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Programmings with constraints of convex processes

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§ 1. Introduction

Let X and Y be real linear spaces which are in duality with respect to a bilinear functional $((\ ,\ ))_1$ and let Z and W be real linear spaces which are in duality with respect to a bilinear functional $((\ ,\ ))_2$. A standard linear programming problem is:

(P_1) Find $M_1 = \inf\{((x, y_0))_1;\ x \in P,\ Ax - z_0 \in Q\}$,

where $y_0 \in Y$, $z_0 \in Z$, P and Q are weakly closed convex cones in X and Z respectively and A is a weakly continuous linear mapping from X to Z.

A dual problem of (P_1) is:

(D_1) Find $M_1^* = \sup\{((z_0, w))_2;\ w \in Q^\circ,\ y_0 - A^*w \in P^\circ\}$,

where P° and Q° are the polar sets of P and Q respectively and A^* is the adjoint mapping of A.

Kretschmer showed

THEOREM 0 ([4; Theorem 3]). (a) *If the set* $H = \{(Ax - z, r + ((x, y_0))_1);\ x \in P,\ z \in Q,\ r \geq 0\}$ *is weakly closed in* $Z \times R$ *and* M_1 *or* M_1^* *is finite, then* $M_1 = M_1^*$ *and there exists an* $x_0 \in P$ *such that* $Ax_0 - z_0 \in Q$ *and* $((x_0, y_0))_1 = M_1$.

(b) *If there exists an element* $w_0 \in Q^\circ$ *such that* $y_0 - A^*w_0$ *is contained in the interior of* P° *with respect to the Mackey topology, then* H *is weakly closed.*

Later on, Fan [2 and 3] dealt with the case where one of P and Q is merely closed convex. Under some conditions, he showed a duality between (P_1) and (D_2):

(D_2) $\sup\{(((z_0, w))_2 - 1)/r;\ r > 0,\ w \in Q^\circ,\ ry_0 - A^*w \in P^\circ\}$.

Furthermore, Levin-Pomerol [5] and Zălinescu [9] were interested in the problems which contain a positively homogeneous functional:

(P_3) $\inf\{((x, y_0))_1;\ x \in P,\ (Ax - C) \cap Q \neq \emptyset\}$,

where C is a weakly compact convex subset of Z,

(D_3) $\sup\{(g_C(w) - 1)/r;\ r > 0,\ w \in Q^\circ,\ ry_0 - A^*w \in P^\circ\}$, where $g_C(w)=\inf\{((z, w))_2;\ z\in C\}$.

On the other hand, programming problems with set-valued mappings have been discussed by several authors. Tagawa considered the following (P_4) in [8; Kapitel 8]:

$$(P_4)\quad \inf\{f(x);\ x\in P,\ Bx\cap Q\neq\emptyset\},$$

where f is a convex functional on X, P is a quasi-convex subset of X, Q is a convex cone in Z which has a nonempty interior with respect to the Mackey topology and B is a set-valued mapping from X to Z whose graph is convex.

Here we are concerned with the problem (P_4). If we assume that f and B are positively homogeneous, then a dual problem of (P_4) is similar to (D_1)$\sim$(D_3). In the present paper, based on the ideas in linear programming problems, we deal with the problem (P_4) under some additional assumptions.

In § 2, we define a programming problem and its dual problem, and prove a duality theorem, which is a generalization of Zălinescu [9; Theorem 4]. In § 3, we are concerned with sufficient conditions for our duality theorem. Fan [3; Theorem 6] and Nakamura-Yamasaki [6; Theorem 3.1] improved Theorem 0, (b), by using Dieudonné's proposition [1; Proposition 1]. Here we also use the proposition and give conditions similar to the ones in Fan [3; Theorems 4 and 6] and Theorem 0, (b).

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§ 2. A duality theorem

Let X, Y, Z and W be as in the preceding section. By $\sigma(X, Y)$ we denote the weak topology on X, and by $\tau(X, Y)$ the Mackey topology on X with respect to the above duality. Throughout this paper, we assume that each one of the paired spaces is assigned the weak topology unless otherwise stated. We always assume also that the cones considered have their vertices at the origin of the space.

Let P be a closed convex subset of X. The polar set P° of P is defined by

$$P^\circ = \{y\in Y;\ ((x, y))_1 \geq -1 \quad \text{for all} \quad x\in P\}.$$

If P is a convex cone, then $P^\circ=\{y\in Y;\ ((x, y))_1\geq 0 \text{ for all } x\in P\}$. The polar set $P^{\circ\circ}$ of P° is equal to P if and only if P is a closed convex set which contains the origin. We define Q° and $Q^{\circ\circ}$ for a closed convex subset Q of Z similarly.

We introduce the definitions of a convex process and its adjoints which are due to Rockafellar [7].

DEFINITION 1. A convex process from X to Z is a set-valued mapping $A: x \to Ax$ such that

(1) $A(x_1 + x_2) \supset Ax_1 + Ax_2$ for all $x_1, x_2 \in X$,
(2) $A(tx) = tAx$ for all $x \in X$ and $t > 0$,
(3) $0 \in A0$.

We set $\operatorname{dom} A = \{x;\ Ax \neq \emptyset\}$, where $\emptyset$ denotes the empty set. A set-valued mapping A is a convex process if and only if $\operatorname{graph} A = \{(x, z);\ x \in \operatorname{dom} A$ and $z \in Ax\}$ is a convex cone which contains the origin of $X \times Z$. If $\operatorname{graph} A$ is a closed convex cone, A is called a closed convex process.

We shall define two kinds of adjoints.

DEFINITION 2. Let A be a convex process from X to Z. We say that A is supremum oriented if its adjoint convex process A^* from W to Y is defined by

$$A^*w = \{y \in Y;\ ((z, w))_2 \leq ((x, y))_1 \quad \text{for all} \quad x \in \operatorname{dom} A \quad \text{and} \quad z \in Ax\}.$$

An infimum oriented convex process and its adjoint are defined in the same way with the reversed inequality. A convex process which is supremum or infimum oriented is called an oriented convex process.

These adjoints are always closed convex processes. If A is a supremum oriented (resp. infimum oriented) closed convex process, then we understand that A^* is infimum (resp. supremum) oriented, so that $A = A^{**}$.

An extended real valued function f on X is called positively homogeneous if $f(tx) = tf(x)$ for all $t > 0$ and $x \in X$. Let f be a lower semicontinuous positively homogeneous convex function on X and g be an upper semicontinuous positively homogeneous concave function on W. In this paper, we always assume that $-\infty < f(x) \leq \infty$, $-\infty \leq g(w) < \infty$ and $f(0) = g(0) = 0$. We set $\partial f(0) = \{y \in Y;\ f(x) \geq ((x, y))_1$ for all $x \in X\}$ and $\partial g(0) = \{z \in Z;\ g(w) \leq ((z, w))_2$ for all $w \in W\}$.

We prove

LEMMA 1. *Both $\partial f(0)$ and $\partial g(0)$ are nonempty closed convex sets and $f(x) = \sup\{((x, y))_1;\ y \in \partial f(0)\}$ for all $x \in X$ and $g(w) = \inf\{((z, w))_2;\ z \in \partial g(0)\}$ for all $w \in W$. Furthermore if f is $\tau(X, Y)$-continuous on X, then $\partial f(0)$ is $\sigma(Y, X)$-compact. Similarly if g is $\tau(W, Z)$-continuous, then $\partial g(0)$ is $\sigma(Z, W)$-compact.*

PROOF. We set $\operatorname{dom} f = \{x \in X;\ f(x) < \infty\}$ and $\operatorname{epi} f = \{(x, r) \in X \times R;\ x \in \operatorname{dom} f$ and $r \geq f(x)\}$. Since f is lower semicontinuous, $\operatorname{epi} f$ is a nonempty closed convex subset of $X \times R$. First we show $\partial f(0) \neq \emptyset$. Since $(0, -1) \notin \operatorname{epi} f$, by the separation theorem, there exists $y_1 \in Y$ such that $((x, y_1))_1 + r \geq 0$ for all $(x, r) \in \operatorname{epi} f$. This yields that $-y_1 \in \partial f(0)$. Hence $\partial f(0) \neq \emptyset$. Next we show that $f(x) = \sup\{((x, y))_1;\ y \in \partial f(0)\}$ for all $x \in X$. Let $x_0 \in X$ and $r_0 < f(x_0)$. Since

$(x_0, r_0) \notin \operatorname{epi} f$, by the separation theorem, there exists $(y_1, t) \in Y \times R$ such that $((x_0, y_1))_1 + r_0 t = -1$ and $((x, y_1))_1 + rt \geq 0$ for all $(x, r) \in \operatorname{epi} f$. From this, $t \geq 0$. If $t > 0$, then we see that $-t^{-1} y_1 \in \partial f(0)$ and $((x_0, -t^{-1} y_1))_1 > r_0$. This means that $f(x_0) = \sup\{((x_0, y))_1;\ y \in \partial f(0)\}$. If $t = 0$, then $((x, y_1))_1 \geq 0$ for all $x \in \operatorname{dom} f$ and $((x_0, y_1))_1 = -1$. It follows that $x_0 \notin \operatorname{dom} f$. Let $y_0 \in \partial f(0)$. Then $y_0 - s y_1 \in \partial f(0)$ for all $s > 0$, since $((x, y_0 - s y_1))_1 \leq ((x, y_0))_1 \leq f(x)$ for all $x \in X$ and all $s > 0$. Thus $f(x_0) = \sup\{((x_0, y))_1;\ y \in \partial f(0)\} = \infty$, because $((x_0, y_0 - s y_1))_1 = ((x_0, y_0))_1 + s$.

It is easy to check that $\partial f(0)$ is closed convex. Finally we show that $\partial f(0)$ is $\sigma(Y, X)$-compact if f is $\tau(X, Y)$-continuous on X. By considering $f(x) - ((x, y_0))_1$ for $y_0 \in \partial f(0)$, we may assume $f \geq 0$. For $U = \{x; f(x) \leq 1\}$, we easily see that $\partial f(0) = -U^\circ$. Since the $\tau(X, Y)$-interior of U contains the origin, we infer that $\partial f(0)$ is $\sigma(Y, X)$-compact by Alaoglu-Bourbaki's theorem.

By considering $-g$, we see that the assertions on the functional g also hold. This completes the proof.

Hereafter let A be a supremum oriented closed convex process from X to Z, and P, Q be closed convex subsets of X and Z respectively. We assume that at least one of P and Q is a cone. Let f and g be as above. For (A, P, Q, f, g), we consider the following programming problem (2.1) and its dual problem (2.2):

(2.1) Find $M = \inf\{f(x);\ x \in S\}$,

where $S = \{x \in P;\ (Ax - \partial g(0)) \cap Q \neq \emptyset\}$.

(2.2) Find $\tilde{M}^* = \sup\{(g(w) - 1)/r;\ (w, r) \in \tilde{S}^*\}$,

where $\tilde{S}^* = \{(w, r);\ r > 0,\ w \in Q^\circ,\ (r\partial f(0) - A^* w) \cap P^\circ \neq \emptyset\}$. If both P and Q are cones, then $\tilde{M}^*$ is equal to M^* in the problem

(2.2′) Find $M^* = \sup\{g(w);\ w \in S^*\}$,

where $S^* = \{w \in Q^\circ;\ (\partial f(0) - A^* w) \cap P^\circ \neq \emptyset\}$.
Here we use the convention that the infimum of a real function on the empty set is equal to ∞. It is easy to check that $\tilde{M}^* \leq M$.

Now we state

THEOREM 1. *We assume that f is $\tau(X, Y)$-continuous and the following two conditions are satisfied:*

(2.3) *There exists an element $\tilde{x} \in P$ such that $A\tilde{x} \cap Q \neq \emptyset$ and $f(\tilde{x}) \leq 0$.*

(2.4) *The set $G = \{(x, -z, f(x) + r);\ x \in \operatorname{dom} A,\ z \in Ax,\ r \geq 0\} + (-P) \times Q \times \{0\} + \{0\} \times \partial g(0) \times \{0\}$ is a closed subset of $X \times Z \times R$.*

Then $M=\tilde{M}^*$ *or* $M=-\tilde{M}^*=\infty$. *Furthermore if* M *is finite, then there exists an element* $x_0\in S$ *such that* $M=f(x_0)$.

For the proof of Theorem 1 we shall use

LEMMA 2. *We set*

$$G_0=\{(x,\ -z,f(x)+r);\ x\in\mathrm{dom}\,A\cap\mathrm{dom}f,\ z\in Ax,\ r\geq 0\},$$

$$H_0=\{(y_1-ty_2,\ w,\ t);\ w\in\mathrm{dom}\,A^*,\ y_1\in A^*w,\ y_2\in\partial f(0),\ t\geq 0\}.$$

Then the polar set G_0° *of* G_0 *with respect to the dual pair* $\{X\times Z\times R,\ Y\times W\times R\}$ *is equal to the closure* $\bar{H}_0$ *of* H_0. *In particular, if* f *is* $\tau(X, Y)$*-continuous, then* $G_0^\circ=H_0$.

PROOF. It is easy to check that $H_0\subset G_0^\circ$. We show that $H_0^\circ\subset G_0$. Let $(x_1, z_1, r_1)\in H_0^\circ$. Then $((x_1, y_1-ty_2))_1+((z_1, w))_2+tr_1\geq 0$ for all $w\in\mathrm{dom}\,A^*$, $y_1\in A^*w$, $y_2\in\partial f(0)$ and $t\geq 0$. We set $t=0$. Then $((x_1, y_1))_1+((z_1, w))_2\geq 0$ for all $w\in\mathrm{dom}\,A^*$ and $y_1\in A^*w$. Since $y_1\in A^*w$ if and only if $(y_1, -w)\in[\mathrm{graph}\,A]^\circ$, we see that $(x_1, -z_1)\in[\mathrm{graph}\,A]^{\circ\circ}=\mathrm{graph}\,A$. Thus $-z_1\in Ax_1$. Next we set $w=0$, $y_1=0$ and $t=1$. Then $r_1-((x_1, y_2))_1\geq 0$ for all $y_2\in\partial f(0)$. By Lemma 1, we see that $r_1\geq f(x_1)$. It follows that $(x_1, z_1, r_1)\in G_0$. Thus $\bar{H}_0=H_0^{\circ\circ}=G_0^\circ$.

If f is $\tau(X, Y)$-continuous, then $\partial f(0)$ is $\sigma(Y, X)$-compact by Lemma 1 so that H_0 is closed. Thus $H_0=G_0^\circ$. This completes the proof.

PROOF OF THEOREM 1. We assume that $M\neq\infty$ or $\tilde{M}^*\neq-\infty$. Let μ be a real number. First we show that if $(0, 0, \mu)\notin G$, then there exist $(y_0, w_0, t_0)\in Y\times W\times R$ and $\alpha_0\in R$ such that $t_0>0$ and $\alpha_0<-1$ and the following two conditions are satisfied:

$$((x-p,\ y_0))_1+((-z+q,\ w_0))_2+t_0f(x)\geq -1 \tag{2.5}$$

for all $x\in\mathrm{dom}\,A$, $z\in Ax$, $p\in P$ and $q\in Q$.

$$-((\hat{z},\ w_0))_2+\mu t_0\leq\alpha_0 \tag{2.6}$$

for all $\hat{z}\in\partial g(0)$.
Since G is a closed convex set, by the separation theorem, there exist $(y, w, t)\in Y\times W\times R$ and $\alpha>0$ such that

$$((x-p,\ y))_1+((-z+q+\hat{z},\ w))_2+t(f(x)+r)\geq\alpha+t\mu$$

for all $x\in\mathrm{dom}\,A$, $z\in Ax$, $p\in P$, $q\in Q$, $\hat{z}\in\partial g(0)$ and $r\geq 0$. We set $\beta_1=\inf\{((x-p,\ y))_1+((-z+q,\ w))_2+t(f(x)+r);\ x\in\mathrm{dom}\,A,\ z\in Ax,\ p\in P,\ q\in Q,\ r\geq 0\}$ and $\beta_2=\sup\{t\mu-((\hat{z},\ w))_2;\ \hat{z}\in\partial g(0)\}$. Then $\beta_1\geq\alpha+\beta_2>\beta_2$. In the definition of β_1, as x and p take $\tilde{x}$ which satisfies (2.3), as z and q take $\tilde{z}$ in $A\tilde{x}\cap Q$

and take $r=-f(\bar{x})$. Then we see that $\beta_1 \leq 0$. Choose α_1 such that $\beta_1 > \alpha_1 > \beta_2$. We note that $t \geq 0$ since β_1 is finite. We see that $y_0 = -\alpha_1^{-1} y$, $w_0 = -\alpha_1^{-1} w$, $t_0 = -t/\alpha_1$ and $\alpha_0 = -\beta_2/\alpha_1$ satisfy (2.5) and (2.6).

We shall show that $t_0 > 0$ if $M \neq \infty$. We assume that $t_0 = 0$ and $M \neq \infty$. Then there exist $\bar{x} \in P$, $\bar{z} \in A\bar{x}$ and $\bar{z}_1 \in \partial g(0)$ such that $\bar{z} - \bar{z}_1 \in Q$. We substitute $x = p = \bar{x}$, $z = \bar{z}$ and $q = \bar{z} - \bar{z}_1$ in (2.5), and $\hat{z} = \bar{z}_1$ in (2.6). Then we have $-((\bar{z}_1, w_0))_2 \geq -1$ and $-((\bar{z}_1, w_0))_2 \leq \alpha_0 < -1$. This is a contradiction. Thus $t_0 > 0$ if $M \neq \infty$.

Next we assume $t_0 = 0$ and $\tilde{M}^* \neq -\infty$. Then there exist $\bar{w} \in Q^\circ$, $\bar{y}_1 \in A^* \bar{w}$, $\bar{y}_2 \in \partial f(0)$ and $\bar{t} > 0$ such that $\bar{t}\bar{y}_2 - \bar{y}_1 \in P^\circ$ and $g(\bar{w}) \neq -\infty$. Setting $\bar{y} = \bar{y}_1 - \bar{t}\bar{y}_2$, we obtain

$$(2.7) \qquad ((x-p, \bar{y}))_1 + ((-z+q, \bar{w}))_2 + \bar{t}f(x) \geq -1$$

for all $x \in \operatorname{dom} A$, $z \in Ax$, $p \in P$ and $q \in Q$. We set $y(s) = (1-s)y_0 + s\bar{y}$, $w(s) = (1-s)w_0 + s\bar{w}$ and $t(s) = s\bar{t}$ for $0 < s < 1$. From (2.5) and (2.7), we derive $((x-p, y(s)))_1 + ((-z+q, w(s)))_2 + t(s)f(x) \geq -1$ for all $x \in \operatorname{dom} A$, $z \in Ax$, $p \in P$, $q \in Q$ and $0 < s < 1$. From (2.6), the inequality

$$-((\hat{z}, w(s)))_2 + \mu t(s) \leq (1-s)\alpha_0 - sg(\bar{w}) + s\bar{t}\mu$$

follows for all $\hat{z} \in \partial g(0)$. We set $\alpha(s) = (1-s)\alpha_0 - sg(\bar{w}) + s\bar{t}\mu$. Then there exists a real number s_0 such that $0 < s_0 < 1$ and $\alpha(s_0) < -1$. Thus $y(s_0)$, $w(s_0)$, $t(s_0)$ and $\alpha(s_0)$ satisfy (2.5) and (2.6).

In (2.5) we fix p and q. If $((x, y_0))_1 + ((-z, w_0))_2 + t_0 f(x) = \gamma < 0$ for some $x \in \operatorname{dom} A$ and $z \in Ax$, then $((sx, y_0))_1 + ((-sz, w_0))_2 + t_0 f(sx) = s\gamma < -1 + ((p, y_0))_1 - ((q, w_0))_2$ for sufficiently large $s > 0$. Since $sz \in A(sx)$, this is a contradiction. Thus $((x, y_0))_1 + ((-z, w_0))_2 + t_0 f(x) \geq 0$ for all $x \in \operatorname{dom} A$ and $z \in Ax$. It follows that $(y_0, w_0, t_0) \in G_0^\circ$. Since at least one of P and Q is a cone, similarly we see that $y_0 \in -P^\circ$ and $w_0 \in Q^\circ$. Using Lemma 2, we observe that $(w_0, t_0) \in \tilde{S}^*$. By (2.6), $\mu \leq (\alpha_0 + ((\hat{z}, w_0))_2)/t_0$ for all $\hat{z} \in \partial g(0)$. It follows from Lemma 1 that $\mu < (g(w_0) - 1)/t_0 \leq \tilde{M}^*$. Thus we see that $\mu < \tilde{M}^*$ if $(0, 0, \mu) \notin G$.

Next we take an arbitrary real number μ such that $\mu < M$. Then it is easy to check that $(0, 0, \mu) \notin G$. Thus $\mu < \tilde{M}^*$ and we have $M \leq \tilde{M}^*$. Since $M \geq \tilde{M}^*$, we see that $M = \tilde{M}^*$.

Finally we assume that M is finite. If $(0, 0, M) \notin G$, then $M < \tilde{M}^*$ by the above argument. This is a contradiction. Thus $(0, 0, M) \in G$. From this, we conclude that there exists an element $x_0 \in S$ such that $f(x_0) = M$. This completes the proof.

As a dual statement, for the problem (2.2′) and the following problem (2.1′), we have a similar result:

$$(2.1') \quad \text{Find } \tilde{M} = \inf\{(f(x)+1)/r;\ (x, r) \in \tilde{S}\},$$

where $\tilde{S}=\{(x, r);\ x\in P,\ r>0,\ (Ax-r\partial g(0))\cap Q\neq\emptyset\}$.

COROLLARY. *We assume that g is $\tau(W, Z)$-continuous, that the convex hulls of $\{0\}\cup P$ and $\{0\}\cup Q$ are closed and that condition* (2.3) *and the following condition* (2.8) *are satisfied*:

(2.8) $G^* = \{(-w,\ y,\ -g(w)+r);\ r\geq 0,\ w\in \operatorname{dom} A^*,\ y\in A^*w\} + Q^\circ \times P^\circ \times \{0\} - \{0\}\times \partial f(0)\times\{0\}$ *is closed.*

Then $M^*=\tilde{M}$ or $M^*=-\tilde{M}=-\infty$. Furthermore if M^* is finite, then there exists an element $w_0\in S^*$ such that $M^*=g(w_0)$.

PROOF. We set $Bw=-A^*(-w)$ if $w\in -\operatorname{dom} A^*$, and $Bw=\emptyset$ if $w\notin -\operatorname{dom} A^*$, and set $\tilde{g}(w)=g(-w)$ for $w\in W$. We regard B as a supremum oriented convex process from W to Y. Then

$$-M^* = \inf\{-\tilde{g}(w);\ w\in -Q^\circ,\ (Bw-(-\partial f(0)))\cap P^\circ\neq\emptyset\}.$$

We apply Theorem 1 to $(B,\ -Q^\circ,\ P^\circ,\ -\tilde{g},\ -f)$. The condition corresponding to (2.3) is satisfied with $\tilde{w}=0$ and the one corresponding to (2.4) is equivalent to (2.8). Since $\partial\tilde{g}(0)=-\partial g(0)$ and $B^*x=Ax$ for all $x\in X$, $M^*=M'$ or $M^*=-M'=-\infty$ follows from Theorem 1, where

$$M' = \inf\{(f(x)+1)/r;\ x\in P^{\circ\circ},\ r>0,\ (Ax-r\partial g(0))\cap Q^{\circ\circ}\neq\emptyset\}.$$

If M^* is finite, then there exists $w_0\in S^*$ such that $M^*=g(w_0)$.

Let us show that $M'=\tilde{M}$. Since $P^{\circ\circ}\supset P$ and $Q^{\circ\circ}\supset Q$, $M'\leq\tilde{M}$. Let $x\in P^{\circ\circ}$ and $r>0$ be such that $(Ax-r\partial g(0))\cap Q^{\circ\circ}\neq\emptyset$. First we assume that P is a cone. Then $P=P^{\circ\circ}$ and $Q^{\circ\circ}$ is the convex hull of $\{0\}\cup Q$. Let $z\in(Ax-r\partial g(0))\cap Q^{\circ\circ}$. Then there exist a real number t and an element $\bar{z}\in Q$ such that $0\leq t\leq 1$ and $z=t\bar{z}$. If $t>0$, then $t^{-1}x\in P$ and $(A(t^{-1}x)-(r/t)\partial g(0))\cap Q\ni\bar{z}$ since P is a cone. Thus $(t^{-1}x,\ r/t)\in\tilde{S}$ and $(f(x)+1)/r\geq(f(x)+t)/r=(f(t^{-1}x)+1)/(rt^{-1})\geq\tilde{M}$. If $t=0$, then $Ax-r\partial g(0)\ni 0$. For $\tilde{x}\in P$ in (2.3), $(x+\tilde{x},\ r)\in\tilde{S}$ and $(f(x)+1)/r\geq(f(x)+f(\tilde{x})+1)/r\geq(f(x+\tilde{x})+1)/r\geq\tilde{M}$ since $f(\tilde{x})\leq 0$ by (2.3). It follows that $M'\geq\tilde{M}$. If Q is a cone, we can similarly prove that $M'\geq\tilde{M}$. This completes the proof.

REMARK. In Theorem 0, the set $H=\{(Ax-z,\ r+f(x));\ x\in P,\ z\in Q,\ r\geq 0\}$ was considered instead of G under the condition that A, f and g are continuous and linear. We note that the closedness of G and the closedness of H are equivalent if A, f and g are continuous and linear; we omit the proof.

§3. Closedness of the set G

In this section we assume that f is $\tau(X, Y)$-continuous and g is $\tau(W, Z)$-

continuous, and give some sufficient conditions for the closedness of the set G in Theorem 1.

Let C be a closed convex subset of X. We denote the $\tau(X, Y)$-interior of C by $\operatorname{int} C$, and set $\operatorname{cone} C=\{tx;\ t>0,\ x\in C\}$. An asymptotic cone $\operatorname{ac} C$ of C is defined by $\operatorname{ac} C=\bigcap_{t>0} t(C-x)$, where $x\in C$. The following proposition is well-known (cf. [7; § 8]), so we omit the proof.

PROPOSITION. *$\operatorname{ac} C$ is a closed convex cone independent of the choice of $x\in C$ and $C+\operatorname{ac} C=C$.*

The following Lemma 3 is also well-known. See Dieudonné [1; Proposition 1], Zărinescu [9; Theorem 12] and Nakamura-Yamasaki [6; Corollary of Theorem 2.2].

LEMMA 3. *Let C and D be closed convex subsets of X.*

(1) *If $\operatorname{ac} C\cap(-\operatorname{ac} D)$ is a linear subspace and at least one of C and D is locally compact, then $C+D$ is closed.*

(2) *If $C^\circ\cap\operatorname{int} D^\circ\neq\emptyset$, then $\operatorname{ac} C\cap(-\operatorname{ac} D)=\{0\}$, and therefore $C+D$ is closed.*

We note that if $\operatorname{int} D^\circ\neq\emptyset$, then D is weakly locally compact. See Fan [2; Theorem 1]. As in Fan [3], we apply this lemma to our problem.

Let $x_0\in X$. Suppose for every open set U which contains Ax_0, there exists a neighborhood V of x_0 such that $U\supset Ax$ for all $x\in V$. Then we say that A is upper semicontinuous at x_0. If A is upper semicontinuous at every $x_0\in X$, A is called upper semicontinuous on X. If Ax is compact for all $x\in\operatorname{dom} A$, then A is called compact valued. We note that if A is upper semicontinuous on X, then $\operatorname{dom} A$ is closed.

We shall prove

THEOREM 2. *Assume that $L=\{x\in\operatorname{ac} P\cap\operatorname{dom} A;\ Ax\cap\operatorname{ac} Q\neq\emptyset, f(x)\leq 0\}$ is a linear subspace of X. If any one of the following conditions is satisfied, then G is closed:*

(3.1) *Both P and Q are locally compact and $A0=\{0\}$.*

(3.2) *P is locally compact, both the space X with $\tau(X, Y)$ and the space Z with $\tau(Z, W)$ are metrizable, and A is upper semicontinuous and compact valued with respect to the Mackey topologies on X and Z.*

PROOF. Let G_0 be as in Lemma 2, and set $G_1=G_0+(-P)\times Q\times\{0\}$. If it is shown that G_1 is closed, then $G=G_1+\{0\}\times\partial g(0)\times\{0\}$ is closed because g is $\tau(W, Z)$-continuous by assumption so that $\partial g(0)$ is $\sigma(Z, W)$-compact by Lemma 1.

Case 1. We assume (3.1). First we show that A is regarded as a linear mapping on L. Let $x \in L$. Then $-x \in L$ and $\{0\} = A0 \supset Ax + A(-x)$, and therefore each one of Ax and $A(-x)$ consists of one point and $A(-x) = -Ax$, since Ax and $A(-x)$ are nonempty. From Definition 1 it follows that $A(t_1x_1 + t_2x_2) = t_1Ax_1 + t_2Ax_2$ for all $t_1, t_2 \in R$ and $x_1, x_2 \in L$.

Next we show that $G_0 \cap [\mathrm{ac}\, P \times (-\mathrm{ac}\, Q) \times \{0\}]$ is a linear subspace of $X \times Z \times R$. Let $(x_1, -z_1, 0), (x_2, -z_2, 0) \in G_0 \cap [\mathrm{ac}\, P \times (-\mathrm{ac}\, Q) \times \{0\}]$ and $t_1, t_2 \in R$. We observe $G_0 \cap [\mathrm{ac}\, P \times (-\mathrm{ac}\, Q) \times \{0\}] = \{(x, -z, 0);\ x \in \mathrm{ac}\, P \cap \mathrm{dom}\, A,\ z \in \mathrm{ac}\, Q \cap Ax,\ f(x) \leq 0\}$ and infer $x_1, x_2 \in L$. Since L is a linear subspace, $t_1x_1 + t_2x_2 \in L$. By the above argument, $A(t_1x_1 + t_2x_2) = \{t_1z_1 + t_2z_2\}$, and we see that $t_1z_1 + t_2z_2 \in \mathrm{ac}\, Q \cap A(t_1x_1 + t_2x_2)$. It follows that $(t_1x_1 + t_2x_2, -t_1z_1 - t_2z_2, 0) \in G_0 \cap [\mathrm{ac}\, P \times (-\mathrm{ac}\, Q) \times \{0\}]$ and therefore $G_0 \cap [\mathrm{ac}\, P \times (-\mathrm{ac}\, Q) \times \{0\}]$ is a linear subspace.

We now apply (1) of Lemma 3 to G_0 and $(-P) \times Q \times \{0\}$ and conclude that G_1 is closed.

Case 2. We assume (3.2). We show that $F = G_0 + \{0\} \times Q \times \{0\}$ is closed. Since F is convex, it is sufficient to show that it is $\tau(X \times Z \times R, Y \times W \times R)$-closed. Since the space with $\tau(X \times Z \times R, Y \times W \times R)$ is metrizable, it is sufficient to show that if $\{(x_n, \bar{z}_n, r_n)\}$ is a sequence in F which converges to (x_0, z_0, r_0), then $(x_0, z_0, r_0) \in F$. There exist $z_n \in Ax_n$ and $q_n \in Q$ such that $\bar{z}_n = -z_n + q_n$. We denote by $d(\cdot, \cdot)$ a distance which defines $\tau(Z, W)$. Since A is upper semicontinuous at x_0, there exists a subsequence $\{z_{n_k}\}$ such that $d(z_{n_k}, Ax_0) < k^{-1}$. We may assume that $d(z_k, Ax_0) < k^{-1}$ for every k. Then there exists $\bar{z}_k \in Ax_0$ such that $d(z_k, \bar{z}_k) \in k^{-1}$. Since Ax_0 is compact, there exists a subsequence of $\{\bar{z}_k\}$ which converges to an element $\bar{z} \in Ax_0$. We may assume that $\{\bar{z}_k\}$ converges to $\bar{z}$. Then $\{z_k\}$ also converges to $\bar{z}$. Since $\{-z_k + q_k\}$ converges to z_0, $\{q_k\}$ converges to $\bar{z} + z_0$. Since Q is closed, $\bar{z} + z_0 \in Q$. Thus $(x_0, z_0, r_0) = (x_0, -\bar{z} + (z_0 + \bar{z}), f(x_0) + (r_0 - f(x_0))) \in F$. It is now shown that F is closed.

Next we show that $\mathrm{ac}\, F = G_0 + \{0\} \times \mathrm{ac}\, Q \times \{0\}$. We may assume that Q contains the origin. Let $(x, -z + q, f(x) + r) \in G_0 + \{0\} \times \mathrm{ac}\, Q \times \{0\}$, where $x \in \mathrm{dom}\, A$, $z \in Ax$, $q \in \mathrm{ac}\, Q$ and $r \geq 0$. Then for every $t > 0$, $(t^{-1}x, -t^{-1}z, (f(x) + r)/t) \in G_0$ and $t^{-1}q \in Q$. Thus $(t^{-1}x, t^{-1}(-z + q), (f(x) + r)/t) \in F$ and therefore $(x, -z + q, f(x) + r) \in \bigcap_{t>0} tF = \mathrm{ac}\, F$. It follows that $G + \{0\} \times \mathrm{ac}\, Q \times \{0\} \subset \mathrm{ac}\, F$.

Conversely let $(x_0, z_0, r_0) \in \mathrm{ac}\, F$. Then for every $t > 0$, $(t^{-1}x_0, t^{-1}z_0, r_0/t) \in F$. For every positive integer n, there exist $x_n \in \mathrm{dom}\, A$, $z_n \in Ax_n$, $q_n \in Q$ and $r_n \geq 0$ such that $(x_n, -z_n + q_n, f(x_n) + r_n) = (nx_0, nz_0, nr_0)$. Using $n^{-1}x_n = x_0$, we have $n^{-1}z_n \in n^{-1}Ax_n = A(n^{-1}x_n) = Ax_0$. Since Ax_0 is $\tau(Z, W)$-compact, there exists a subsequence of $\{n^{-1}z_n\}$ which converges to an element $\bar{z} \in Ax_0$. We may assume $\{n^{-1}z_n\}$ converges to $\bar{z}$. On account of the equality $-z_n + q_n = nz_0$, $\{n^{-1}q_n\}$ converges to $z_0 + \bar{z}$. Since Q is a convex set which contains the origin, $(tn)^{-1}q_n \in Q$ for $t > 0$ such that $(tn)^{-1} \leq 1$. Thus for every fixed $t > 0$, $t^{-1}(z_0 + \bar{z}) = t^{-1} \lim_{n \to \infty} \cdot n^{-1}q_n = \lim_{n \to \infty} (tn)^{-1}q_n \in Q$, because Q is closed. Thus $z_0 + \bar{z} \in \bigcap_{t>0} tQ = \mathrm{ac}\, Q$ and $(x_0, z_0, r_0) = (x_0, -\bar{z} + (z_0 + \bar{z}), r_0) \in G_0 + \{0\} \times \mathrm{ac}\, Q \times \{0\}$. It follows that

$G_0+\{0\}\times \mathrm{ac}\, Q\times\{0\}=\mathrm{ac}\, F$.

We note that $\mathrm{ac}\, F\cap \mathrm{ac}\,[P\times\{0\}\times\{0\}]=[G_0+\{0\}\times \mathrm{ac}\, Q\times\{0\}]\cap[\mathrm{ac}\, P\times\{0\}\times\{0\}]=\{(x, 0, 0);\, x\in \mathrm{ac}\, P\cap \mathrm{dom}\, A,\, Ax\cap \mathrm{ac}\, Q\neq\emptyset,\, f(x)\leq 0\}=L\times\{0\}\times\{0\}$ and L is a linear subspace. We apply (1) of Lemma 3 to F and $(-P)\times\{0\}\times\{0\}$ and conclude that G_1 is closed.

In the proof of the following theorem we shall apply (2) of Lemma 3.

THEOREM 3. (1) *If $\partial f(0)\cap \mathrm{cone}\,[A^*(\mathrm{int}\, Q^\circ)+\mathrm{int}\, P^\circ]\neq\emptyset$, then G is closed.*

(2) *Under condition* (3.2), *if $\partial f(0)\cap \mathrm{cone}\,[A^*(Q^\circ)+\mathrm{int}\, P^\circ]\neq\emptyset$, then G is closed.*

PROOF. (1) It is sufficient to show that $G_0^\circ\cap \mathrm{int}\,[(-P)\times Q\times\{0\}]^\circ\neq\emptyset$. By Lemma 2, $G_0^\circ=H_0$. Hence $G_0^\circ\cap \mathrm{int}\,[(-P)\times Q\times\{0\}]^\circ=H_0\cap[(-\mathrm{int}\, P^\circ)\times(\mathrm{int}\, Q^\circ)\times R]$. Since $\partial f(0)\cap \mathrm{cone}\,[A^*(\mathrm{int}\, Q^\circ)+\mathrm{int}\, P^\circ]\neq\emptyset$, there exist $t>0$, $w\in \mathrm{int}\, Q^\circ\cap \mathrm{dom}\, A^*$, $y_1\in A^*w$, $y_2\in\partial f(0)$ and $y\in \mathrm{int}\, P^\circ$ such that $y_2=t(y_1+y)$. Then $(y_1-t^{-1}y_2, w, t^{-1})\in H_0\cap[(-\mathrm{int}\, P^\circ)\times(\mathrm{int}\, Q^\circ)\times R]$.

(2) As in the proof of Theorem 3, we see by condition (3.2) that $G_0+\{0\}\times Q\times\{0\}$ is closed. Thus it is sufficient to show that $[G_0+\{0\}\times Q\times\{0\}]^\circ\cap \mathrm{int}\,[(-P)\times\{0\}\times\{0\}]^\circ\neq\emptyset$. Since $[G_0+\{0\}\times Q\times\{0\}]^\circ=G_0^\circ\cap(Y\times Q^\circ\times R)=H_0\cap(Y\times Q^\circ\times R)$, we see in a similar manner that $[G_0+\{0\}\times Q\times\{0\}]^\circ\cap \mathrm{int}\,[(-P)\times\{0\}\times\{0\}]^\circ=H_0\cap[(-\mathrm{int}\, P^\circ)\times Q^\circ\times R]$ is nonempty.

REMARK. In (3.2), if A is a continuous linear mapping, then we do not need the condition that both the spaces X and Z are metrizable.

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