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Ideally finite Lie algebras

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Introduction

Recently Stitzinger [4] presented several equivalent conditions for a subalgebra to be an ω-step ascendant subalgebra in a locally solvable, ideally finite Lie algebra. On the other hand, Tôgô [5] introduced the concept of weakly ascendant subalgebras generalizing that of ascendant subalgebras and Kawamoto [2] considered E_∞-pairs of subalgebras to study ascendancy in Lie algebras.

The purpose of this paper is first to generalize and sharpen the results of Stitzinger [4] by using the concepts of weakly ascendant subalgebras, E_∞-pairs of subalgebras and others, and secondly to characterize the class of locally solvable, ideally finite Lie algebras and similar classes.

Section 2 is devoted to searching several equivalent conditions for a subalgebra to be a weakly ascendant subalgebra in a certain Lie algebra (Theorems 2.1, 2.2, 2.3 and Corollary 2.4). In Section 3 we shall show that if L is a locally solvable, ideally finite Lie algebra and H is a subalgebra of L, then the condition $H \lhd^{\omega} L$ is equivalent to each of the following: (a) $H \operatorname{asc} L$; (b) $H \leq^{\omega} L$; (c) $H \operatorname{wasc} L$; (d) (H, L) is an E-pair; (e) (H, L) is an E_∞-pair; (f) $L = H + L_0(h)$ for all $h \in H$; (g) $L_1(h) \subseteq H$ for all $h \in H$; (h) $H \lhd^{\omega} K$ for any subalgebra K of L containing H; (i) $H \lhd^{\omega} \langle H, x\rangle$ for any $x \in L$; (j) $H \lhd^{\omega} \langle H, [x, H]\rangle$ for any $x \in L$; (k) For any $x \in L$, there exists an $n = n(x)$ such that $H \lhd^{\omega} \langle H, [x, {}_nH]\rangle$ (Theorems 3.1 and 3.2). This sharpens [4, Theorems 1 and 3]. We shall also give a simple proof of [4, Theorem 2] in Proposition 3.3.

In Section 4 we shall show that all ideally finite Lie algebras belonging to a class $\mathfrak{X}$ are precisely locally solvable, ideally finite Lie algebras, if $\mathfrak{X}$ is a class of Lie algebras being between the class $\acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \cap \acute{\mathrm{E}}_{\omega}(\lhd)\mathfrak{A}$ and the class $\mathfrak{X}_0$ of all Lie algebras in which every non-zero finite-dimensional subalgebra is non-perfect (Theorem 4.3). Finally in Section 5 we shall show that (a) the class of locally nilpotent, subideally finite Lie algebras coincides with the class of Baer algebras and (b) the class of locally nilpotent, ascendantly finite Lie algebras is contained in the class of Gruenberg algebras (Theorem 5.3). We shall also give two examples showing that there are no inclusions between the class of locally solvable, ideally finite Lie algebras and the class of locally nilpotent, subideally finite Lie algebras (Example 5.7).

1.

Throughout the paper, Lie algebras are not necessarily finite-dimensional over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified.

Let L be a Lie algebra over $\mathfrak{k}$ and let H be a subalgebra of L. For an ordinal σ, H is a σ-step ascendant subalgebra of L, denoted by $H \lhd^{\sigma} L$, if there is a series $(H_{\alpha})_{\alpha \leq \sigma}$ of subalgebras of L such that

(1) $H_0 = H$ and $H_{\sigma} = L$,

(2) $H_{\alpha} \lhd H_{\alpha+1}$ for any ordinal $\alpha < \sigma$,

(3) $H_{\lambda} = \bigcup_{\alpha < \lambda} H_{\alpha}$ for any limit ordinal $\lambda \leq \sigma$.

H is a σ-step weakly ascendant subalgebra of L, denoted by $H \leq^{\sigma} L$, if there is an ascending series $(M_{\alpha})_{\alpha \leq \sigma}$ of subspaces of L such that

(1) $M_0 = H$ and $M_{\sigma} = L$,

(2) $[M_{\alpha+1}, H] \subseteq M_{\alpha}$ for any ordinal $\alpha < \sigma$,

(3) $M_{\lambda} = \bigcup_{\alpha < \lambda} M_{\alpha}$ for any limit ordinal $\lambda \leq \sigma$.

The series $(M_{\alpha})_{\alpha \leq \sigma}$ is called a σ-step weakly ascending series from H to L. H is an ascendant subalgebra (resp. a weakly ascendant subalgebra) of L, denoted by $H \operatorname{asc} L$ (resp. $H \operatorname{wasc} L$), if $H \lhd^{\sigma} L$ (resp. $H \leq^{\sigma} L$) for some ordinal σ. When σ is finite, H is a subideal (resp. a weak subideal) of L and denoted by $H \operatorname{si} L$ (resp. $H \operatorname{wsi} L$).

Let us recall some classes of Lie algebras:

$L \in \mathfrak{A}$ iff L is abelian.

$L \in \mathfrak{F}$ iff L is finite-dimensional.

$L \in \mathfrak{N}$ iff L is nilpotent.

$L \in \mathfrak{E}$ iff for any $x, y \in L$ there exists a positive integer $n = n(x, y)$ such that $[x, {}_n y] = 0$.

$L \in \mathfrak{F}\mathfrak{t}$ iff L is the sum of the nilpotent ideals of L.

$L \in \mathfrak{B}$ (resp. $\mathfrak{Gr}$) iff $\langle x \rangle \operatorname{si} L$ (resp. $\langle x \rangle \operatorname{asc} L$) for any $x \in L$.

$L \in \mathfrak{Z}_{\omega}$ iff $L = \bigcup_{n \in N} \zeta_n(L)$ where $(\zeta_n(L))_{n \in N}$ is the upper central series of L.

Let $\mathfrak{X}$, $\mathfrak{Y}$ be any classes of Lie algebras.

$L \in \mathrm{R}\mathfrak{X}$ iff L has a family $(I_{\alpha})_{\alpha \in A}$ of ideals such that $L/I_{\alpha} \in \mathfrak{X}$ for all $\alpha \in A$ and $\bigcap_{\alpha \in A} I_{\alpha} = 0$.

$L \in \mathrm{L}\mathfrak{X}$ iff every finite subset of L is contained in an $\mathfrak{X}$-subalgebra of L. When $L \in \mathrm{L}\mathfrak{N}$, L is called a locally nilpotent Lie algebra.

$L \in \acute{\mathrm{E}}_{\sigma}\mathfrak{X}$ (resp. $\acute{\mathrm{E}}_{\sigma}(\lhd)\mathfrak{X}$) iff L has an ascending series $(K_{\alpha})_{\alpha \leq \sigma}$ of subalgebras (resp. ideals) such that

(1) $K_0 = 0$ and $K_{\sigma} = L$,

(2) $K_{\alpha} \lhd K_{\alpha+1}$ and $K_{\alpha+1}/K_{\alpha} \in \mathfrak{X}$ for any ordinal $\alpha < \sigma$,

(3) $K_{\lambda} = \bigcup_{\alpha < \lambda} K_{\alpha}$ for any limit ordinal $\lambda \leq \alpha$.

$L \in \acute{\mathrm{E}}\mathfrak{X}$ (resp. $\acute{\mathrm{E}}(\lhd)\mathfrak{X}$) iff $L \in \acute{\mathrm{E}}_{\sigma}\mathfrak{X}$ (resp. $\acute{\mathrm{E}}_{\sigma}(\lhd)\mathfrak{X}$) for some ordinal σ.

$L \in \text{E}\mathfrak{X}$ iff $L \in \acute{\text{E}}_n\mathfrak{X}$ for some finite ordinal $n < \omega$. When $L \in \text{E}\mathfrak{A}$, L is solvable.
$\mathfrak{X}$ is R (resp. L, É, E)-closed iff $\mathfrak{X} = \text{R}\mathfrak{X}$ (resp. $\text{L}\mathfrak{X}$, $\acute{\text{E}}\mathfrak{X}$, $\text{E}\mathfrak{X}$).
$L \in \mathfrak{X}\mathfrak{Y}$ iff L has an ideal $I \in \mathfrak{X}$ such that $L/I \in \mathfrak{Y}$.

Now we introduce the following notations: Let Δ be any of the relations $\lhd$, si, asc, wsi, wasc. We say a Lie algebra L to lie in $\text{L}(\Delta)\mathfrak{X}$ if for any finite subset Y of L there exists an $\mathfrak{X}$-subalgebra K of L such that $Y \subseteq K \,\Delta\, L$. When $L \in \text{L}(\lhd)\mathfrak{F}$ (resp. $\text{L}(\text{si})\mathfrak{F}$, $\text{L}(\text{asc})\mathfrak{F}$), L is called an ideally finite (resp. a subideally finite, an ascendantly finite) Lie algebra. For a subset X of L, we say a Lie algebra L to lie in $\text{L}(X)\mathfrak{X}$ if every finite subset of L is contained in an X-invariant $\mathfrak{X}$-subalgebra of L. We say a Lie algebra L to lie in $\text{L}(X\text{-inv}\,\Delta)\mathfrak{X}$ if for any finite subset Y of L there exists an X-invariant $\mathfrak{X}$-subalgebra K of L such that $Y \subseteq K \,\Delta\, L$.

The Hirsch-Plotkin radical $\rho(L)$ of L is a unique maximal locally nilpotent ideal of L. If char $\mathfrak{k} = 0$, the Baer radical $\beta(L)$ of L is the subalgebra generated by all nilpotent subideals of L and the Gruenberg radical $\gamma(L)$ of L is the subalgebra generated by all nilpotent ascendant subalgebras of L.

For each $x \in L$, we use the following notations in [4]:

$$L_0(x) = \{z \in L \mid z\,(\text{ad}\,x)^k = 0 \text{ for some } k \in \boldsymbol{N}\},$$
$$L_1(x) = \textstyle\bigcap_{k=1}^{\infty} \text{Im}\,(\text{ad}\,x)^k.$$

For a subalgebra H of L, (H, L) is an E_∞-pair [2] iff for each $x \in L$ there exists a positive integer $n = n(x)$ such that $[x, {}_nh] \in H$ for any $h \in H$. As a generalization of an E_∞-pair we introduce an E-pair. Namely, we say that (H, L) is an E-pair if for any $x \in L$ and any $h \in H$ there exists a positive integer $n = n(x, h)$ such that $[x, {}_nh] \in H$.

In [3] L was called ideally (resp. subideally, ascendantly) finite if L has a collection $\{L_i\}_{i \in I}$ of finite-dimensional ideals (resp. subideals, ascendant subalgebras) which generate L and have the property that whenever $i, j \in I$ there exists $k \in I$ such that $\langle L_i, L_j \rangle \leq L_k$.

LEMMA 1.1. *Let L be a Lie algebra over a field $\mathfrak{k}$. Then $L \in \text{L}(\lhd)\mathfrak{F}$ (resp. $\text{L}(\text{si})\mathfrak{F}$, $\text{L}(\text{asc})\mathfrak{F}$) if and only if L is ideally (resp. subideally, ascendantly) finite in the sense of* [3].

PROOF. Let $L \in \text{L}(\lhd)\mathfrak{F}$ and let Γ be the collection of all finite subsets of L. For any $X \in \Gamma$ there exists L_X minimal with respect to $X \subseteq L_X \lhd L$ and $L_X \in \mathfrak{F}$. Evidently $L = \langle L_X \mid X \in \Gamma \rangle$ and for any X, $Y \in \Gamma$

$$\langle L_X, L_Y \rangle \leq L_{X \cup Y},$$

by the minimality of L_X and L_Y. Therefore L is ideally finite in the sense of [3]. The converse is evident. The other assertions are similarly proved.

Hence our definitions of ideally finite, subideally finite and ascendantly finite

Lie algebras don't conflict with those in [3].

2.

We begin this section with

THEOREM 2.1. *Let L be a Lie algebra over a field $\mathfrak{k}$ and let H be a subalgebra of L. Assume that $L \in \mathrm{L}(H)\mathfrak{F}$. Then the following conditions are equivalent*:

(1) H wasc L.

(2) $H \leq^{\omega} L$.

(3) *(H, L) is an E-pair.*

(4) *(H, L) is an E_{∞}-pair.*

(5) *$L = H + L_0(h)$ for all $h \in H$.*

PROOF. By hypothesis we have

$$L = \cup_{\lambda \in \Lambda} A(\lambda),$$

where $A(\lambda)$ is an H-invariant finite-dimensional subalgebra of L for each $\lambda \in \Lambda$.

(1)$\Rightarrow$(2): Let $(H_{\alpha})_{\alpha \leq \sigma}$ be a weakly ascending series from H to L. For any $n \in \boldsymbol{N}$, let $\mu(n)$ be the first ordinal such that $[A(\lambda), {}_nH] \subseteq H_{\mu(n)}$. Since $A(\lambda)$ is H-invariant, $[A(\lambda), {}_nH]$ is finite-dimensional. Hence $\mu(n)$ is not a limit ordinal. Therefore $\mu(n+1) < \mu(n)$ unless $\mu(n) = 0$. Since the ordinals $\leq \sigma$ are well-ordered, it follows that $\mu(n) = 0$ for some $n \in \boldsymbol{N}$. Hence $[A(\lambda), {}_nH] \subseteq H_0 = H$. As $L = \cup_{\lambda \in \Lambda} A(\lambda)$, for any $x \in L$ there exists an $n = n(x) \in \boldsymbol{N}$ such that $[x, {}_nH] \subseteq H$. Thus we put

$$M_n = \{x \in L \mid [x, {}_nH] \subseteq H\} \qquad \text{for each } n \in \boldsymbol{N},$$
$$M_{\omega} = \cup_{n < \omega} M_n.$$

Then we see that $(M_{\alpha})_{\alpha \leq \omega}$ is an ω-step weakly ascending series from H to L.

(2)$\Rightarrow$(1) is clear.

(2)$\Rightarrow$(4): For any $x \in L$ there exists an $n = n(x) \in \boldsymbol{N}$ such that $[x, {}_nH] \subseteq H$. In particular $[x, {}_nh] \in H$ for all $h \in H$. Therefore (H, L) is an E_{∞}-pair.

(4)$\Rightarrow$(3) is clear.

(3)$\Rightarrow$(2): Let $\lambda \in \Lambda$ and for each $n \in \boldsymbol{N}$ put

$$H_n = \{x \in L \mid [x, {}_nH] \subseteq H\}.$$

Now suppose that $A(\lambda) \nsubseteq H_n$ for all $n \in \boldsymbol{N}$. Since $A(\lambda)$ and H_n are H-invariant, $(A(\lambda) + H_n)/H_n$ is a non-zero finite-dimensional H-module. By using the fact that (H, L) is an E-pair, ad h induces a nilpotent transformation on this space for each $h \in H$. Hence owing to Engel's theorem, there exists

$$z \in A(\lambda) \backslash H_n \quad \text{such that} \quad [z, H] \subseteq H_n.$$

Hence $z \in H_{n+1}$. Therefore

$$A(\lambda) \cap H_n \subsetneqq A(\lambda) \cap H_{n+1} \qquad \text{for all } n \in \boldsymbol{N}.$$

This is a contradiction since $A(\lambda) \in \mathfrak{F}$. Thus there exists an $n = n(x) \in \boldsymbol{N}$ such that $A(\lambda) \subseteq H_n$. We put

$$H_\omega = \cup_{n<\omega} H_n.$$

Then we have $H_\omega = L$, whence $(H_\alpha)_{\alpha \leq \omega}$ is a weakly ascending series from H to L.

(2)⇒(5): Let $h \in H$. For any $x \in L$, there is an H-invariant finite-dimensional subalgebra $A(\lambda)$ of L containing x. Since $H \leq^\omega L$ and $A(\lambda) \in \mathfrak{F}$, there exists an $n \in \boldsymbol{N}$ such that $[A(\lambda), {}_nH] \subseteq H$. Especially, $A(\lambda)(\operatorname{ad} h)^n \subseteq H$. Therefore

$$A(\lambda) = A(\lambda)_1(h) + A(\lambda)_0(h) \subseteq H + L_0(h).$$

It follows that $L = H + L_0(h)$.

(5)⇒(3): For any $x \in L$ and any $h \in H$, there exists $k \in H$ and $y \in L_0(h)$ such that $x = k + y$. Then we have $y(\operatorname{ad} h)^n = 0$ for some $n \in \boldsymbol{N}$. Hence

$$x(\operatorname{ad} h)^n = k(\operatorname{ad} h)^n + y(\operatorname{ad} h)^n \in H,$$

that is, there exists an $n = n(x, h) \in \boldsymbol{N}$ such that $[x, {}_nh] \in H$. Therefore (H, L) is an E-pair.

By making use of Theorem 2.1, we have

THEOREM 2.2. *Let L be a Lie algebra over a field $\mathfrak{k}$ and let H be a subalgebra of L. Assume that $L \in \mathrm{L}(H)\mathfrak{F}$. Then for any ordinals $\alpha_i \geq \omega\,(1 \leq i \leq 5)$ the following conditions are equivalent*:

(1) $H \leq^{\alpha_1} L$.

(2) $H \leq^{\alpha_2} K$ *for any subalgebra K of L containing H.*

(3) $H \leq^{\alpha_3} \langle H, x\rangle$ *for any $x \in L$.*

(4) $H \leq^{\alpha_4} \langle H, [x, H]\rangle$ *for any $x \in L$.*

(5) *For any $x \in L$, there exists an $n = n(x) \in \boldsymbol{N}$ such that $H \leq^{\alpha_5} \langle H, [x, {}_nH]\rangle$.*

PROOF. By Theorem 2.1 it suffices to show the statement in the case that $\alpha_1 = \cdots = \alpha_5 = \omega$.

(1)⇒(2)⇒(3) is clear.

(3)⇒(4) is clear since $\langle H, [x, H]\rangle \leq \langle H, x\rangle$.

(4)⇒(5): Put $n = n(x) = 1$ for all $x \in L$.

(5)⇒(1): Let $x \in L$ and let $(H_\beta(x))_{\beta \leq \omega}$ be an ω-step weakly ascending series from H to $\langle H, [x, {}_nH]\rangle$. We define

$$H_{\omega+i}(x) = H_\omega(x) + \textstyle\sum_{k=1}^{i} [x, {}_{n-k}H] \qquad \text{for } 1 \leq i \leq n,$$

$$H_\beta(x) = H_{\omega+n}(x) \qquad \text{for } \omega + n < \beta < \omega 2.$$

Then we have

$$[H_{\beta+1}(x), H] \subseteq H_\beta(x) \qquad \text{for any } \beta < \omega 2.$$

Thus we define

$$H_\beta = \textstyle\sum_{x\in L} H_\beta(x) \qquad \text{for any } \beta < \omega 2,$$
$$H_{\omega 2} = \textstyle\bigcup_{\beta<\omega 2} H_\beta .$$

Then

$$[H_{\beta+1}, H] \subseteq H_\beta \qquad \text{for any } \beta < \omega 2,$$
$$H_\omega = \textstyle\bigcup_{\beta<\omega} H_\beta ,$$
$$H_{\omega 2} = L.$$

Therefore

$$H \leq^{\omega 2} L.$$

By Theorem 2.1 we obtain

$$H \leq^{\omega} L.$$

We strengthen the hypothesis of Theorem 2.1 to obtain the following

THEOREM 2.3. *Let L be a Lie algebra over a field $\mathfrak{k}$ and let H be a subalgebra of L. Assume that $L \in$ L(H-inv wsi)$\mathfrak{F}$. Then the following conditions are equivalent:*

(1) H wasc L.
(2) $H \leq^{\omega} L$.
(3) *(H, L) is an E-pair.*
(4) *(H, L) is an E_∞-pair.*
(5) *$L = H + L_0(h)$ for all $h \in H$.*
(6) *$L_1(h) \subseteq H$ for all $h \in H$.*

PROOF. In Theorem 2.1 we have already shown that all conditions from (1) to (5) are equivalent.

(1)$\Rightarrow$(6): For any $h \in H$, there exists a finite-dimensional weak subideal $A(h)$ of L containing h. Then

$$H \cap A(h) \text{ wasc } A(h) \in \mathfrak{F}$$

and so

$$H \cap A(h) \text{ wsi } A(h) \text{ wsi } L.$$

It follows that

$$H \cap A(h) \leq^{n} L \qquad \text{for some } n \in \boldsymbol{N},$$

that is,

$$[L, {}_nH \cap A(h)] \subseteq H \cap A(h).$$

In particular,

$$[L, {}_nh] \subseteq H.$$

Therefore we have

$$L_1(h) \subseteq \operatorname{Im}(\operatorname{ad} h)^n \subseteq H.$$

(6)⇒(5): Let $h \in H$. For any $x \in L$, there exists an H-invariant finite-dimensional subalgebra $A(x)$ of L containing x. Since $A(x)$ is stable by ad h, for all $x \in L$

$$A(x) = A(x)_1(h) + A(x)_0(h) \subseteq L_1(h) + L_0(h).$$

Therefore

$$L \subseteq L_1(h) + L_0(h) \subseteq H + L_0(h).$$

As a special case of Theorems 2.2 and 2.3, we have

COROLLARY 2.4. *Let L be a Lie algebra over a field $\mathfrak{k}$ belonging to* $\mathrm{L}(\lhd)\mathfrak{F}$, *and let H be a subalgebra of L. Then for any ordinals $\alpha_i \geq \omega$ $(1 \leq i \leq 5)$ the following conditions are equivalent*:

(1) H wasc L.

(2) $H \leq^{\alpha_1} L$.

(3) *(H, L) is an E-pair.*

(4) *(H, L) is an E_∞-pair.*

(5) *$L = H + L_0(h)$ for all $h \in H$.*

(6) *$L_1(h) \subseteq H$ for all $h \in H$.*

(7) *$H \leq^{\alpha_2} K$ for any subalgebra K of L containing H.*

(8) *$H \leq^{\alpha_3} \langle H, x\rangle$ for any $x \in L$.*

(9) *$H \leq^{\alpha_4} \langle H, [x, H]\rangle$ for any $x \in L$.*

(10) *For any $x \in L$, there exists an $n = n(x) \in \boldsymbol{N}$ such that $H \leq^{\alpha_5} \langle H, [x, {}_nH]\rangle$.*

Finally we show the following

PROPOSITION 2.5. *Let L be a Lie algebra over a field $\mathfrak{k}$ and let H be a subalgebra of L. Then there exists the largest subalgebra M of L such that $H \leq^{\omega} M$.*

PROOF. Let Γ be the collection of all subalgebras K of L such that $H \leq^{\omega} K$. We put

$$M = \langle K \mid K \in \Gamma\rangle.$$

By [5, Lemma 6] we have

$$H \leq^{\omega} M.$$

3.

In this section we shall investigate the class $\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ of Lie algebras, which is equal to $\mathfrak{I} = \mathrm{LE}\mathfrak{A} \cap \mathrm{L}(\lhd)\mathfrak{F}$ in [4]. We first show the following theorem, which sharpens [4, Theorem 1] where only the equivalence of (2), (7) and (8) is proved.

THEOREM 3.1. *Let L be a Lie algebra over a field $\mathfrak{k}$ belonging to $\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ and let H be a subalgebra of L. Then the following conditions are equivalent*:

(1) H asc L.
(2) $H \lhd^{\omega} L$.
(3) H wasc L.
(4) $H \leq^{\omega} L$.
(5) *(H, L) is an E-pair.*
(6) *(H, L) is an E_{∞}-pair.*
(7) *$L = H + L_0(h)$ for all $h \in H$.*
(8) *$L_1(h) \subseteq H$ for all $h \in H$.*

PROOF. In Theorem 2.3 we have already shown that all conditions from (3) to (8) are equivalent.

(2)$\Rightarrow$(1)$\Rightarrow$(3) is evident.

(7)$\Rightarrow$(2): This is shown in [4]. For completeness sake, we give an outline of that proof. By hypothesis we have

$$L = \cup_{\lambda \in \Lambda} A(\lambda)$$

where $A(\lambda)$ is a finite-dimensional ideal of L for each $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, there exists an ascending chain

$$0 = A(\lambda, 0) \lneqq \cdots \lneqq A(\lambda, n(\lambda)) = A(\lambda)$$

where $A(\lambda, i) \lhd L$ and $A(\lambda, i)/A(\lambda, i-1)$ is an irreducible L-module for $1 \leq i \leq n(\lambda)$. We first define

$$\Omega(0) = 0.$$

If $A(\lambda) \nsubseteq H$, then we put $r_1(\lambda) = \min\{r \in \boldsymbol{N} \mid A(\lambda, r) \nsubseteq H\}$. If $A(\lambda) \subseteq H$, then we put $r_1(\lambda) = n(r)$. We define

$$W(\lambda, 1) = \{z \in A(\lambda, r_1(\lambda)) \mid [z, H] \subseteq H\},$$

$$\Omega(1) = \langle W(\lambda, 1) \mid \lambda \in \Lambda \rangle.$$

Then

$$H + \Omega(0) \lhd H + \Omega(1).$$

Next let $k \geq 1$ and suppose that we have constructed $\Omega(i)$ for $i \leq k$ so that

$$H = H + \Omega(0) \lhd H + \Omega(1) \lhd \cdots \lhd H + \Omega(k).$$

If $A(\lambda) \nsubseteq H + \Omega(k)$, then we put $r_{k+1}(\lambda) = \min\{r \in \boldsymbol{N} \mid A(\lambda, r) \nsubseteq H + \Omega(k)\}$. If $A(\lambda) \subseteq H + \Omega(k)$, then we put $r_{k+1}(\lambda) = n(\lambda)$. We define

$$W(\lambda, k+1) = \{z \in A(\lambda, r_{k+1}(\lambda)) \mid [z, H] \subseteq H + \Omega(k)\},$$
$$\Omega(k+1) = \langle W(\lambda, k+1) \mid \lambda \in \Lambda \rangle.$$

Then

$$H + \Omega(k) \lhd H + \Omega(k+1).$$

Since we have

$$L = \cup_{i<\omega} \Omega(i) = \cup_{i<\omega} (H + \Omega(i)),$$

we conclude that

$$H \lhd^{\omega} L.$$

As a consequence of Corollary 2.4 and Theorem 3.1 we have the following theorem. This sharpens [4, Theorem 3] which shows only the equivalence of (1), (3), (5) and (7) in the case that $\alpha_1 = \alpha_3 = \alpha_5 = \alpha_7 = \omega$.

THEOREM 3.2. *Let L be a Lie algebra over a field $\mathfrak{k}$ belonging to* $\mathrm{L}(\lhd)$ $(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ *and let H be a subalgebra of L. Then for any ordinals $\alpha_i \geq \omega$ $(1 \leq i \leq 10)$ the following conditions are equivalent*:

(1) $H \lhd^{\alpha_1} L$.

(2) $H \leq^{\alpha_2} L$.

(3) $H \lhd^{\alpha_3} K$ *for any subalgebra K of L containing H.*

(4) $H \leq^{\alpha_4} K$ *for any subalgebra K of L containing H.*

(5) $H \lhd^{\alpha_5} \langle H, x \rangle$ *for any $x \in L$.*

(6) $H \leq^{\alpha_6} \langle H, x \rangle$ *for any $x \in L$.*

(7) $H \lhd^{\alpha_7} \langle H, [x, H] \rangle$ *for any $x \in L$.*

(8) $H \leq^{\alpha_8} \langle H, [x, H] \rangle$ *for any $x \in L$.*

(9) *For any $x \in L$, there exists an $n = n(x) \in \boldsymbol{N}$ such that $H \lhd^{\alpha_9} \langle H, [x, {}_nH] \rangle$.*

(10) *For any $x \in L$, there exists an $n = n(x) \in \boldsymbol{N}$ such that $H \leq^{\alpha_{10}} \langle H, [x, {}_nH] \rangle$.*

As an immediate consequence of Proposition 2.5 and Theorem 3.1 we have the following proposition ([4, Theorem 2]).

PROPOSITION 3.3. *Let L be a Lie algebra over a field $\mathfrak{k}$ belonging to* $\mathrm{L}(\lhd)$

$(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ *and let* H *be a subalgebra of* L. *Then there exists the largest subalgebra* M *of* L *such that* $H \lhd^{\omega} M$.

Furthermore we shall show the following propositions.

PROPOSITION 3.4. *Let* L *be a Lie algebra over a field* $\mathfrak{k}$ *belonging to* $(\mathrm{E}\mathfrak{A})(\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}))$ *and let* H *be a solvable subalgebra of* L. *If* H wsi L, *then* $H \lhd^{\omega} L$.

PROOF. We shall show the proposition following the proof of [6, Lemma 6]. By hypothesis there exists a solvable ideal K of L such that $L/K \in \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$. Then

$$H \text{ wsi } H+K \quad \text{and} \quad (H+K)/K \text{ wsi } L/K.$$

Since $H, K \in \mathrm{E}\mathfrak{A}$, we have $H+K \in \mathrm{E}\mathfrak{A}$. By making use of [5, Theorem 1] it follows that $H \operatorname{si} H+K$. On the other hand, by Theorem 3.1 we have $(H+K)/K \lhd^{\omega} L/K$. Thus we have $H \lhd^{\omega} L$.

PROPOSITION 3.5. *Let* L *be a Lie algebra over a field* $\mathfrak{k}$ *belonging to* $\mathrm{L}(\mathrm{si})(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ *and let* H *be a finitely generated subalgebra of* L. *Then the following conditions are equivalent*:

(1) $H \operatorname{asc} L$.
(2) $H \operatorname{si} L$.
(3) $H \operatorname{wasc} L$.
(4) $H \operatorname{wsi} L$.

PROOF. It is sufficient to show that (3) implies (2). Since $H \in \mathfrak{G}$, there exists a solvable and finite-dimensional subideal K of L containing H. Then we have $H \operatorname{wasc} K$. Since $K \in \mathrm{E}\mathfrak{A} \cap \mathfrak{F}$, by using [5, Theorem 1] we have $H \operatorname{si} K$. Therefore $H \operatorname{si} L$.

PROPOSITION 3.6. *Let* L *be a Lie algebra over a field* $\mathfrak{k}$ *belonging to* $(\mathrm{E}\mathfrak{A})(\mathrm{L}(\mathrm{si})(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}))$ *and let* H *be a solvable and finitely generated subalgebra of* L. *Then*

(1) $H \operatorname{wasc} L$ *if and only if* $H \operatorname{asc} L$,
(2) $H \operatorname{wsi} L$ *if and only if* $H \operatorname{si} L$.

PROOF. Assume that $H \operatorname{wasc} L$. By hypothesis there exists a solvable ideal K of L such that $L/K \in \mathrm{L}(\mathrm{si})(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$. Hence we have

$$H \text{ wasc } H+K \quad \text{and} \quad (H+K)/K \text{ wasc } L/K.$$

Obviously $H+K \in \mathrm{E}\mathfrak{A}$. By using [5, Theorem 1], we have $H \operatorname{asc} H+K$. Since $(H+K)/K \in \mathfrak{G}$, by Proposition 3.5 $(H+K)/K \operatorname{si} L/K$. It follows that $H+K \operatorname{si} L$.

Thus we have H asc L. The converse is evident and (1) is proved. (2) is similarly proved.

4.

In this section we shall give some characterizations of the class $\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ of locally solvable, ideally finite Lie algebras. We here need the following two lemmas.

LEMMA 4.1. $\mathrm{L}(\lhd)\mathfrak{F} \leq \acute{\mathrm{E}}(\lhd)\mathfrak{F}$.

PROOF. Let $L \in \mathrm{L}(\lhd)\mathfrak{F}$. Define $L_0 = 0$ and suppose that $(L_\beta)_{\beta<\alpha}$ has been defined for an ordinal $\alpha > 0$. If α is a limit ordinal, define $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$. Assume that α is not a limit ordinal and $L_{\alpha-1} \lneqq L$. Then we can find an $x \in L \setminus L_{\alpha-1}$. Define $L_\alpha = L_{\alpha-1} + \langle x^L \rangle$. Since $L \in \mathrm{L}(\lhd)\mathfrak{F}$,

$$L_{\alpha-1} \lneqq L_\alpha \lhd L \quad \text{and} \quad L_\alpha/L_{\alpha-1} \in \mathfrak{F}.$$

Now by set-theoretical considerations we can find an ordinal σ such that $L = L_\sigma$. Therefore we have $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}$.

LEMMA 4.2. $\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}) \leq \acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \cap \acute{\mathrm{E}}_\omega(\lhd)\mathfrak{A}$.

PROOF. Let $L \in \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$. By Lemma 4.1 we have an ascending $\mathfrak{F}$-series $(L_\alpha)_{\alpha\leq\sigma}$ of ideals of L. Then for each $\alpha < \sigma$

$$L_{\alpha+1}/L_\alpha \in \mathfrak{F} \cap \mathrm{LE}\mathfrak{A} \leq \mathrm{E}\mathfrak{A}.$$

Therefore we can find an $n(\alpha) \in \boldsymbol{N}$ such that $L_{\alpha+1}^{(n(\alpha)+1)} \leq L_\alpha$. Furthermore

$$L_{\alpha+1}^{(i)} \lhd L \quad \text{and} \quad L_{\alpha+1}^{(i)}/L_{\alpha+1}^{(i+1)} \in \mathfrak{A} \cap \mathfrak{F} \qquad \text{for } 0 \leq i \leq n(\alpha).$$

Hence we refine each factor $L_{\alpha+1}/L_\alpha$ by $(L_{\alpha+1}^{(i)})_{i=0}^{n(\alpha)}$ to obtain $L \in \acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F})$.

On the other hand, by putting $H = 0$ in the proof (7)$\Rightarrow$(2) of Theorem 3.1, we have

$$\Omega(i) \lhd L,\ \Omega(i+1)/\Omega(i) \in \mathfrak{A}\ (i < \omega) \quad \text{and} \quad L = \bigcup_{i<\omega} \Omega(i).$$

Therefore $L \in \acute{\mathrm{E}}_\omega(\lhd)\mathfrak{A}$.

Before stating the main theorem of this section, we introduce a new class of Lie algebras. We denote by

$$\mathfrak{X}_0$$

the class of all Lie algebras in which every non-zero finite-dimensional subalgebra is non-perfect. It is easy to show that

$$\{\mathrm{S}, \mathrm{L}, \mathrm{R}, \acute{\mathrm{E}}\}\mathfrak{X}_0 = \mathfrak{X}_0 .$$

Therefore we also have $\mathrm{D}\mathfrak{X}_0 = \mathfrak{X}_0$. On the other hand, by making use of Engel's theorem we obtain

$$\mathfrak{E} \leq \mathfrak{X}_0 .$$

Hence this class is fairly large. In fact, the classes $\mathrm{RLE}\mathfrak{A}$, $\acute{\mathrm{E}}\mathrm{LE}\mathfrak{A}$, $\mathrm{R}\mathfrak{E}$ and $\acute{\mathrm{E}}\mathfrak{E}$ are contained in $\mathfrak{X}_0$.

Now we have the characterizations of the class $\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$.

Theorem 4.3. *For any class $\mathfrak{X}$ of Lie algebras such that*

$$\acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \cap \acute{\mathrm{E}}_\omega(\lhd)\mathfrak{A} \leq \mathfrak{X} \leq \mathfrak{X}_0 ,$$

we have

$$\mathfrak{X} \cap \mathrm{L}(\lhd)\mathfrak{F} = \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}).$$

Proof. Let $L \in \mathfrak{X}_0 \cap \mathrm{L}\mathfrak{F}$ and let H be a finitely generated subalgebra of L. Then $H \in \mathfrak{F}$. So we use induction on $n = \dim H$ to show that $H \in \mathrm{E}\mathfrak{A}$. It is trivial for $n \leq 1$. Let $n \geq 2$ and suppose that every $\mathfrak{F}_{n-1}$-subalgebra of L is solvable. H is a non-zero finite-dimensional subalgebra of L. Since $L \in \mathfrak{X}_0$, we have $H^{(1)} \lneqq H$. Hence $\dim H^{(1)} \leq n-1$. By inductive hypothesis we have $H^{(1)} \in \mathrm{E}\mathfrak{A}$, whence $H \in \mathrm{E}\mathfrak{A}$. It follows that $L \in \mathrm{L}(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$. Therefore we obtain

$$\mathfrak{X}_0 \cap \mathrm{L}\mathfrak{F} = \mathrm{L}(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}).$$

Now let $\mathfrak{X}$ be any class of Lie algebras such that

$$\acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \cap \acute{\mathrm{E}}_\omega(\lhd)\mathfrak{A} \leq \mathfrak{X} \leq \mathfrak{X}_0 .$$

Then by Lemma 4.2 we have

$$\begin{aligned} \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}) &\leq \acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \cap \acute{\mathrm{E}}_\omega(\lhd)\mathfrak{A} \cap \mathrm{L}(\lhd)\mathfrak{F} \\ &\leq \mathfrak{X} \cap \mathrm{L}(\lhd)\mathfrak{F} \leq \mathfrak{X}_0 \cap \mathrm{L}(\lhd)\mathfrak{F} = \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}). \end{aligned}$$

Thus

$$\mathfrak{X} \cap \mathrm{L}(\lhd)\mathfrak{F} = \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}).$$

As a special case of Theorem 4.3 we have the following result.

Corollary 4.4. *If $\mathfrak{X}$ is one of the classes*

$$\acute{\mathrm{E}}\mathfrak{A}, \quad \acute{\mathrm{E}}(\lhd)\mathfrak{A}, \quad \acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}), \quad \mathrm{RLE}\mathfrak{A}, \quad \acute{\mathrm{E}}\mathrm{LE}\mathfrak{A}, \quad \mathrm{R}\acute{\mathrm{E}}\mathfrak{E}, \quad \acute{\mathrm{E}}\mathfrak{E},$$

then $\mathfrak{X} \cap \mathrm{L}(\lhd)\mathfrak{F} = \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$.

It is well known that if a Lie algebra L is locally solvable, then every minimal ideal of L is abelian. In an ideally finite Lie algebra the converse is also true under a certain condition. That is, we show

PROPOSITION 4.5. *If $\mathfrak{X}$ is a* Q*-closed class consisting of Lie algebras in which every minimal ideal is abelian, then*

$$\mathfrak{X} \cap \mathrm{L}(\lhd)\mathfrak{F} \leq \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}).$$

PROOF. Let $0 \neq L \in \mathfrak{X} \cap \mathrm{L}(\lhd)\mathfrak{F}$. Since L is ideally finite,

$$L = \bigcup_{\lambda\in\Lambda} A(\lambda)$$

where each $A(\lambda)$ is a non-zero finite-dimensional ideal of L. Since $A(\lambda)$ is a finite-dimensional L-module for each $\lambda \in \Lambda$, $A(\lambda)$ has a composition series

$$0 = A(\lambda, 0) \lneqq A(\lambda, 1) \lneqq \cdots \lneqq A(\lambda, n(\lambda)) = A(\lambda)$$

where $A(\lambda, i)/A(\lambda, i-1)$ is an irreducible L-module for $1 \leq i \leq n(\lambda)$. Then $A(\lambda, i)/A(\lambda, i-1)$ is a minimal ideal of $L/A(\lambda, i-1)$. Since $L/A(\lambda, i-1) \in \mathrm{Q}\mathfrak{X} = \mathfrak{X}$, we have $A(\lambda, i)/A(\lambda, i-1) \in \mathfrak{A}$. It follows that $A(\lambda) \in \mathrm{E}\mathfrak{A}$. Therefore

$$L \in \mathrm{LE}\mathfrak{A} \cap \mathrm{L}(\lhd)\mathfrak{F} = \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}).$$

Thus we have

$$\mathfrak{X} \cap \mathrm{L}(\lhd)\mathfrak{F} \leq \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}).$$

5.

In this section we shall study the relation between locally nilpotent, subideally finite (resp. ascendantly finite) Lie algebras and Baer algebras (resp. Gruenberg algebras). We need the following lemma.

LEMMA 5.1. *Let L be a Lie algebra over a field $\mathfrak{k}$ belonging to* $\mathrm{L}(\mathrm{si})\mathfrak{F}$ *(resp.* $\mathrm{L}(\mathrm{asc})\mathfrak{F}$*), and let H be an ascendant locally nilpotent subalgebra of L. Then for any finite subset X of H we have* $\langle X \rangle \operatorname{si} L$ *(resp.* $\langle X \rangle \operatorname{asc} L$*).*

PROOF. Let X be a finite subset of H. Then there exists a finite-dimensional subideal (resp. ascendant subalgebra) K of L containing X. $H \operatorname{asc} L$ implies $H \cap K \operatorname{asc} K$. Since $K \in \mathfrak{F}$, we have $H \cap K \operatorname{si} K$. On the other hand, we have $H \cap K \in \mathfrak{N}$ since $H \in \mathrm{L}\mathfrak{N}$. Therefore $\langle X \rangle \operatorname{si} H \cap K$. Hence

$$\langle X \rangle \operatorname{si} H \cap K \operatorname{si} K \operatorname{si} L \qquad (\text{resp. } K \operatorname{asc} L).$$

Thus we have $\langle X \rangle \operatorname{si} L$ (resp. $\langle X \rangle \operatorname{asc} L$).

COROLLARY 5.2. *Over a field of characteristic zero,*

(1) *if* $L \in \mathrm{L(si)}\mathfrak{F}$ *then* $\beta(L) = \gamma(L) = \rho(L)$, *and*
(2) *if* $L \in \mathrm{L(asc)}\mathfrak{F}$ *then* $\gamma(L) = \rho(L)$.

Proof. Let $L \in \mathrm{L(si)}\mathfrak{F}$ (resp. $L \in \mathrm{L(asc)}\mathfrak{F}$). Using [1, Theorem 6.2.1 and Corollary 6.3.5] we obtain

$$\beta(L) \leq \gamma(L) \leq \rho(L).$$

Let $x \in \rho(L)$. Since $\rho(L)$ is a locally nilpotent ideal of L, by Lemma 5.1 we have $\langle x \rangle \operatorname{si} L$ (resp. $\langle x \rangle \operatorname{asc} L$). It follows that $x \in \beta(L)$ (resp. $x \in \gamma(L)$). Hence

$$\rho(L) \leq \beta(L) \quad (\text{resp. } \rho(L) \leq \gamma(L)).$$

Thus we have

$$\beta(L) = \gamma(L) = \rho(L) \quad (\text{resp. } \gamma(L) = \rho(L)).$$

We now have the following main result of this section.

Theorem 5.3. (1) $\mathfrak{E} \cap \mathrm{L(si)}\mathfrak{F} = \mathrm{L(si)}(\mathfrak{N} \cap \mathfrak{F}) = \mathfrak{B}$.
(2) $\mathfrak{E} \cap \mathrm{L(asc)}\mathfrak{F} = \mathrm{L(asc)}(\mathfrak{N} \cap \mathfrak{F}) \leq \mathfrak{Gr}$.

Proof. By Engel's theorem $\mathfrak{E} \cap \mathrm{L}\mathfrak{F} = \mathrm{L}\mathfrak{N}$. Therefore we have

$$\mathfrak{E} \cap \mathrm{L(si)}\mathfrak{F} = \mathrm{L(si)}(\mathfrak{N} \cap \mathfrak{F}) \quad \text{and}$$
$$\mathfrak{E} \cap \mathrm{L(asc)}\mathfrak{F} = \mathrm{L(asc)}(\mathfrak{N} \cap \mathfrak{F}).$$

By using Lemma 5.1 we obtain

$$\mathrm{L(si)}(\mathfrak{N} \cap \mathfrak{F}) \leq \mathfrak{B} \quad \text{and}$$
$$\mathrm{L(asc)}(\mathfrak{N} \cap \mathfrak{F}) \leq \mathfrak{Gr}.$$

Finally we shall show that

$$\mathfrak{B} \leq \mathrm{L(si)}(\mathfrak{N} \cap \mathfrak{F}).$$

Let $L \in \mathfrak{B}$ and let X be a finite subset of L. Put $H = \langle X \rangle$. By [1, Theorem 7.1.5] we have $L \in \mathfrak{B} \leq \mathrm{L}\mathfrak{N}$, and so $H \in \mathfrak{N} \cap \mathfrak{F}$. We use induction on $n = \dim H$ to show that $H \operatorname{si} L$. It is trivial for $n \leq 1$ by the definition of $\mathfrak{B}$. Let $n \geq 2$ and assume that the result is true for $n-1$. Since $0 \neq H \in \mathfrak{N}$, we have $H^2 \lneqq H$. Therefore we can find an ideal H_0 of H such that $H^2 \leq H_0$ and $\dim H/H_0 = 1$. It follows that $\dim H_0 = n-1$, and we obtain $H_0 \operatorname{si} L$ by inductive hypothesis. Now we can find an $x \in L$ such that $H = H_0 + \langle x \rangle$. Since $L \in \mathfrak{B}$, $\langle x \rangle \operatorname{si} L$. On the other hand, H_0 permutes with $\langle x \rangle$. Therefore by [1, Lemma 2.1.4] we have

$$H = H_0 + \langle x \rangle \operatorname{si} L.$$

Hence $H \operatorname{si} L$, as was desired. Thus $H = \langle X \rangle$ is a finite-dimensional nilpotent subideal of L, which shows that $L \in \mathrm{L}(\mathrm{si})(\mathfrak{N} \cap \mathfrak{F})$. Therefore we obtain $\mathfrak{B} \leq \mathrm{L}(\mathrm{si})(\mathfrak{N} \cap \mathfrak{F})$. This completes the proof.

REMARK 5.4. Using the results described in [1, p. 258], we obtain

$$\mathrm{L}(\mathrm{si})(\mathfrak{N} \cap \mathfrak{F}) = \mathrm{L}\mathfrak{N} \cap \mathrm{N}\mathfrak{F} = \mathfrak{B} \quad \text{and}$$
$$\mathrm{L}(\mathrm{asc})(\mathfrak{N} \cap \mathfrak{F}) = \mathrm{L}\mathfrak{N} \cap \acute{\mathrm{N}}\mathfrak{F} = \mathfrak{Gr}$$

over a field of characteristic zero. On the other hand, by [3, Theorem 3.6] we have

$$\mathrm{L}(\lhd)(\mathfrak{N} \cap \mathfrak{F}) \leq \mathfrak{Z}_\omega \leq \mathfrak{Ft}.$$

But in general

$$\mathrm{L}(\lhd)(\mathfrak{N} \cap \mathfrak{F}) \lneqq \mathfrak{Z}_\omega.$$

In fact, let V be a vector space with basis $\{v_0, v_1, \dots\}$ and think of V as an abelian Lie algebra. Then there is a derivation σ of V such that

$$v_{2k}\sigma = 0, \ v_{2k+1}\sigma = v_{2k} \qquad \text{for } k \geq 0.$$

Form the split extension $L = V \dotplus \langle\sigma\rangle$. Then

$$L^2 = V\sigma = \langle v_{2k} \mid k \geq 0\rangle.$$

Hence

$$L^3 = \langle v_{2k} \mid k \geq 0\rangle\sigma = 0.$$

Therefore we have

$$L \in \mathfrak{N}_2 \leq \mathfrak{Z}_\omega.$$

However

$$\langle\sigma^L\rangle = \langle\sigma\rangle + [\sigma, L] = \langle\sigma\rangle + L^2 \notin \mathfrak{F}.$$

It follows that $L \notin \mathrm{L}(\lhd)\mathfrak{F}$.

Next we shall study the relations between Engel conditions and Gruenberg algebras.

We denote by $\mathfrak{e}(L)$ the set of left Engel elements of a Lie algebra L as usual. Then we have the following

LEMMA 5.5. *Let L be a Lie algebra over a field $\mathfrak{k}$. For any $x \in L$, the following conditions are equivalent*:

(1) $x \in \mathfrak{e}(L)$.

(2) $\langle x\rangle$ wasc L.

(3) $\langle x\rangle \leq^\omega L$.

PROOF. (1)⇔(3) is evident. (2)⇔(3) follows from [5, Theorem 4].

PROPOSITION 5.6. (1) $(\bigcup_{n\geq 1}\mathfrak{E}_n) \cap (\mathrm{E}\mathfrak{A})(\mathrm{L}(\mathrm{si})(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})) \leq \mathfrak{B}$.
(2) $\mathfrak{E} \cap (\mathrm{E}\mathfrak{A})(\mathrm{L}(\mathrm{asc})\acute{\mathrm{E}}(\lhd)\mathfrak{A}) \leq \mathfrak{Gr}$.

PROOF. (1) Let $L \in (\bigcup_{n\geq 1}\mathfrak{E}_n) \cap (\mathrm{E}\mathfrak{A})(\mathrm{L}(\mathrm{si})(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}))$ and let $x \in L$. Then $L \in \mathfrak{E}_n$ for some $n \geq 1$. It follows that $[L, {}_nx]=0$. Hence $\langle x\rangle \leq^n L$. By Proposition 3.6 we have $\langle x\rangle$ si L. Therefore $L \in \mathfrak{B}$.

(2) Let $L \in \mathfrak{E} \cap (\mathrm{E}\mathfrak{A})(\mathrm{L}(\mathrm{asc})\acute{\mathrm{E}}(\lhd)\mathfrak{A})$ and let $x \in L$. Since $L=\mathfrak{e}(L)$, we have $\langle x\rangle$ wasc L by Lemma 5.5. Now there exists a solvable ideal K of L such that $L/K \in \mathrm{L}(\mathrm{asc})\acute{\mathrm{E}}(\lhd)\mathfrak{A}$. Then

$$\langle x\rangle \text{ wasc } \langle x\rangle+K \quad \text{and} \quad (\langle x\rangle + K)/K \text{ wasc } L/K.$$

Clearly $\langle x\rangle+K \in \mathrm{E}\mathfrak{A}$. Hence $\langle x\rangle$ asc $\langle x\rangle+K$ by [5, Theorem 1]. On the other hand, we can find an ascendant subalgebra H/K of L/K which contains $(\langle x\rangle+K)/K$ and belongs to $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$. Then

$$(\langle x\rangle + K)/K \text{ wasc } H/K\cdot$$

By [5, Theorem 1] we have

$$(\langle x\rangle + K)/K \text{ asc } H/K.$$

Hence

$$(\langle x\rangle + K)/K \text{ asc } L/K.$$

It follows that

$$\langle x\rangle + K \text{ asc } L.$$

Therefore

$$\langle x\rangle \text{ asc } L.$$

Thus we have $L \in \mathfrak{Gr}$.

Finally we shall give two examples to show that $\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ doesn't coincide with $\mathrm{L}(\mathrm{si})(\mathfrak{N} \cap \mathfrak{F})$. One of the examples also shows that a subideally finite Lie algebra is not necessarily ideally finite.

EXAMPLE 5.7. Here we show that there are no inclusions between $\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ and $\mathrm{L}(\mathrm{si})(\mathfrak{N} \cap \mathfrak{F})$. That is,

(1) $\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}) \nleq \mathrm{L}(\mathrm{si})(\mathfrak{N} \cap \mathfrak{F})$,

(2) $\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}) \ngeq \mathrm{L}(\mathrm{si})(\mathfrak{N} \cap \mathfrak{F})$.

Let H be a two-dimensional non-abelian Lie algebra. Then H is a trivial example for (1). On the other hand, let H_i $(i \in \boldsymbol{Z})$ be an isomorphic copy of H and put $L=\bigoplus_{i\in \boldsymbol{Z}} H_i$. Then $L \in \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ but $L \notin \mathrm{L}(\mathrm{si})(\mathfrak{N} \cap \mathfrak{F})$. Hence L is an infinite-dimensional example for (1).

Next let M be the McLain Lie algebra of type $\boldsymbol{Z}$, i.e., M has a basis $\{e_{ij} \mid i, j \in \boldsymbol{Z}, i<j\}$ with multiplication

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}.$$

Then for each e_{ij} with $i<j$

$$\langle e_{ij}^M \rangle = \langle e_{kl} \mid k \leq i,\ l \geq j \rangle \in \mathfrak{A}.$$

Therefore

$$M = \textstyle\sum_{i<j} \langle e_{ij}^M \rangle \in \mathfrak{F}\mathrm{t} \leq \mathfrak{B} = \mathrm{L}(\mathrm{si})(\mathfrak{N} \cap \mathfrak{F}).$$

But $\langle e_{ij}^M \rangle \notin \mathfrak{F}$. Therefore $M \notin \mathrm{L}(\lhd)\mathfrak{F}$. Thus M is an example showing (2).

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