma(X_\gamma - \sigma X_\gamma).$$

(Note that if $\gamma\in Q_{i,1}$ then $-\gamma\circ\sigma$ is also in $Q_{i,1}$.) So if X commutes with $X^{\gamma_i}+X^{-\gamma_i}=X_{\gamma_i}+X_{-\gamma_i}$, then $\sum_{\gamma\in Q_{i,1}}c_\gamma(X_\gamma-\sigma X_\gamma)$ also commutes with $X_{\gamma_i}+X_{-\gamma_i}$. We have to prove that $\sum_{\gamma\in Q_{i,1}}c_\gamma(X_\gamma-\sigma X_\gamma)\in\mathfrak{g}_{i+1}$. Put $Q_{i,2}=\{\gamma\in Q_{i,1}\colon c_\gamma\neq 0\}$. It is enough to show that $Q_{i,2}\subset Q_{i+1}$. Now using Lemma 3(1), we have

$$0 = [\sum_{\gamma\in Q_{i,1}} c_\gamma(X_\gamma - \sigma X_\gamma), X_{\gamma_i} + X_{-\gamma_i}] = \sum_{\gamma\in Q_{i,2}} c_\gamma([X_\gamma, X_{-\gamma_i}] - [\sigma X_\gamma, X_{\gamma_i}]).$$

Since γ_i is the lowest root of Q_i, if $\gamma-\gamma_i$ $(\gamma\in Q_{i,2})$ is a root then it is positive. Moreover if $\gamma\circ\sigma+\gamma_i$ is a root, then it is negative (for, $\gamma\in Q_{i,1}\Rightarrow-\gamma\circ\sigma\in Q_{i,1}\Rightarrow-\gamma\circ\sigma>\gamma_i\Rightarrow\gamma\circ\sigma+\gamma_i<0$). Therefore the above equality implies that $\sum_{\gamma\in Q_{i,2}}c_\gamma[X_\gamma, X_{-\gamma_i}]=0$, and hence $\gamma-\gamma_i$ is not a root for any γ in $Q_{i,2}$. This means $Q_{i,2}\subset Q_{i+1}$.

Now consider the second case. Then

$$X^{\gamma_i} + X^{-\gamma_i} = X_{\gamma_i} - \sigma X_{-\gamma_i} + X_{-\gamma_i} - \sigma X_{\gamma_i}.$$

If $X\in\mathfrak{g}_i\cap(\mathfrak{q}^C\cap\mathfrak{p}^C)$,

$$X = c_{\gamma_i}(X_{\gamma_i} - \sigma X_{\gamma_i}) + c_{-\gamma_i}(X_{-\gamma_i} - \sigma X_{-\gamma_i}) + \sum_{\gamma\in Q_{i,1}} c_\gamma(X_\gamma - \sigma X_\gamma).$$

Put $Q_{i,2}=\{\gamma\in Q_{i,1}\colon c_\gamma\neq 0\}$. If X commutes with $X^{\gamma_i}+X^{-\gamma_i}$,

$$\begin{aligned} 0 &= [X, X_{\gamma_i} - \sigma X_{-\gamma_i} + X_{-\gamma_i} - \sigma X_{\gamma_i}] \\ &= (c_{\gamma_i} - c_{-\gamma_i})(H_{\gamma_i} + \sigma H_{\gamma_i}) \\ &\quad + \sum_{\gamma\in Q_{i,2}} c_\gamma([X_\gamma, X_{-\gamma_i}] - [X_\gamma, \sigma X_{\gamma_i}] - [\sigma X_\gamma, X_{\gamma_i}] + [\sigma X_\gamma, \sigma X_{-\gamma_i}]). \end{aligned}$$

Since $\gamma_i | \mathfrak{t}_{\mathfrak{h}}^{\mathbf{C}} \neq 0$, we have $H_{\gamma_i} + \sigma H_{\gamma_i} \neq 0$ and so $c_{\gamma_i} = c_{-\gamma_i}$. Hence it is enough to prove that $Q_{i,2} \subset Q_{i+1}$. Let $Q_{i,3} = \{\gamma \in Q_{i,2} : \gamma | \mathfrak{t}_{\mathfrak{q}}^{\mathbf{C}} = \gamma_i | \mathfrak{t}_{\mathfrak{q}}^{\mathbf{C}}\}$ and let $Q_{i,4} = \{\gamma \in Q_{i,2} : \gamma | \mathfrak{t}_{\mathfrak{q}}^{\mathbf{C}} \neq \gamma_i | \mathfrak{t}_{\mathfrak{q}}^{\mathbf{C}}\}$. If $\gamma \in Q_{i,3}$, then $[X_\gamma, X_{-\gamma_i}]$ and $[X_\gamma, \sigma X_{\gamma_i}]$ are both in $\mathfrak{h}^{\mathbf{C}}$. On the other hand if $\gamma \in Q_{i,4}$, then $\gamma - \gamma_i$, $\gamma \circ \sigma + \gamma_i$ are not identically zero on $\mathfrak{t}_{\mathfrak{q}}$ and so $\gamma \circ \sigma - \gamma_i \circ \sigma < 0$, $\gamma + \gamma_i \circ \sigma > 0$ (note that γ_i is the lowest root of Q_i, therefore $\gamma - \gamma_i > 0 > \gamma \circ \sigma + \gamma_i$). Combining these facts with the above equation, we find that

$$\sum_{\gamma \in Q_{i,3}} 2c_\gamma [X_\gamma, X_{-\gamma_i}] + \sum_{\gamma \in Q_{i,4}} c_\gamma ([X_\gamma, X_{-\gamma_i}] - [X_\gamma, \sigma X_{\gamma_i}]) = 0 .$$

Now suppose $Q_{i,2} \not\subset Q_{i+1}$. Then the above says that there exist two roots $\beta, \gamma \in Q_{i,2}$ such that $\beta - \gamma_i = \gamma + \gamma_i \circ \sigma \in \Phi$. But this contradicts Lemma 14. Hence the proof is now complete.

As a straightforward consequence of Lemmas 12 and 15, one can prove

COROLLARY. $\mathfrak{b}$ *is a maximal abelian subspace of* $\mathfrak{q} \cap \mathfrak{p}$.

LEMMA 16. $\gamma_i(H^{\gamma_i}) = 2$, $[X^{\gamma_i}, X^{-\gamma_i}] = H^{\gamma_i}$, $[H^{\gamma_i}, X^{\gamma_i}] = 2X^{\gamma_i}$, *and* $[H^{\gamma_i}, X^{-\gamma_i}] = -2X^{-\gamma_i}$ $(1 \leqq i \leqq s)$.

PROOF. If $\gamma_i | \mathfrak{t}_{\mathfrak{h}}^{\mathbf{C}} = 0$, then the required relations follow from the definitions. Suppose that $\gamma_i | \mathfrak{t}_{\mathfrak{h}}^{\mathbf{C}} \neq 0$. Then it follows from Lemmas 3 and 13, that $\gamma_i \pm \gamma_i \circ \sigma$ is not a root nor zero. Therefore $\gamma_i(\sigma H_{\gamma_i}) = 0$ (for, $\sigma H_{\gamma_i} = H_{\gamma_i \circ \sigma}$), and so $\gamma_i(H^{\gamma_i}) = \gamma_i(H_{\gamma_i} - \sigma H_{\gamma_i}) = 2$. Moreover $[X^{\gamma_i}, X^{-\gamma_i}] = [X_{\gamma_i} - \sigma X_{-\gamma_i}, X_{-\gamma_i} - \sigma X_{\gamma_i}] = H_{\gamma_i} - \sigma H_{\gamma_i} = H^{\gamma_i}$. Since $H^{\gamma_i} \in \mathfrak{t}_{\mathfrak{q}}^{\mathbf{C}}$, the last two equations are verified by a simple calculation.

LEMMA 17. *Let* ν *denote the automorphism of* $\mathfrak{g}^{\mathbf{C}}$ *given by*

$$\nu = \exp(\pi/4) \operatorname{ad} \left(\sum_{i=1}^{s} (X^{\gamma_i} - X^{-\gamma_i})\right).$$

Then $\nu(X^{\gamma_i} + X^{-\gamma_i}) = H^{\gamma_i}$ $(1 \leqq i \leqq s)$. *Moreover for any* t *in* $\boldsymbol{R}$ *we have*

$$\exp t(X^{\gamma_i} + X^{-\gamma_i}) = \exp((\tanh t) X^{-\gamma_i}) \exp(\log(\cosh t) H^{\gamma_i}) \exp((\tanh t) X^{\gamma_i}).$$

This lemma follows from Lemma 16 and [8, Lemma 9].

LEMMA 18. *Let* $\mathfrak{a}_{\mathfrak{k}} = \{H \in \mathfrak{t} : B(H, H_{\gamma_i}) = B(H, \sigma H_{\gamma_i}) = 0$ *for all* $i\}$ *and let* $\mathfrak{a}_{\mathfrak{p}} = \sum_i \boldsymbol{R}(X_{\gamma_i} + X_{-\gamma_i}) + \sum_i \boldsymbol{R}(\sigma X_{\gamma_i} + \sigma X_{-\gamma_i})$. *Put* $\mathfrak{a} = \mathfrak{a}_{\mathfrak{k}} + \mathfrak{a}_{\mathfrak{p}}$. *Then* $\mathfrak{a}$ *is a Cartan subalgebra of* $\mathfrak{g}$ *and* $\nu(\mathfrak{a}^{\mathbf{C}}) = \mathfrak{t}^{\mathbf{C}}$. *Moreover* $\mathfrak{a}_{\mathfrak{k}}$, $\mathfrak{a}_{\mathfrak{p}}$ *are both* σ*-stable.* $\mathfrak{a}_{\mathfrak{p}} \cap \mathfrak{q} = \mathfrak{b}$.

PROOF. If $H \in \mathfrak{a}_{\mathfrak{k}}$, $\gamma_i(H) = \gamma_i(\sigma H) = 0$ $(1 \leqq i \leqq s)$ and therefore $\nu(H) = H$. On the other hand, for any index i such that $\gamma_i | \mathfrak{t}_{\mathfrak{h}}^{\mathbf{C}} \neq 0$ we have $[X_{\gamma_i} - X_{-\gamma_i}, \sigma X_{\gamma_i} - \sigma X_{-\gamma_i}] = 0$ (Lemma 13). So we can write

$$\nu = \prod_{i\in I} \exp(\pi/4)\,\mathrm{ad}\,(X_{\gamma_i} - X_{-\gamma_i})$$
$$\times \prod_{i\in J} \exp(\pi/4)\,\mathrm{ad}\,(X_{\gamma_i} - X_{-\gamma_i}) \prod_{i\in J} \exp(\pi/4)\,\mathrm{ad}\,(\sigma X_{\gamma_i} - \sigma X_{-\gamma_i}).$$

(Here, $I=\{i: \gamma_i | \mathfrak{t}_{\mathfrak{h}}^{C}=0\}$ and $J=\{i: \gamma_i | \mathfrak{t}_{\mathfrak{h}}^{C}\neq 0\}$). Hence it follows from [8, Lemma 9] that (1) if $\gamma_i | \mathfrak{t}_{\mathfrak{h}}^{C}=0$ then $\nu(X_{\gamma_i}+X_{-\gamma_i})=H_{\gamma_i}$, (2) if $\gamma_i | \mathfrak{t}_{\mathfrak{h}}^{C}\neq 0$ then $\nu(X_{\gamma_i}+X_{-\gamma_i})=H_{\gamma_i}$ and $\nu(\sigma X_{\gamma_i}+\sigma X_{\gamma_i})=\sigma H_{\gamma_i}$. This implies that $\nu(\mathfrak{a}_{\mathfrak{p}}^{C})=\sum_i \boldsymbol{C}H_{\gamma_i}+\sum_i \boldsymbol{C}\sigma H_{\gamma_i}$. Since $H_{\gamma_i}\in i\mathfrak{t}$ $(1\leqq i\leqq s)$, from the definition of $\mathfrak{a}_{\mathfrak{k}}$ we find that $\mathfrak{a}_{\mathfrak{k}}^{C}$ is the orthogonal complement of $\nu(\mathfrak{a}_{\mathfrak{p}}^{C})$ in $\mathfrak{t}^{C}$ with respect to the positive definite Hermitian form $-B(X, \tilde{\theta}Y)$ $(X, Y\in\mathfrak{g}^{C})$. The above arguments imply that

$$\mathfrak{t}^{C} = \mathfrak{a}_{\mathfrak{k}}^{C} + \nu(\mathfrak{a}_{\mathfrak{p}}^{C}) = \nu(\mathfrak{a}_{\mathfrak{k}}^{C}) + \nu(\mathfrak{a}_{\mathfrak{p}}^{C}) = \nu(\mathfrak{a}^{C}).$$

As ν is an automorphism and $\mathfrak{a}$ is θ-stable, it follows that $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Clearly $\mathfrak{a}_{\mathfrak{k}}$ and $\mathfrak{a}_{\mathfrak{p}}$ are σ-stable. Moreover from the definition of $\mathfrak{b}$ we conclude $\mathfrak{a}_{\mathfrak{p}}\cap\mathfrak{q}=\mathfrak{b}$.

LEMMA 19. *γ_i $(1\leqq i\leqq s)$ are linearly independent on $\nu(\mathfrak{b}^{C})$.*

PROOF. Since $\nu(X^{\gamma_i}+X^{-\gamma_i})=H^{\gamma_i}$, $\{H^{\gamma_1},\dots, H^{\gamma_s}\}$ is a basis of $\nu(\mathfrak{b}^{C})$. Moreover $\gamma_i(H^{\gamma_j})=2\delta_{ij}$, and so γ_i $(1\leqq i\leqq s)$ are linearly independent on $\nu(\mathfrak{b}^{C})$.

Let Δ be the set of non zero roots of $(\mathfrak{g}^{C}, \mathfrak{a}^{C})$. Since $\nu(\mathfrak{a}^{C})=\mathfrak{t}^{C}$ and ν is an automorphism of $\mathfrak{g}^{C}$, it is obvious that

$$\Delta = \{\alpha\circ\nu : \alpha\in\Phi\}.$$

On the other hand $\mathfrak{b}$ is a maximal abelian subspace of $\mathfrak{q}\cap\mathfrak{p}$ (Corollary to Lemma 15), and therefore we can define $\Delta_{\mathfrak{b}}$ as in Section 4. Then

$$\Delta_{\mathfrak{b}} = \{\beta|_{\mathfrak{b}} : \beta\in\Delta,\ \beta|_{\mathfrak{b}}\neq 0\}.$$

Finally

$$\Delta_{\mathfrak{b}} = \{\alpha\circ\nu|_{\mathfrak{b}} : \alpha\in\Phi,\ \alpha|_{\nu(\mathfrak{b})}\neq 0\}.$$

LEMMA 20. *Let α be any root in Φ such that $\alpha|_{\nu(\mathfrak{b})}=0$. Then α is compact.*

PROOF. We may assume that $\alpha>0$. Suppose that α is not compact. Then Lemma 3(1) says that $\alpha+\gamma_i$ and $\alpha-\gamma_i\circ\sigma$ are not roots for any i $(1\leqq i\leqq s)$. Therefore $\alpha(H_{\gamma_i})\geqq 0$ and $\alpha(\sigma H_{\gamma_i})\leqq 0$ $(1\leqq i\leqq s)$, and so $\alpha(H_{\gamma_i})=\alpha(\sigma H_{\gamma_i})=0$ (note: $\alpha(H^{\gamma_i})=0$). This implies that $\alpha-\gamma_i$, $\alpha+\gamma_i\circ\sigma$ can never be a root nor zero. Hence for any i we have $[X_\alpha-\sigma X_\alpha, X^{\gamma_i}+X^{-\gamma_i}]=0$, and so $X_\alpha-\sigma X_\alpha\in\mathfrak{b}^{C}$. This implies $X_\alpha-\sigma X_\alpha=0$. However $X_\alpha-\sigma X_\alpha=X_\alpha+X_{\alpha\circ\sigma}\neq 0$ (Lemma 1). Hence the lemma follows.

Let λ and μ be two linear functions on $\mathfrak{t}^{C}$. We write $\lambda\sim\mu$ if $\lambda-\mu$ vanishes identically on $\nu(\mathfrak{b}^{C})=\sum_i \boldsymbol{C}H^{\gamma_i}$. For any index i $(1\leqq i\leqq s)$, set $C_i=\{\alpha\in P_k:$

$\alpha\sim-2^{-1}\gamma_i\}$, $P_i=\{\gamma\in P_n: \gamma\sim 2^{-1}\gamma_i\}$, and $P^i=\{\gamma\in P_n: \gamma\sim\gamma_i\}$. For any pair of indices i, j $(1\leqq i<j\leqq s)$, set $C_{ij}=\{\alpha\in P_k: \alpha\sim 2^{-1}(\gamma_j-\gamma_i)\}$, $P_{ij}=\{\gamma\in P_n: \gamma\sim 2^{-1}(\gamma_j+\gamma_i)\}$. Put $C_0=\{\alpha\in P_k: \alpha\sim 0\}$.

LEMMA 21. *P_k is the disjoint union of C_0, C_i, C_{ij} $(1\leqq i<j\leqq s)$.*

PROOF. The disjointness is a consequence of Lemma 19. Let α be a root in P_k so that $\alpha\nsim 0$. Then $X_\alpha\notin\mathfrak{g}_{s+1}$. Let i denote the least index $(1\leqq i\leqq s)$ such that $X_\alpha\notin\mathfrak{g}_{i+1}$. Since $X_{-\alpha}(=-\tilde{\theta}X_\alpha)\in\mathfrak{g}_i$, if $\gamma_i-\alpha$ is a root then $\gamma_i-\alpha\in Q_i$ and $\gamma_i-\alpha<\gamma_i$. But this contradicts the definition of γ_i, and so $\gamma_i-\alpha$ is not a root. Similarly $\gamma_i+\alpha\circ\sigma$ is not a root. Therefore $\alpha(H_{\gamma_i})\leqq 0$ and $\alpha(\sigma H_{\gamma_i})\geqq 0$, and so $\alpha(H^{\gamma_i})\leqq 0$. If $\alpha(H^{\gamma_i})=0$, then it follows from the above inequalities that $\alpha(H_{\gamma_i})=\alpha(\sigma H_{\gamma_i})=0$. But this implies that $X_\alpha\in\mathfrak{g}_{i+1}$ which contradicts the choice of the index i. So $\alpha(H^{\gamma_i})<0$. Now we claim that $\gamma_j+\alpha$, $\gamma_j-\alpha\circ\sigma$ are not roots for any $j\neq i$. If $\gamma_j+\alpha$ is a root, then

$$(\gamma_j+\alpha)(H_{\gamma_i}-\sigma H_{\gamma_i})=\alpha(H_{\gamma_i}-\sigma H_{\gamma_i})<0.$$

On the other hand since γ_j is totally positive, $\gamma_j+\alpha$ is non compact positive, and therefore it follows from Lemma 3(1) that

$$(\gamma_j+\alpha)(H_{\gamma_i}-\sigma H_{\gamma_i})=(\gamma_j+\alpha)(H_{\gamma_i})-(\gamma_j+\alpha)(\sigma H_{\gamma_i})\geqq 0,$$

which conflicts with our conclusion above. By a similar method we can show that $\gamma_j-\alpha\circ\sigma$ is not a root. So we have the following two cases: (a) $\gamma_j\pm\alpha$, $\gamma_j\pm\alpha\circ\sigma$ is never a root for $j\neq i$, (b) either $\gamma_j-\alpha$ or $\gamma_j+\alpha\circ\sigma$ is a root for some $j\neq i$.

In the first case $\alpha(H_{\gamma_j})=\alpha(\sigma H_{\gamma_j})=0$ for all $j\neq i$. Moreover we have the following three possibilities: (1) $\gamma_i+\alpha\in\Phi$ and $\gamma_i-\alpha\circ\sigma\notin\Phi$, (2) $\gamma_i+\alpha\notin\Phi$ and $\gamma_i-\alpha\circ\sigma\in\Phi$, (3) $\gamma_i+\alpha\in\Phi$ and $\gamma_i-\alpha\circ\sigma\in\Phi$. We consider the case (1). Then since γ_i and $\gamma_i+\alpha$ are both in P_n, $\alpha+2\gamma_i$ is not a root (Lemma 3), and so $\alpha(H_{\gamma_i})=-1$. Since $\alpha(\sigma H_{\gamma_i})=0$, $\alpha(H^{\gamma_i})=-1=-2^{-1}\gamma_i(H^{\gamma_i})$. Noting that $\alpha(H^{\gamma_j})=0$ for all $j\neq i$, we get $\alpha\sim-2^{-1}\gamma_i$. In the case (2), $-\gamma_i\circ\sigma$ and $\alpha-\gamma_i\circ\sigma$ are both non compact positive, and therefore $\alpha-2\gamma_i\circ\sigma$ is not a root. This implies that $\alpha(\sigma H_{\gamma_i})=1$. Since $\alpha(H_{\gamma_i})=0$, $\alpha(H^{\gamma_i})=-1=-2^{-1}\gamma_i(H^{\gamma_i})$. This means that $\alpha\sim-2^{-1}\gamma_i$. Now we consider the case (3). Then $\alpha(H_{\gamma_i})=-1$ and $\alpha(\sigma H_{\gamma_i})=1$. We claim that $\gamma_i|\mathfrak{t}_{\mathfrak{h}}^{\mathbf{C}}=0$. For otherwise if $\gamma_i|\mathfrak{t}_{\mathfrak{h}}^{\mathbf{C}}\neq 0$, then $\alpha(H^{\gamma_i})=\alpha(H_{\gamma_i})-\alpha(\sigma H_{\gamma_i})=-2=-\gamma_i(H^{\gamma_i})$. This means that $\alpha+\gamma_i\sim 0$. But $\alpha+\gamma_i$ is non compact and so we get a contradiction with Lemma 20. Hence $\gamma_i|\mathfrak{t}_{\mathfrak{h}}^{\mathbf{C}}=0$. This shows that $\alpha(H^{\gamma_i})=\alpha(H_{\gamma_i})=-1=-2^{-1}\gamma_i(H^{\gamma_i})$, and therefore $\alpha\sim-2^{-1}\gamma_i$.

Now we come to the second case (b). Let j be the least index such that either $\gamma_j-\alpha$ or $\gamma_j+\alpha\circ\sigma$ is a root. Then $j\neq i$ and in view of our definition of i, $j>i$. Moreover $\alpha(H_{\gamma_j})-\alpha(\sigma H_{\gamma_j})>0$. First we show that $\gamma_k\pm\alpha$ and $\gamma_k\pm\alpha\circ\sigma$ can never be roots for any index k $(1\leqq k\leqq s)$ other than i, j. We have already seen this for $\gamma_k+\alpha$ and $\gamma_k-\alpha\circ\sigma$. Suppose $\gamma_k-\alpha$ is a root. Then $(\gamma_k-\alpha)(H_{\gamma_j}-\sigma H_{\gamma_j})=-\alpha(H_{\gamma_j})+\alpha(\sigma H_{\gamma_j})<0$. On the other hand since $\gamma_k-\alpha$ is non compact positive, $(\gamma_k-\alpha)(H_{\gamma_j}-\sigma H_{\gamma_j})=(\gamma_k-\alpha)(H_{\gamma_j})-(\gamma_k-\alpha)(\sigma H_{\gamma_j})\geqq 0$, which gives a contradiction. Hence $\gamma_k-\alpha$ is not a root. Similarly, $\gamma_k+\alpha\circ\sigma$ is not a root. Therefor we find that $\alpha(H^{\gamma_k})=0$ for any index k $(1\leqq k\leqq s)$ other than i, j. Now we distinguish four cases: (1) there is exactly one root in $\{\alpha+\gamma_i, \alpha-\gamma_i\circ\sigma\}$, and similarly there is only one root in $\{\alpha-\gamma_j, \alpha+\gamma_j\circ\sigma\}$, (2) $\alpha+\gamma_i$, $\alpha-\gamma_i\circ\sigma$ are both roots, and only one in $\{\alpha-\gamma_j, \alpha+\gamma_j\circ\sigma\}$ is a root, (3) there is only one root in $\{\alpha+\gamma_i, \alpha-\gamma_i\circ\sigma\}$, and $\alpha-\gamma_j$, $\alpha+\gamma_j\circ\sigma$ are both roots, (4) $\alpha+\gamma_i$, $\alpha-\gamma_i\circ\sigma$, $\alpha-\gamma_j$, $\alpha+\gamma_j\circ\sigma$ are all roots. In the case (1), we have that $\alpha(H^{\gamma_i})=-1$ and $\alpha(H^{\gamma_j})=1$. Since $\alpha(H^{\gamma_k})=0$ $(k\neq i, j)$, it is easy to see that $\alpha(H^{\gamma_k})=2^{-1}(\gamma_j-\gamma_i)(H^{\gamma_k})$ $(1\leqq k\leqq s)$ and therefore $\alpha\sim 2^{-1}(\gamma_j-\gamma_i)$. In the case (2), we assert that $\gamma_i\,|\,\mathfrak{t}_{\mathfrak{h}}^{\mathbf{C}}=0$. For otherwise suppose $\gamma_i\,|\,\mathfrak{t}_{\mathfrak{h}}^{\mathbf{C}}\neq 0$, then $\alpha(H^{\gamma_i})=\alpha(H_{\gamma_i})-\alpha(\sigma H_{\gamma_i})=-2=-\gamma_i(H^{\gamma_i})$, that is, $(\alpha+\gamma_i)(H^{\gamma_i})=0$. On the other hand since γ_i, $-\gamma_i\circ\sigma$, $\alpha+\gamma_i$ are all non compact positive roots, neither $(\alpha+\gamma_i)+\gamma_i$ nor $(\alpha+\gamma_i)-\gamma_i\circ\sigma$ is a root (Lemma 3). But $(\alpha+\gamma_i)-\gamma_i$ is a root. Therefore

$$(\alpha+\gamma_i)(H^{\gamma_i})=(\alpha+\gamma_i)(H_{\gamma_i})-(\alpha+\gamma_i)(\sigma H_{\gamma_i})>0,$$

which is a contradiction. So $\gamma_i\,|\,\mathfrak{t}_{\mathfrak{h}}^{\mathbf{C}}=0$. Hence $\alpha(H^{\gamma_i})=\alpha(H_{\gamma_i})=-1$. Since $\alpha(H^{\gamma_j})=1$ and $\alpha(H^{\gamma_k})=0$ $(k\neq i, j)$, we have $\alpha(H^{\gamma_k})=2^{-1}(\gamma_j-\gamma_i)(H^{\gamma_k})$ $(1\leqq k\leqq s)$. This means that $\alpha\sim 2^{-1}(\gamma_j-\gamma_i)$. In the case (3), we can show that $\gamma_j\,|\,\mathfrak{t}_{\mathfrak{h}}^{\mathbf{C}}=0$. (If $\gamma_j\,|\,\mathfrak{t}_{\mathfrak{h}}^{\mathbf{C}}\neq 0$, then $\alpha(H^{\gamma_j})=\alpha(H_{\gamma_j})-\alpha(\sigma H_{\gamma_j})=2=\gamma_j(H^{\gamma_j})$, that is, $(\alpha-\gamma_j)(H^{\gamma_j})=0$. However we know that $(\alpha-\gamma_j)-\gamma_j\notin\Phi$, $(\alpha-\gamma_j)+\gamma_j\circ\sigma\notin\Phi$, $(\alpha-\gamma_j)+\gamma_j\in\Phi$. Therefore $(\alpha-\gamma_j)(H^{\gamma_j})=(\alpha-\gamma_j)(H_{\gamma_j})-(\alpha-\gamma_j)(\sigma H_{\gamma_j})<0$. This is a contradiction.) Hence $\alpha(H^{\gamma_j})=\alpha(H_{\gamma_j})=1$. Since $\alpha(H^{\gamma_i})=-1$ and $\alpha(H^{\gamma_k})=0$ $(k\neq i, j)$, we get $\alpha\sim 2^{-1}(\gamma_j-\gamma_i)$. In the case (4), we can show, as in the cases (2) and (3), that $\gamma_i\,|\,\mathfrak{t}_{\mathfrak{h}}^{\mathbf{C}}=\gamma_j\,|\,\mathfrak{t}_{\mathfrak{h}}^{\mathbf{C}}=0$. Hence $\alpha(H^{\gamma_i})=\alpha(H_{\gamma_i})=-1$ and $\alpha(H^{\gamma_j})=\alpha(H_{\gamma_j})=1$, and therefore $\alpha\sim 2^{-1}(\gamma_j-\gamma_i)$. The proof is now complete.

LEMMA 22. *P_n is the disjoint union of P^i, P_i, P_{ij} $(1\leqq i<j\leqq s)$.*

PROOF. The disjointness follows from Lemma 19. Suppose $\gamma\in P_n$. We assume that $\gamma\notin P^j$ for all j $(1\leqq j\leqq s)$. Since $X_\gamma\in\mathfrak{p}^{\mathbf{C}}$, $X_\gamma\notin\mathfrak{g}_{s+1}$. So we can choose the least index i such that $X_\gamma\notin\mathfrak{g}_{i+1}$. As $\gamma\notin P^i$ (especially, $\gamma\neq\gamma_i$), $\gamma>\gamma_i$. Moreover it follows from Lemma 3 that $\gamma+\gamma_i$ and $\gamma-\gamma_i\circ\sigma$ are not roots. Therefore, since $X_\gamma\notin\mathfrak{g}_{i+1}$, either $\gamma-\gamma_i$ or $\gamma+\gamma_i\circ\sigma$ is a root. We distinguish two cases: (1) $\gamma-\gamma_i$ is a root, (2) $\gamma+\gamma_i\circ\sigma$ is a root.

Consider the first case. Then $\alpha=\gamma-\gamma_i$ is a compact positive root. Since $\gamma\notin P^i$, $\alpha\notin C_0$. Hence either $\alpha\sim -2^{-1}\gamma_j$ or $\alpha\sim 2^{-1}(\gamma_k-\gamma_j)$ for some j or (j, k). If $\alpha\sim -2^{-1}\gamma_j$ then $\gamma\sim\gamma_i-2^{-1}\gamma_j$ and so $\gamma(H^{\gamma_j})=2\delta_{ij}-1$. On the other hand Lemma 3(1) tells us that $\gamma(H_{\gamma_j})\geqq 0$ and $\gamma(\sigma H_{\gamma_j})\leqq 0$, which implies that $\gamma(H^{\gamma_j})\geqq 0$.

Therefore $i=j$, $\gamma\sim 2^{-1}\gamma_i$ and $\gamma\in P_i$. If $\alpha\sim 2^{-1}(\gamma_k-\gamma_j)$ then $\gamma\sim\gamma_i+2^{-1}(\gamma_k-\gamma_j)$ and $\gamma(H^{\gamma_j})=2\delta_{ij}-1$. Since $\gamma(H^{\gamma_j})\geqq 0$, we conclude that $i=j$ and so $\gamma\in P_{ik}$.

Now consider the second case (2). The definition of the index i implies that $X_\gamma\in\mathfrak{g}_i$, and so $\sigma X_\gamma\in\mathfrak{g}_i$. Hence $-\gamma\circ\sigma\in Q_i$. Moreover since γ is not in P^i, $-\gamma\circ\sigma\neq\gamma_i$. Therefore we find that $-\gamma\circ\sigma>\gamma_i$. On the other hand $-\gamma_i-\gamma\circ\sigma$ is not identically zero on $\mathfrak{t}_\mathfrak{q}^C$ (for, $\gamma\notin P^i$). This implies that $\gamma+\gamma_i\circ\sigma$ is positive. Set $\alpha=\gamma+\gamma_i\circ\sigma$. Then α is a positive compact root such that $\alpha\notin C_0$. Hence Lemma 21 is applicable. Suppose that $\alpha\in C_j$ for some j. Then $\gamma\sim-\gamma_i\circ\sigma-2^{-1}\gamma_j\sim\gamma_i-2^{-1}\gamma_j$ (for, $\nu(\mathfrak{b}^C)\subset\mathfrak{q}^C$), and therefore we can show, as in the case (1), that $i=j$, But this implies that $\gamma\sim 2^{-1}\gamma_i$, that is, $\gamma\in P_i$. Next we suppose that $\alpha\in C_{jk}$ for some (j,k) $(1\leqq j<k\leqq s)$. Then $\gamma\sim-\gamma_i\circ\sigma+2^{-1}(\gamma_k-\gamma_j)\sim\gamma_i+2^{-1}(\gamma_k-\gamma_j)$. This implies that $i=j$. Hence $\gamma\sim 2^{-1}(\gamma_k+\gamma_i)$, that is, $\gamma\in P_{ik}$.

LEMMA 23. *For each index i, there exists a one-one mapping of C_i onto P_i.*

PROOF. Lemma 19 and the proof of Lemma 21 imply: (a) Let α be in C_i. Then i equals the least index such that $X_\alpha\notin\mathfrak{g}_{i+1}$. (b) Put $C_i'=\{\alpha\in C_i\colon \alpha+\gamma_i$ is a root $\}$ and put $C_i''=\{\alpha\in C_i\colon \alpha-\gamma_i\circ\sigma$ is a root$\}$. Then $C_i=C_i'\cup C_i''$. Moreover if $\gamma_i\,|\,\mathfrak{t}_\mathfrak{h}\neq 0$ then $C_i'\cap C_i''=\phi$. (Clearly, if $\gamma_i\,|\,\mathfrak{t}_\mathfrak{h}=0$ then $C_i'=C_i''$).

On the other hand, the proof of Lemma 22 tells us: (c) If γ is in P_i, then i is the least index such that $X_\gamma\notin\mathfrak{g}_{i+1}$. (d) $P_i=P_i'\cup P_i''$ where $P_i'=\{\gamma\in P_i\colon \gamma-\gamma_i$ is a root$\}$ and $P_i''=\{\gamma\in P_i\colon \gamma+\gamma_i\circ\sigma$ is a root$\}$.

We assert that if $\gamma_i\,|\,\mathfrak{t}_\mathfrak{h}\neq 0$ then $P_i'\cap P_i''=\phi$. Let γ be an element in $P_i'\cap P_i''$. Lemma 3(1) says that $\gamma+\gamma_i$, $\gamma-\gamma_i\circ\sigma$ are not roots. Moreover $\gamma-2\gamma_i\sim-(3/2)\gamma_i$ and $\gamma+2\gamma_i\circ\sigma\sim-(3/2)\gamma_i$. But these relations combined with Lemma 22 imply that $\gamma-2\gamma_i$, $\gamma+2\gamma_i\circ\sigma$ are not roots. Hence $\gamma(H_{\gamma_i})=1$ and $\gamma(\sigma H_{\gamma_i})=-1$, and therefore $\gamma(H^{\gamma_i})=2\neq 2^{-1}\gamma_i(H^{\gamma_i})$. This contradicts with $\gamma\in P_i$. Thus $P_i'\cap P_i''=\phi$ as asserted. We note that if $\gamma_i\,|\,\mathfrak{t}_\mathfrak{h}=0$ then $P_i'=P_i''$. Now it is obvious that the mapping $\alpha\to\alpha+\gamma_i$ (resp. $\alpha\to\alpha-\gamma_i\circ\sigma$) is a bijective correspondence between C_i' and P_i' (resp. C_i'' and P_i''). Hence the lemma follows.

LEMMA 24. *There exists a one-one mapping of C_{ij} onto P_{ij} $(1\leqq i<j\leqq s)$.*

PROOF. From the proof of Lemma 21 we get: (a) Let $\alpha\in C_{ij}$. Then i equals the least index such that $X_\alpha\notin\mathfrak{g}_{i+1}$. (b) Set $C_{ij}'=\{\alpha\in C_{ij}\colon \alpha+\gamma_i\in\Phi\}$, and set $C_{ij}''=\{\alpha\in C_{ij}\colon \alpha-\gamma_i\circ\sigma\in\Phi\}$. Then $C_{ij}=C_{ij}'\cup C_{ij}''$. Moreover if $\gamma_i\,|\,\mathfrak{t}_\mathfrak{h}\neq 0$ then $C_{ij}'\cap C_{ij}''=\phi$.

Similarly the proof of Lemma 22 implies: (c) Let $\gamma\in P_{ij}$. Then i is the least index such that $X_\gamma\notin\mathfrak{g}_{i+1}$. (d) $P_{ij}=P_{ij}'\cup P_{ij}''$ where $P_{ij}'=\{\gamma\in P_{ij}\colon \gamma-\gamma_i\in\Phi\}$ and $P_{ij}''=\{\gamma\in P_{ij}\colon \gamma+\gamma_i\circ\sigma\in\Phi\}$. Moreover we can show, as in the proof of Lemma 23, that if $\gamma_i\,|\,\mathfrak{t}_\mathfrak{h}\neq 0$ then $P_{ij}'\cap P_{ij}''=\phi$. Now it is easy to check that the mapping $\alpha\to\alpha+\gamma_i$ (resp. $\alpha\to\alpha-\gamma_i\circ\sigma$) is a bijective correspondence between C_{ij}'

and P'_{ij} (resp. C''_{ij} and P''_{ij}). The lemma now follows.

Let r_i, r_{ij}, p^i be the number of roots in C_i, C_{ij}, P^i respectively. Then Lemmas 23 and 24 say that r_i and r_{ij} are also the numbers of roots in P_i and P_{ij} respectively. Now we recall the maximal abelian subspace $\mathfrak{b}=\sum_{i=1}^{s}\boldsymbol{R}(X^{\gamma_i}+X^{-\gamma_i})$ and we retain the notation of Section 4. Moreover we fix the Haar measure dk (on K) and dH (on $\mathfrak{b}$) such that

$$\int_{G/H_\sigma} f(x)dx = \int_K \int_{\mathfrak{b}^+} f(k\exp H)D(\exp H)dkdH \qquad \text{for all} \quad f\in L^2(G/H_\sigma).$$

LEMMA 25. *Let* $H=\sum_i t_i(X^{\gamma_i}+X^{-\gamma_i})$ $(t_i\in\boldsymbol{R})$. *Then*

$$D(\exp H) \leqq 2^{\varepsilon}\prod_i (\cosh t_i)^{2\rho^i+2r_i+2s_i}.$$

Here $\varepsilon=\sum_{i=1}^{s}p^i$, $\rho^i=\rho(H^{\gamma_i})$ *and* $s_i=\sum_{i<j}r_{ij}$.

PROOF. Clearly $D(\exp H)\leqq\{\prod_{\beta\in\Delta_{\mathfrak{b}}}(\cosh\beta(H))^{p_\beta+q_\beta}\}^{1/2}$. On the other hand we have already seen that

$$\Delta_{\mathfrak{b}} = \{\alpha\circ\nu\,|_{\mathfrak{b}} : \alpha\in\Phi,\ \alpha\,|_{\nu(\mathfrak{b})}\neq 0\}.$$

Moreover for each β in $\Delta_{\mathfrak{b}}$, $p_\beta+q_\beta=\dim_{\boldsymbol{R}}\mathfrak{g}^\beta=$the number of those roots in Φ which coincide with α on $\nu(\mathfrak{b}^C)$, where α is a root in Φ so that $\beta=\alpha\circ\nu\,|_{\mathfrak{b}}$. Noting that $\nu(H)=\sum_i t_iH^{\gamma_i}$, we have

$$\begin{aligned}
D(\exp H) &\leqq \{\textstyle\prod_{\beta\in\Delta_{\mathfrak{b}}}(\cosh\beta(H))^{p_\beta+q_\beta}\}^{1/2}\\
&= \textstyle\prod_{1\leqq i\leqq s}(\cosh t_i)^{2r_i}\prod_{1\leqq i<j\leqq s}(\cosh(t_i-t_j)\cosh(t_i+t_j))^{r_{ij}}\\
&\quad\times\textstyle\prod_{1\leqq i\leqq s}(\cosh 2t_i)^{p^i}\\
&= \textstyle\prod_i(\cosh t_i)^{2r_i}\prod_{i<j}\{(\cosh t_i\cdot\cosh t_j)^2-(\sinh t_i\cdot\sinh t_j)^2\}^{r_{ij}}\\
&\quad\times\textstyle\prod_i\{(\cosh t_i)^2+(\sinh t_i)^2\}^{p^i}\\
&\leqq 2^{\varepsilon}\textstyle\prod_i(\cosh t_i)^{2r_i}\prod_{i<j}(\cosh t_i\cdot\cosh t_j)^{2r_{ij}}\prod_i(\cosh t_i)^{2p^i}\\
&= 2^{\varepsilon}\textstyle\prod_i(\cosh t_i)^{2r_i+2s_i+2s^i+2p^i}. \qquad (s^i=\sum_{j<i}r_{ji}).
\end{aligned}$$

A simple calculation shows that

$$2\rho(H^{\gamma_i}) = \textstyle\sum_{\alpha\in P}\alpha(H^{\gamma_i}) = 2s^i+2p^i.$$

Hence the lemma is true.

Recall that in Section 3 we constructed a C^∞ function f_Λ $(\Lambda\in L)$ on G/H_σ. We are now in a position to compute the norm of f_Λ. From the definition (Section 3), we have $\tau_0(H)\phi_0=\Lambda_0(H)\phi_0$ $(H\in\mathfrak{t}^C)$ and $\tau_0(X_\alpha)\phi_0=0$ for all α in P_k. Moreover we assert that for each α in P_k, $\tau_0(X_{-\alpha}^r)\phi_0=0$ for r sufficiently large.

For the nonzero vectors among $\tau_0(X_{-\alpha}^r)\phi_0$ $(r\geqq 0)$ are linearly independent since they belong to the distinct weight $\Lambda_0-r\alpha$. But the dimension of V_0 is finite. This implies our assertion. Therefore Lemma 2 of [6] tells us that the subspace V_0' of V_0 spaned by $\tau_0(X)\phi_0$ $(X\in U(\mathfrak{k}^C))$ is irreducible under K^C. Let τ_0' be the corresponding representation of K^C on V_0'.

For each element $H=\sum_i t_i(X^{\gamma_i}+X^{-\gamma_i})$ in $\mathfrak{b}$, we put $H'=\sum_i \log(\cosh t_i)H^{\gamma_i}$.

LEMMA 26. *Let Λ be an element of L. Then there exists a positive constant c_Λ so that*

$$|f_\Lambda(k\exp H)| \leqq c_\Lambda\|\tau_0'(\exp H')\|e^{\Lambda_-(H')} \qquad (k\in K,\ H\in\mathfrak{b})$$

where $\|\tau_0'(\exp H')\|$ is the operator norm of $\tau_0'(\exp H')$.

PROOF. Recall, $f_\Lambda(k\exp H)=(\phi_0,\ \tau_0(k\exp H)\psi_0)\prod_{j=1}^p(\phi_j,\ \tau_j(k\exp H)\psi_j)^{m_j}$. Combining Lemma 17, Lemma 3(2) and the fact that $\tilde{\theta}(X^{-\gamma_i})\in\mathfrak{p}_+$ (for the notation, see Lemma 3), we find

$$(\phi_0,\ \tau_0(k\exp H)\psi_0) = (\tau_0(\exp H'\cdot k^{-1})\phi_0,\ \tau_0(\exp \textstyle\sum_i(\tanh t_i)X^{\gamma_i})\psi_0).$$

Therefore we conclude that if $c_0=\mathrm{Max}_{-1\leqq x_i\leqq 1}\|\tau_0(\exp\sum_i x_iX^{\gamma_i})\psi_0\|$ then

$$|(\phi_0,\ \tau_0(k\exp H)\psi_0)| \leqq c_0\|\tau_0'(\exp H'\cdot k^{-1})\phi_0\| \leqq c_0\|\tau_0'(\exp H')\|.$$

Now we fix the index j $(1\leqq j\leqq p)$. We have shown in the proof of Lemma 10 that $\tau_j(X)\phi_j=0$ for any X in $\mathfrak{k}'=[\mathfrak{k},\mathfrak{k}]$. So using Lemma 17, we have

$$\begin{aligned}|(\phi_j,\ \tau_j(k\exp H)\psi_j)| &= |(\tau_j(\exp H')\phi_j,\ \tau_j(\exp \textstyle\sum_i(\tanh t_i)X^{\gamma_i})\psi_j)| \\ &= e^{\Lambda_j(H')}|(\phi_j,\ \tau_j(\exp\textstyle\sum_i(\tanh t_i)X^{\gamma_i})\psi_j)|.\end{aligned}$$

We recall the subgroups G^0, H^0, N_-^0 of G^C (see the proof of Lemma 8). It is easy to see that $\sigma\circ\tilde{\theta}(X^{\gamma_i})=X^{\gamma_i}(1\leqq i\leqq s)$. This implies that $\exp(\sum_i x_iX^{\gamma_i})\in G^0$ for any x_i $(1\leqq i\leqq s)$ in $\boldsymbol{R}$. Also we have already seen that $G^0=N_-^0\exp(i\mathfrak{t}_\mathfrak{q})H^0$ is an Iwasawa decomposition of G^0 (see the proof of Lemma 8). We define the element $H(x)$ in $i\mathfrak{t}_\mathfrak{q}$ for $x=(x_1,\dots,x_s)$ in $\boldsymbol{R}^s$ by

$$\exp(\textstyle\sum_i x_iX^{\gamma_i})\in N_-^0\exp H(x)H^0.$$

Then

$$|(\phi_j,\ \tau_j(\exp\textstyle\sum_i(\tanh t_i)X^{\gamma_i})\psi_j)| = |(\phi_j,\ \tau_j(\exp H(x))\psi_j)| = e^{\Lambda_j(H(x))}$$

where $x=(\tanh t_1,\dots,\tanh t_s)$. This shows that if $c_j=\mathrm{Min}_{-1\leqq x_i\leqq 1}e^{\Lambda_j(H(x))}$ then

$$|(\phi_j,\ \tau_j(k\exp H)\psi_j)^{m_j}| \leqq c_j^{m_j}\cdot e^{m_j\Lambda_j(H')}.$$

Noting that $\Lambda_-=\sum_{j=1}^p m_j\Lambda_j$ we have shown that

$$|f_\Lambda(k \exp H)| \leqq c_\Lambda \|\tau_0'(\exp H')\| e^{\Lambda_-(H')}$$

where $c_\Lambda = c_0 \prod_{j=1}^{p} c_j^{m_j}$.

LEMMA 27. *Let W_k denote the Weyl group of $(\mathfrak{k}^C, \mathfrak{t}^C)$. Set $2\rho_k = \sum_{\alpha \in P_k} \alpha$. Then for all H in $i\mathfrak{t}$ we have*

$$\|\tau_0'(\exp H)\|^2 \leqq \sum_{s \in W_k} e^{2s(\Lambda_0 + \rho_k)(H) - 2\rho_k(H)}.$$

PROOF. Since the representation τ_0' is irreducible $\tau_0'(\exp H) = e^{\Lambda_0(H)} \cdot I$ for any H in $\mathfrak{c}^C$ (by Schur's lemma). Moreover $s \,|\, \mathfrak{c}^C = I$ for all s in W_k. Hence we have only to prove the lemma for H in $i\mathfrak{t}'$ $(\mathfrak{t}' = \mathfrak{t} \cap \mathfrak{k}')$. Set $(i\mathfrak{t}')_0 = \{H \in i\mathfrak{t}' : \alpha(H) \neq 0$ for all α in $P_k\}$, and set $(i\mathfrak{t}')_+ = \{H \in i\mathfrak{t}' : \alpha(H) > 0$ for all α in $P_k\}$. Then $(i\mathfrak{t}')_0$ is dense in $i\mathfrak{t}'$, and $(i\mathfrak{t}')_0 = \bigcup_{s \in W_k} s(i\mathfrak{t}')_+$.

Since $\tilde{\theta} \,|\, i\mathfrak{t} = -I$, for each H in $i\mathfrak{t}$ $\tau_0'(\exp H)$ is a self-adjoint operator in V_0'. We remark that if T is a positive self-adjoint operator in a finite dimensional vector space over $\boldsymbol{C}$, then the operator norm $\|T\|$ of T cannot exceed the largest eigenvalue in its spectra. Also every weight of τ_0' is of the form $\Lambda_0 - \sum_{\alpha \in P_k} n_\alpha \alpha$ where n_α $(\alpha \in P_k)$ are nonnegative integers (see, [6, Lemma 2]). Let μ be a weight of τ_0'. Then $s^{-1}\mu$ $(s \in W_k)$ is also a weight (note that $s^{-1}\mu = \mu \circ s$), and so we can write

$$s^{-1}\mu = \Lambda_0 - \sum_{\alpha \in P_k} n_\alpha \alpha \qquad (n_\alpha \geqq 0).$$

Hence we find that if $H \in s(i\mathfrak{t}')_+$ then

$$\mu(H) = s\Lambda_0(H) - \sum_{\alpha \in P_k} n_\alpha (s\alpha)(H) \leqq s\Lambda_0(H).$$

Combining this with the above remark we find that

$$\|\tau_0'(\exp H)\|^2 \leqq e^{2s\Lambda_0(H)} \qquad (s \in W_k,\ H \in s(i\mathfrak{t}')_+).$$

On the other hand we know that $\rho_k(H_\alpha) \geqq 0$ for any α in P_k. This implies that $s\rho_k \leqq \rho_k$ (i.e. $s\rho_k(H) \leqq \rho_k(H)$ for all H in $(i\mathfrak{t}')_+$) for any s in W_k. Therefore $s\rho_k(H) - \rho_k(H) \geqq 0$ $(s \in W_k,\ H \in s(i\mathfrak{t}')_+)$. We therefore see that

$$\|\tau_0'(\exp H)\|^2 \leqq e^{2s(\Lambda_0 + \rho_k)(H) - 2\rho_k(H)} \qquad (s \in W_k,\ H \in s(i\mathfrak{t}')_+).$$

But this implies that

$$\|\tau_0'(\exp H)\|^2 \leqq \sum_{s \in W_k} e^{2s(\Lambda_0 + \rho_k)(H) - 2\rho_k(H)} \qquad (H \in (i\mathfrak{t}')_0).$$

Since $(i\mathfrak{t}')_0$ is dense in $i\mathfrak{t}'$ the result is true for any H in $i\mathfrak{t}'$.

THEOREM 1. *Assume the assumptions AI$\sim$AIV. Then there is a real constant $c\,(\leqq 0)$ so that if $(\Lambda + \rho)(H_\gamma) < c$ for all γ in P_n then $f_\Lambda \in L^2(G/H_\sigma)$.*

PROOF. Let $H = \sum_i t_i (X^{\gamma_i} + X^{-\gamma_i}) \in \mathfrak{b}$. Then $H' = \sum \log(\cosh t_i) H^{\gamma_i}$ is in *it*. Hence combining Lemmas 26 and 27 we find

$$|f_\Lambda(k\exp H)|^2 \leqq c_\Lambda^2\{\textstyle\sum_{s\in W_k} e^{2s(\Lambda_0+\rho_k)(H')-2\rho_k(H')}\}e^{2\Lambda_-(H')}.$$

On the other hand Lemma 5 says that $\Lambda_-(H_\alpha)=0$ for all α in P_k, and so $s\Lambda_-=\Lambda_-$ for any s in W_k. Also it follows from Lemma 3(3) that $s\rho_n=\rho_n$ for s in W_k. Here $2\rho_n=\sum_{\gamma\in P_n}\gamma$. Therefore

$$\begin{aligned}|f_\Lambda(k\exp H)|^2 &\leqq c_\Lambda^2\{\textstyle\sum_{s\in W_k} e^{2s(\Lambda+\rho)(H')}\}e^{-2\rho(H')}\\ &= c_\Lambda^2\{\textstyle\sum_{s\in W_k}\prod_i(\cosh t_i)^{2\{s(\Lambda+\rho)\}^i}\}\prod_i(\cosh t_i)^{-2\rho^i}.\end{aligned}$$

Here $\{s(\Lambda+\rho)\}^i=s(\Lambda+\rho)(H^{\gamma_i})$. This inequality combined with Lemma 25 implies that

$$|f_\Lambda(k\exp H)|^2D(\exp H) \leqq c_\Lambda^2 2^\varepsilon\{\textstyle\sum_{s\in W_k}\prod_i(\cosh t_i)^{2\{s(\Lambda+\rho)^i\}}\}\prod_i(\cosh t_i)^{2r_i+2s_i}.$$

Now let $c=-\mathrm{Max}_i(r_i+s_i)$. Then noting Lemma 3(3), we find that if $(\Lambda+\rho)(H_\gamma)<c$ for all γ in P_n then $f_\Lambda\in L^2(G/H_\sigma)$.

§6. Irreducibility of H_Λ

In this paragraph we assume: (1) $\Lambda\in L$, (2) $f_\Lambda\in L^2(G/H_\sigma)$. Let H_Λ be the smallest closed subspace of $L^2(G/H_\sigma)$ containing f_Λ which is invariant under $\pi(G)$. (π is the left regular representation of G on $L^2(G/H_\sigma)$). Let $\mathfrak{n}$, $\bar{\mathfrak{n}}$, $\bar{N}$, T, T^C, ξ_Λ, Γ_Λ be as in Section 2.

LEMMA 28 (Harish-Chandra [7, Lemma 6]). *There exists a function* $\phi^\Lambda\in\Gamma_\Lambda$ *such that*

$$\int_T\phi(xhwh^{-1})dh = \phi(x)\phi^\Lambda(w) \qquad (x\in G,\ w\in GT^C\bar{N})$$

for every $\phi\in\Gamma_\Lambda$, *(dh is the normalized Haar measure on T). This function is unique and* $\phi^\Lambda(1)=1$.

Let E_Λ be the subspace of H_Λ spanned by $\pi(k)f_\Lambda$ $(k\in K)$. We have seen in the proof of Lemma 10 that $\tau_j(X)\phi_j=0$ for all X in $\mathfrak{k}'^C$ $(1\leqq j\leqq p)$. But this implies that $\tau_j(k)\phi_j\in \boldsymbol{C}\phi_j$ for every k in K $(1\leqq j\leqq p)$, and therefore the definition of f_Λ tells us that E_Λ is finite dimensional.

LEMMA 29. *E_Λ is irreducible under $\pi(k)$ $(k\in K)$.*

PROOF. Let $E_\Lambda=\sum_{i=1}^n E_i$ be a decomposition of E_Λ into the irreducible components. Let $f_\Lambda=\sum_i f_i$ $(f_i\in E_i)$. We can write $f_i=\sum_j c_{ij}f_\Lambda(k_{ij}^{-1}\cdot)$ where c_{ij} are complex numbers and k_{ij} are in K. From Lemma 11, for each element h of T we have

$$\sum_i f_i = f_\Lambda = \xi_\Lambda(h^{-1})\pi(h)f_\Lambda = \sum_i \xi_\Lambda(h^{-1})\pi(h)f_i,$$

and therefore $f_i = \xi_\Lambda(h^{-1})\pi(h)f_i$ $(1 \leqq i \leqq n,\ h \in T)$. Now let F_Λ be as in the proof of Lemma 11. Since $f_\Lambda(x) = \{F_\Lambda(x^{-1})\}^-$ $(x \in G)$ and $F_\Lambda \in \Gamma_\Lambda$, we have for x in G

$$\begin{aligned} f_i(x) &= \xi_\Lambda(h^{-1})f_i(h^{-1}x) && (h \in T) \\ &= \sum_j c_{ij} f_\Lambda(k_{ij}^{-1}h^{-1}x)\xi_\Lambda(h^{-1}) = \sum_j c_{ij}\{F_\Lambda(x^{-1}hk_{ij})\xi_\Lambda(h)\}^- \\ &= \sum_j c_{ij}\{F_\Lambda(x^{-1}hk_{ij}h^{-1})\}^- = \sum_j c_{ij}\int_T \{F_\Lambda(x^{-1}hk_{ij}h^{-1})\}^- dh. \end{aligned}$$

We apply Lemma 28 and find

$$\int_T F_\Lambda(x^{-1}hk_{ij}h^{-1})dh = F_\Lambda(x^{-1})\phi^\Lambda(k_{ij}).$$

Thus

$$f_i(x) = [\sum_j c_{ij}\{\phi^\Lambda(k_{ij})\}^-]f_\Lambda(x) \qquad (1 \leqq i \leqq n,\ x \in G).$$

But this means that there is an index i_0 so that $f_\Lambda \in \boldsymbol{C}f_{i_0} \subset E_{i_0}$, that is, $E_\Lambda = E_{i_0}$. This implies the lemma.

Let $\hat{K}$ denote the set of all equivalence classes of finite dimensional irreducible unitary representations of K. For each δ in $\hat{K}$, let χ_δ denote the character of δ, $d(\delta)$ the degree of δ. Put $P(\delta) = d(\delta)\int_K \bar{\chi}_\delta(k)\pi(k)dk$, dk normalized Haar measure on K. Let $H_\Lambda(\delta) = P(\delta)H_\Lambda$. Then $H_\Lambda(\delta)$ consists of those vectors in H_Λ, the linear span of whose K-orbit is finite dimensional and splits into irreducible K-submodules of type δ. Set $H_{\Lambda,K} = \sum_{\delta \in \hat{K}} H_\Lambda(\delta)$. Since $H_\Lambda \cap C^\infty(G/H_\sigma)$ is dense in H_Λ, $H_{\Lambda,K}$ is a dense subspace of H_Λ (cf. [28, Proposition 4.4.3.5]). Set $H_{\Lambda,0} = \pi(U(\mathfrak{g}^C))f_\Lambda$, and set $H_{\Lambda,0}(\delta) = H_{\Lambda,0} \cap H_\Lambda(\delta)$ $(\delta \in \hat{K})$.

LEMMA 30. *$H_{\Lambda,0} \subset H_{\Lambda,K}$. $H_{\Lambda,0}$ is dense in H_Λ. $H_{\Lambda,0}(\delta)$ is a dense subspace of $H_\Lambda(\delta)$ for any δ in $\hat{K}$.*

PROOF. If $Z \in U(\mathfrak{g}^C)$ and $k \in K$,

$$\pi(k)\pi(Z)f_\Lambda = \pi(Z^k)\pi(k)f_\Lambda \in \pi(Z^k)E_\Lambda.$$

Since E_Λ is finite dimensional, this implies the first assertion. G is connected and f_Λ is analytic, so the second assertion is obtained by a standard argument.

For each f in $H_{\Lambda,0}$, the linear span of K-orbit of f is a finite dimensional subspace of $H_{\Lambda,0}$. Therefore $P(\delta)H_{\Lambda,0} \subset H_{\Lambda,0} \cap H_\Lambda(\delta) = H_{\Lambda,0}(\delta)$. Using $P(\delta)| H_\Lambda(\delta) = I$, we get $P(\delta)H_{\Lambda,0} = H_{\Lambda,0}(\delta)$. Now let $f \in H_\Lambda(\delta)$ and suppose that $f_n \to f$ where $f_n \in H_{\Lambda,0}$. Since $P(\delta)$ is continuous, $P(\delta)f_n \to P(\delta)f = f$. Thus $H_{\Lambda,0}(\delta)$ is dense in $H_\Lambda(\delta)$.

LEMMA 31. *Let δ_Λ denote the irreducible unitary representation of K with* *highest weight Λ. Then*

$$H_{\Lambda,0}(\delta_\Lambda) = E_\Lambda.$$

PROOF. Obviously $H_{\Lambda,0}(\delta_\Lambda) \supset E_\Lambda$. So we need only to prove that $[H_{\Lambda,0}(\delta_\Lambda); \delta_\Lambda] \leqq 1$. Define

$$H_{\Lambda,0}(\delta_\Lambda)_h = \{f \in H_{\Lambda,0}(\delta_\Lambda)\colon (1)\ \pi(X_\alpha)f = 0 \quad \text{for all } \alpha \text{ in } P_k, \quad (2)\ \pi(H)f = \Lambda(H)f \text{ for all } H \text{ in } \mathfrak{t}^C\}.$$

It is enough to prove that $\dim H_{\Lambda,0}(\delta_\Lambda)_h \leqq 1$. Put $\mathscr{F} = \{\sum_{\alpha \in P} n_\alpha \alpha\colon n_\alpha$ are non negative integers$\}$. For each ν in $\mathscr{F}$, let

$$U(\bar{\mathfrak{n}})^{-\nu} = \{u \in U(\bar{\mathfrak{n}})\colon [H, u] = -\nu(H)u \text{ for all } H \text{ in } \mathfrak{t}^C\}.$$

Then $U(\bar{\mathfrak{n}}) = \sum_{\nu \in \mathscr{F}} U(\bar{\mathfrak{n}})^{-\nu}$. We remark that $U(\bar{\mathfrak{n}})^0 = \boldsymbol{C}$. Now set $K_\Lambda = \{u \in U(\bar{\mathfrak{n}})\colon uf_\Lambda = 0\}$. Then for each u in $U(\bar{\mathfrak{n}})$, we can choose the elements $u_i \in U(\bar{\mathfrak{n}})$ and $\nu_i \in \mathscr{F}$ $(i = 1,\dots, n)$ such that

(a) $u_i \in U(\bar{\mathfrak{n}})^{-\nu_i}$ $(1 \leqq i \leqq n)$,

(b) $u = \sum_i u_i$ modulo K_Λ, and

(c) $\{u_1 f_\Lambda,\dots, u_n f_\Lambda\}$ is linearly independent.

Let f be in $H_{\Lambda,0}(\delta_\Lambda)_h$. Since $U(\mathfrak{g}^C) = U(\mathfrak{g}^C)\mathfrak{n} + U(\bar{\mathfrak{n}})U(\mathfrak{t}^C)$, we can write $f = uf_\Lambda$ where u is in $U(\bar{\mathfrak{n}})$. For each H in $\mathfrak{t}^C$, we have

$$\begin{aligned} Hf &= H(uf_\Lambda) = H(\textstyle\sum_i u_i f_\Lambda) = \sum_i [H, u_i] f_\Lambda + \sum_i u_i (Hf_\Lambda) \\ &= -\textstyle\sum_i \nu_i(H) u_i f_\Lambda + \Lambda(H) \sum_i u_i f_\Lambda = -\sum_i \nu_i(H) u_i f_\Lambda + \Lambda(H) f. \end{aligned}$$

On the other hand, $Hf = \Lambda(H)f$ since f is in $H_{\Lambda,0}(\delta_\Lambda)_h$. Hence $\sum_i \nu_i(H) u_i f_\Lambda = 0$ for all H in $\mathfrak{t}^C$. This combined with (c) implies that $\nu_i = 0$ $(1 \leqq i \leqq n)$, and so $f \in \boldsymbol{C} f_\Lambda$. Thus $\dim H_{\Lambda,0}(\delta_\Lambda)_h \leqq 1$, as we wished to prove.

THEOREM 2. *Assume the assumptions* AI$\sim$AIV. *Let Λ be an element of L such that f_Λ is in $L^2(G/H_\sigma)$. Let H_Λ be the smallest closed subspace of $L^2(G/H_\sigma)$ containing f_Λ which is invariant under $\pi(G)$. Then H_Λ is irreducible under π.*

PROOF. First we assert that $H_\Lambda(\delta_\Lambda) = E_\Lambda$. Indeed, $H_{\Lambda,0}(\delta_\Lambda)$ is a finite dimensional dense subspace of $H_\Lambda(\delta_\Lambda)$ (Lemmas 30, 31). Hence $H_\Lambda(\delta_\Lambda) = H_{\Lambda,0}(\delta_\Lambda) = E_\Lambda$ as asserted. Now let H_1 be any closed invariant subspace of H_Λ and let H_2 be the orthogonal complement of H_1 in H_Λ. Then $H_\Lambda = H_1 + H_2$, and therefore either $P(\delta_\Lambda)H_1 \neq 0$ or $P(\delta_\Lambda)H_2 \neq 0$. If $P(\delta_\Lambda)H_1 \neq 0$, then $P(\delta_\Lambda)H_1 = E_\Lambda$ since $H_\Lambda(\delta_\Lambda) = E_\Lambda$ is irreducible under $\pi(K)$. But this implies that $H_1 \ni f_\Lambda$, and hence $H_1 = H_\Lambda$. Similar reasoning shows that if $P(\delta_\Lambda)H_2 \neq 0$ then $H_1 = 0$. Hence the theorem follows.

§7. The final result

Let Λ be an element of L such that $f_\Lambda \in L^2(G/H_\sigma)$. Then from Lemmas 11, 29 and Theorem 2, we have

(a) $\pi(X_\alpha)f_\Lambda=0$ for every positive root α in Φ,
(b) $\pi(H)f_\Lambda=\Lambda(H)f_\Lambda$ for all H in $\mathfrak{t}^C$, and
(c) H_Λ is irreducible under $\pi(G)$ and dim $\pi(U(\mathfrak{k}^C))f_\Lambda$ is finite.

Now we recall the holomorphic discrete series representation $(U_\Lambda, \mathscr{H}_\Lambda)$ (see the last of Section 2). Lemma 29 of [8] says that ψ^Λ is in $\mathscr{H}_\Lambda$. (For the definition of ψ^Λ, see Lemma 28.) Moreover the following conditions hold:

(a′) $U_\Lambda(X_\alpha)\psi^\Lambda=0$ for every α in P.
(b′) $U_\Lambda(H)\psi^\Lambda=\Lambda(H)\psi^\Lambda$ for all H in $\mathfrak{t}^C$.
(c′) U_Λ is irreducible and dim $U_\Lambda(U(\mathfrak{k}^C))\psi^\Lambda$ is finite.

(For a proof, see [7, Lemmas 8 and 12].) Therefore H_Λ and $\mathscr{H}_\Lambda$ are infinitesimally equivalent. Consequently H_Λ and $\mathscr{H}_\Lambda$ are unitarily equivalent (see, [6, Theorem 2] and [28, p. 329]).

For a non positive real number c, let L_c denote the set of all Λ in L which satisfy the following condition:

$$(\Lambda+\rho)(H_\gamma)<c \qquad \text{for all} \quad \gamma \quad \text{in} \quad P_n.$$

We remark that L_c is a infinite set for every c (see, Lemma 5).

On the basis of these observations and Theorem 1, we have

THEOREM 3. *Assume the assumptions* AI~AIV. *Then there exists a real constant* $c(\leqq 0)$ *so that if* Λ *is in* L_c, *then* $(U_\Lambda, \mathscr{H}_\Lambda)$ *is a representation of the discrete series for* G/H_σ. *Here* $(U_\Lambda, \mathscr{H}_\Lambda)$, *which was defined at the end of* §2, *is an element of the holomorphic discrete series for* G.

Bibliography

[1] M. Berger, *Les espaces symétriques non-compacts*, Ann. Sci. École Norm. Sup., **74** (1957), 85–177.
[2] H. Doi, *A classification of certain symmetric Lie algebras*, Hiroshima Math. J., **11** (1981), 173–180.
[3] J. Faraut, *Noyaux sphériques sur un hyperboloide a une nappe*, Lecture Notes in Math., **497**, Springer-Verlag, Berlin and New York, 1973.
[4] ———, *Distributions sphériques sur les espaces hyperboliques*, preprint.
[5] I. M. Gelfand, M. I. Graef and N. Ya. Vilenkin, Generalized Functions, vol. 5, Academic Press, New York, 1966.

[6] Harish-Chandra, *Representation of semisimple Lie groups* IV, Amer. J. Math., **77** (1955), 743–777.

[7] ———, *Representations of semisimple Lie groups* V, Amer. J. Math., **78** (1956), 1–41.

[8] ———, *Representations of semisimple Lie groups* VI, Amer. J. Math., **78** (1956), 564–628.

[9] S. Helgason, Differential Geometry and Symmetric Spaces, Academic Press, New York, 1962.

[10] K. Hiraoka, S. Matsumoto and K. Okamoto, *Eigenfunctions of the Laplacian on a real hyperboloid of one sheet*, Hiroshima Math. J., **7** (1977), 855–864.

[11] M. Flensted-Jensen, *Discrete series for semisimple symmetric spaces*, Ann. of Math., **111** (1980), 253–311.

[12] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry I, Interscience Publishers, 1963.

[13] S. S. Koh, *On affine symmetric spaces*, Trans. Amer. Math. Soc., **119** (1965), 291–309.

[14] O. Loos, Symmetric Spaces I, II, W. A. Benjamin, Inc., New York, Amsterdam, 1965.

[15] T. Matsuki, *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups*, J. Math. Soc. Japan, **31** (1979), 331–357.

[16] S. Matsumoto, *The Plancherel formula for a pseudo-Riemannian symmetric space*, Hiroshima Math. J., **8** (1978) 181–193.

[17] V. F. Molchanov, *Harmonic analysis on a hyperboloid of one sheet*, Soviet Math. Dokl., **7** (1966), 1533–1556.

[18] ———, *Analogue of the Plancherel formula for hyperboloid*, Soviet Math., Dokl., **9** (1968), 1382–1385.

[19] ———, *Representations of pseudo-orthogonal group associated with a cone*, Math. USSR Sbornik, **10** (1970), 333–347.

[20] K. Nomizu, *Invariant affine connections on homogeneous spaces*, Amer. J. Math., **76** (1954), 33–65.

[21] ———, *Reduction theorem for connections and its application to the problem of isotropy and holonomy groups of a Riemannian manifold*, Nagoya Math. J., **9** (1955), 57–66.

[22] T. Oshima and T. Matsuki, *Orbits on affine symmetric spaces under the action of the isotropy subgroups*, J. Math. Soc. Japan, **32** (1980), 399–414.

[23] W. Rossmann, *The structure of semisimple symmetric spaces*, Queen's Math., preprint (1977).

[24] ———, *Analysis on real hyperbolic spaces*, J. Functional Analysis, **30** (1978), 448–477.

[25] T. Shintani, *On the decomposition of regular representation of the Lorentz group on a hyperboloid of one sheet*, Proc. Japan Acad., **43** (1967), 1–5.

[26] R. S. Strichartz, *Harmonic analysis on hyperboloids*, J. Functional Analysis, **12** (1973), 341–383.

[27] N. Wallach, Harmonic Analysis on Homogeneous Spaces, Marcel Dekker, Inc., New York, 1973.

[28] G. Warner, Harmonic Analysis on Semisimple Lie Groups I, Springer-Verlag, Berlin and New York, 1972.

Department of Mathematics,
Faculty of Science,
Hiroshima University