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Novikov's Ext² at the prime 2

Katsumi SHIMOMURA

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§1. Introduction

Let BP denote the Brown-Peterson spectrum at a prime p, whose coefficient ring $BP_*=\pi_*(BP)$ is the polynomial ring

$$BP_* = Z_{(p)}[v_1, v_2, \cdots] \qquad (\deg v_i = 2(p^i-1))$$

with Hazewinkel's generators v_i ([2]). Then, we have the Hopf algebroid (BP_*, BP_*BP), where $BP_*BP=BP_*[t_1, t_2, \cdots]$ $(\deg t_i=2(p^i-1))$ ([1; Part II], [5], [8], [6; §1]). For the spectrum BP, we have Novikov's analogue of the Adams spectral sequence converging to the stable homotopy ring $\pi_*(S)$ of the sphere spectrum S. Its E_2-term $E_2^{*,*}$ is the cohomology $\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$ (denoted simply by $\mathrm{Ext}^{*,*}BP_*$) of the Hopf algebroid (BP_*, BP_*BP), (cf. [1; Part III], [7]). The E_2-term $E_2^{1,*}$ is determined by S. P. Novikov [7] for any prime p, and $E_2^{2,*}$ by H. R. Miller, D. C. Ravenel and W. S. Wilson [6] for any odd prime p.

In this paper, we shall determine $E_2^{2,*}$ for the prime 2, and study the nontriviality of some elements in $\pi_*(S)$. We notice that the results for $E_2^{2,*}$ is also obtained by S. A. Mitchell, independently. From now on, we assume that the prime p is 2.

To state our results, we recall the elements $y_i \in v_1^{-1}BP_*$ and $x_i \in v_2^{-1}BP_*$, which are denoted by $x_{1,i}$ and $x_{2,i}$ in [6; (5.11)] respectively, given by

(1.1) $\quad y_0 = v_1,\ y_1 = v_1^2 - 4v_1^{-1}v_2,\ y_i = y_{i-1}^2 \quad (i \geqq 2),$

$$x_0 = v_2,\ x_1 = v_2^2 - v_1^2v_2^{-1}v_3,\ x_2 = x_1^2 - v_1^3v_2^3 - v_1^5v_3,\ x_i = x_{i-1}^2 \quad (i \geqq 3).$$

By using these elements and the universal Greek letter map η (for the definition, cf. [6; (3.6)]), we can define the elements

(1.2) $\quad \alpha_m = \eta(v_1^m/2)$ for odd $m \geqq 1$, $\quad \alpha_{2/2} = \eta(y_1/4)$, and

$$\alpha_{2^im/i+2} = \eta(y_i^m/2^{i+2}) \text{ for } i \geqq 1, \text{ odd } m \geqq 1 \text{ with } 2^im \geqq 4,$$

which generate Ext^1BP_* (cf. [6; Cor. 4.23]). Further, we can define the elements

(1.3.1) $$\beta_{2^ns/j,i+1} = \eta(x_n^s/2^{i+1}v_1^j) \quad \text{in} \quad \mathrm{Ext}^2BP_*$$

for $n \geqq 0$, odd $s \geqq 1$, $j \geqq 1$, $i \geqq 0$ with

$$n \geqq i,\ 2^i \mid j,\ j \leqq a_{n-i}, \text{ and especially } j \leqq 2^n \text{ if } s=1 \text{ and } i=0,$$

by the same way as the case for the odd prime ([6; p. 483]); and moreover we define the elements

$$(1.3.2) \qquad \beta_{2^ns/j,i+2} = \eta(x_n^s/2^{i+2}y_i^m) \quad (m=j/2^i) \qquad \text{in } \mathrm{Ext}^2 BP_*$$

for n, s, j and i of above with the additional conditions that

$$n \geqq i+1 \geqq 2,\ j=2 \text{ and } s \geqq 3 \text{ if } n=2, \text{ and } j \leqq a_{n-i-1} \text{ if } n \geqq 3,$$

(see Lemma 3.8), where a_k ($=a_{2,k}$ in [6; (5.13)]) are given by

$$(1.4) \qquad a_0=1, \quad a_1=2, \quad a_k=3\cdot 2^{k-1} \qquad (k \geqq 2).$$

For β in (1.3.2) and (1.3.1), we shall see the equalities

$$\beta_{2^ns/j,i+2} = \beta_{2^ns/j,(i+1)+1} \text{ under the condition that } 2^{i+1} \mid j,$$

$$2\beta_{2^ns/j,i+2} = \beta_{2^ns/j,i+1} \quad \text{(see Lemma 3.10)}.$$

By using the above elements, our main theorem is stated as follows:

THEOREM 1.5. *The 2-line of Novikov's spectral sequence,* $\mathrm{Ext}^{2,*}BP_* = \mathrm{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*)$, *for the prime 2 is the direct sum of the cyclic* $Z_{(2)}$*-submodules generated by the following elements:*

(1.6) $\alpha_1\alpha_m$ *for odd* $m \geqq 1$, $\alpha_1\alpha_{2^im/i+2}$ *for* $i \geqq 1$ *and odd* $m \geqq 1$ *with* $2^im \geqq 4$;

(1.7.1) $\beta_{2^ns/j,i+1}$ *in* (1.3.1) *with the additional conditions that* $s \geqq 3$ *if* $n=0$, $n>i$ *and either* j *is odd or* $a_{n-i-1}<j$ *if* $n \geqq 1$, *and especially* $s=1$ *if* $n=2$ *and* $i=1$;

(1.7.2) $\beta_{2^ns/j,i+2}$ *in* (1.3.2) *with the additional condition that* $2^{i+1} \nmid j$.

The elements in (1.6), (1.7.1) *and* (1.7.2) *have orders* 2, 2^{i+1} *and* 2^{i+2}, *respectively.*

After proving Theorem 3.3 which determines $\mathrm{Ext}^0 M_0^2$ ($M_0^2 = v_2^{-1}BP_*/(2^\infty, v_1^\infty)$), we prove the above theorem in §4. (For a BP_*BP-comodule M, $\mathrm{Ext}^*_{BP_*BP}(BP_*, M)$ will be denoted simply by Ext^*M.) Furthermore, by using Theorem 3.3, we can prove Theorem 5.2, which gives a partial information on $\mathrm{Ext}^3 BP_*$, and the following theorem in §6 by using the result of M. Mahowald (cf. [9; Cor. 7.6]) that $\beta_{8t/4,2}$ is a permanent cycle converging to the element in $\pi_*(S)$ of order 4.

THEOREM 1.8. *For any* $t \geqq 1$, *the elements* $\alpha_1\beta_{8t/3,1}$, $\alpha_{4/4}\beta_{8t/3,1}$ *and* $\alpha_{4/4}\beta_{8t/4,2}$ *in* $\mathrm{Ext}^3 BP_*$ *are nontrivial permanent cycles converging to the homotopy elements in* $\pi_*(S)$ *of orders* 2, 2 *and* 4, *respectively.*

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§ 2. Preliminaries

For the right unit $\eta_R\colon BP_*\to BP_*BP$, we see easily the following formulae by [6; (1.1), (1.3)] (cf. also [1], [2], [8]):

(2.1) $$\eta_R v_1 = v_1 + 2t_1,$$

(2.2) $$\eta_R v_2 \equiv v_2 - v_1t_1^2 + v_1^2t_1 + 2t_2 \quad \mathrm{mod}(4),$$

(2.3) $$\eta_R v_3 \equiv v_3 + v_2t_1^4 + v_2^2t_1 + v_1t_2^2 + v_1^2v_2t_1^2 + v_1^2t_1^5 \quad \mathrm{mod}(2, v_1^3).$$

η_R induces the map $\eta_R\colon v_n^{-1}BP_*\to v_n^{-1}BP_*/J\otimes_{BP_*}BP_*BP$ for any ideal J of BP_* such that $(2, v_1,\cdots, v_{n-1})^k\subset J$ for some positive integer k. In this section we shall compute the image of the elements $y_i\in v_1^{-1}BP_*$ and $x_i\in v_2^{-1}BP_*$ given in (1.1) by the map

$$d=\eta_R-\mathrm{id}\otimes 1\colon v_n^{-1}BP_* \longrightarrow v_n^{-1}BP_*/J\otimes_{BP_*}BP_*BP \quad (n=1, 2)$$

for some ideal J of above. Here we shall say that $x\in v_n^{-1}BP_*$ is invariant mod J if $dx=0$ (cf. [4]).

For the elements $y_i\in v_1^{-1}BP_*$, we recall the following

(2.4) ([6; Lemma 4.12], cf. [4; p. 504]) *For* $i\geqq 1$, y_i *is invariant* $\mathrm{mod}(2^{i+2})$. *More precisely,*

$$dy_i\equiv 2^{i+2}v_1^{2^i}\rho_1 \quad \mathrm{mod}(2^{i+3}),$$

where $\rho_1\in v_1^{-1}BP_*BP$ *is given by*

(2.5) $$\rho_1 = v_1^{-3}(t_2-t_1^3) + v_1^{-4}v_2t_1 \qquad (\text{cf. [6; Prop. 3.18]}).$$

For the elements $x_i\in v_2^{-1}BP_*$, we have the following lemmas, where a_i is the integer given in (1.4).

LEMMA 2.6 (cf. [6; Prop. 5.14]). *For* $i\geqq 0$, x_i *is invariant* $\mathrm{mod}(2, v_1^{a_i})$. *More precisely, we have the following congruences* $\mathrm{mod}(2, v_1^{2+a_i})$:

$$dx_i\equiv\begin{cases} v_1t_1^2+v_1^2t_1 & (i=0),\\ v_1^2\tilde\rho_1+v_1^3v_2\zeta_2 & (i=1),\\ v_1^{a_i}v_2^{2^{i-1}}\zeta_2^{2^{i-1}} & (i\geqq 2).\end{cases}$$

Here $\tilde\rho_1\in BP_*BP$ *and* $\zeta_2\in v_2^{-1}BP_*BP$ *are given by*

(2.7) $$\tilde\rho_1 = v_1^4\rho_1 = v_1(t_2-t_1^3)+v_2t_1,$$

(2.8) $\zeta_2 = v_2^{-1}t_2 + v_2^{-2}(t_2^2 - t_1^6) - v_2^{-3}v_3t_1^2$ (cf. [6; Prop. 3.18]).

PROOF. The congruence follows from (2.2) for $i=0$ and is given in [6; (5.15)] for $i=1$.

We notice the following which is seen easily:

(2.9) *If* $dx \equiv y \mod (2, u)$, *then* $dx^{2^i} \equiv y^{2^i} \mod (2, u^{2^i})$.

By the congruence for $i=1$ in the lemma and (2.9),

$$dx_1^2 \equiv v_1^4v_2^2t_1^2 + v_1^6t_1^6 + v_1^6t_2^2 + v_1^6v_2^2\zeta_2^2 \mod(2, v_1^8).$$

Also by using (2.1–3), we see the following mod $(2, v_1^8)$:

$$dv_1^3v_2^3 \equiv v_1^4v_2^2t_1^2 + v_1^5v_2t_1^4 + v_1^5v_2^2t_1 + v_1^6t_1^6 + v_1^7t_1^5 + v_1^7v_2t_1^2,$$

$$dv_1^5v_3 \equiv v_1^5v_2t_1^4 + v_1^5v_2^2t_1 + v_1^6t_2^2 + v_1^7v_2t_1^2 + v_1^7t_1^5.$$

Collect terms to obtain the case $i=2$. For $i \geqq 3$, use $x_i = x_{i-1}^2$ in (1.1), (2.9) and the inequality $2^{i-2} \cdot 8 \geqq 2 + 3 \cdot 2^{i-1}$. q.e.d.

LEMMA 2.10. *Let s be any odd integer. Then*

(i) $dx_1^s \equiv 2v_1v_2^{2s-1}t_1^2 \mod(2^2, v_1^2)$,

(ii) $dx_2^s \equiv 0 \mod (2^3, v_1^2)$,

(iii) $dx_n^s \equiv 2^{n-1}v_1^3v_2^{2^ns-1}\zeta_2 \mod(2^n, v_1^4)$ *for* $n = 2, 3, 4$,

(iv) $dx_n^s \equiv 2^{n-i}v_1^{q_i}v_2^{2^ns-2^{i-1}}\zeta_2^{2^{i-1}} \mod(2^{n-i+1}, v_1^{2+a_i})$ *for* $2 \leqq i \leqq n$.

PROOF. (i) and (ii) follow from (1.1), (2.1) and (2.2).

Recall here that

(2.11) ([6; p. 498]) *If L is any ideal with* $y^2 \in L$, *and if* $dx \equiv y \mod(2, L)$, *then* $dx^j \equiv jx^{j-1}y \mod(2^{i+1}, L)$ *for any integer* $j \geqq 1$ *and* $i \geqq 0$ *with* $2^i \mid j$.

(iii) for $n=2$ follows from (1.1), (2.1), (2.2), Lemma 2.6 and (2.11), and (iii) for $n=3$, 4 follows from (iii) for $n=2$, (1.1) and (2.11).

(iv) follows from (1.1), Lemma 2.6 and (2.11). q.e.d.

§3. $\mathrm{Ext}^0 M_0^2$ and $\mathrm{Ext}^0 N_0^2$

In [6; §3], the BP_*BP-comodules N_n^s and M_n^s are defined inductively by setting $N_n^0 = BP_*/(2, v_1, \cdots, v_{n-1})$, $M_n^s = v_{n+s}^{-1}N_n^s$ and by the short exact sequence

(3.1) $$0 \longrightarrow N_n^s \xrightarrow[\subset]{j} M_n^s \xrightarrow{k} N_n^{s+1} \longrightarrow 0.$$

Their coactions are induced from the right unit $\eta_R\colon BP_* \to BP_*BP$. Further the sequence

$$(3.2)\qquad 0 \longrightarrow M_{n+1}^{s-1} \xrightarrow{\ i\ } M_n^s \xrightarrow{\ v_n\ } M_n^s \longrightarrow 0$$

is exact, where $i(x)=x/v_n$ (with $v_0=2$) (cf. [6; (3.10)]).

For the BP_*BP-comodule $M=N_n^s$ or M_n^s, $\mathrm{Ext}^*M=\mathrm{Ext}^*_{BP_*BP}(BP_*, M)$ is the homology of the cobar complex Ω^*M (cf. [6; (1.8–10)]). Especially the differential $d\colon \Omega^0M=M\to\Omega^1M=M\otimes_{BP_*}BP_*BP$ is given by $d=\eta_R-\mathrm{id}\otimes 1$ and $\mathrm{Ext}^0M=\mathrm{Ker}\, d$. In this section we shall determine $\mathrm{Ext}^0M_0^2$ and $\mathrm{Ext}^0N_0^2$. Our results are stated in the following theorems:

THEOREM 3.3. *$\mathrm{Ext}^0M_0^2$ is the direct sum of the cyclic $Z_{(2)}$-submodules generated by the following cycles*:

(3.3.1) $x_n^s/2^{i+1}v_1^j$ *for* $n\geqq 0$, *odd* s, $i\geqq 0$ *and* $j\geqq 1$ *such that*

$j\leqq a_n$ *and either* j *is odd or* $a_{n-1}<j$ *if* $i=0$,

$n-2\geqq i$, $2^i\,|\,j$ *and* $a_{n-i-1}<j\leqq a_{n-i}$ *otherwise*;

(3.3.2) $x_n^s/2^{i+2}y_i^m$ *for* $n\geqq 2$, *odd* s, $i\geqq 1$ *and odd* $m\geqq 1$ *such that*

$i=m=1$ *if* $n=2, 3$, $n-2>i$ *and* $2^im\leqq a_{n-i-1}$ *if* $n\geqq 4$;

(3.3.3) $1/2v_1^m$ *for odd* $m\geqq 1$, *and* $1/2^{i+2}y_i^m$ *for* $i\geqq 1$ *and odd* $m\geqq 1$.

THEOREM 3.4. *$\mathrm{Ext}^0N_0^2$ is the direct sum of the cyclic $Z_{(2)}$-submodules generated by the cycles in* (3.3.3),

(3.4.1) *the ones in* (3.3.1) *with* $s\geqq 1$ *and* $j\leqq 2^n$ *if* $s=1$ *and* $i=0$,

(3.4.2) *the ones in* (3.3.2) *with* $s\geqq 1$ *and especially* $s\geqq 3$ *if* $n=2$, *and the cycle* $x_2/2^2y_1=2(x_2/2^3y_1)=x_2/2^2v_1^2$ (*see Lemma* 3.10).

PROOF OF THEOREM 3.4 FROM THEOREM 3.3. We have the exact sequence

$$(3.5)\qquad 0 \longrightarrow \mathrm{Ext}^0N_0^2 \xrightarrow{\ j\ } \mathrm{Ext}^0M_0^2 \xrightarrow{\ k\ } \mathrm{Ext}^0N_0^3$$

associated to the exact sequence (3.1) for $s=2$, $n=0$. By the definition of N_0^3, the k-image of $z/2^iv_1^j$ $(\neq 0)\in M_0^2$ with $z\in v_2^{-1}BP_*/(2^i, v_1^j)$ $(i, j>0)$ is zero if and only if $z\in BP_*/(2^i, v_1^j)$. We see also by (1.1) that

$$(3.6)\qquad x_n^s \equiv v_2^k + 2^i(v_1^{4l}v_2^{k-6l}v_3^{2l} + v_1^{3l}v_2^{k-l} + v_1^{5l}v_2^{k-4l}v_3^l) \quad \mathrm{mod}(2^{i+1}, v_1^{a_{n-i}})$$
$$(k=2^ns,\ l=2^{n-i-2}) \text{ for } n\geqq 2 \text{ and } n-2\geqq i\geqq 0,$$
$$x_1^s \equiv v_2^{2s} \quad \mathrm{mod}(2, v_1^2), \qquad x_0^s = v_2^s,$$

where s is any odd integer. Further (1.1) implies the equality

$$(3.7)\qquad 1/2^{i+2}y_i^m = (1/2^{i+2}v_1^j) + m(v_2/2v_1^{j+3})\ (j=2^im) \text{ in } M_0^2 \text{ for } i\geqq 1,\ m\geqq 1.$$

Then for any cycle g in (3.3.1–3), we can write $g=\sum_{i,j} z_{ij}/2^i v_1^j$ with $z_{ij}\in v_2^{-1}BP_*/(2^i, v_1^j)$ by using (3.7); and we see by (3.6) that all z_{ij}'s belong to $BP_*/(2^i, v_1^j)$ if and only if g is the one in (3.3.3), (3.4.1–2), except $g=x_2/2^3y_1$. Thus, we see the kernel of k in (3.5) at any degree other than 20 by using Theorem 3.3. On the other hand, we see that $k(x_2/2^3y_1)=v_2^3/2v_1v_2(\neq 0)$ by (1.1) and (3.7). Thus, at the degree 20, $x_2/2^2y_1$ is the generator of $\mathrm{Ext}^0N_0^2$. Therefore Theorem 3.4 follows from Theorem 3.3. q.e.d.

To prove Theorem 3.3, we prepare several lemmas, (cf. [4] for the definition and the properties of invariant ideals).

LEMMA 3.8. *Let $\alpha_0, \alpha_1, \alpha_2$ be given as follows, where s is any integer:*

(1) $\alpha_0 = 2^{i+1},\ \alpha_1 = v_1^j,\ \alpha_2 = x_n^s$ *for* $0\leqq i\leqq n,\ 1\leqq j\leqq a_{n-i}$ *with* $2^i\mid j$;

(2) $\alpha_0 = 2^{i+2},\ \alpha_1 = y_i^m,\ \alpha_2 = x_n^s$ *for* (a) $i=m=1$ *if* $n=2, 3$,

and for (b) $1\leqq i\leqq n-2,\ m\geqq 1$ *with* $2^i m\leqq a_{n-i-1}$ *if* $n\geqq 4$.

Then $\alpha_2/\alpha_0\alpha_1$ is a cycle and hence $\alpha_2/\alpha_0\alpha_1\in \mathrm{Ext}^0M_0^2$. More precisely, if $\alpha_0, \alpha_1, \alpha_2$ are in (1) *or* (2) *for* (b), *then $(\alpha_0, \alpha_1, \alpha_2)$ is an invariant ideal of BP_*.*

PROOF. (1) If $0\leqq n-i\leqq 1$, then $(\alpha_0, \alpha_1, \alpha_2)$ is equal to $(2, v_1, v_2^s)$, $(2, v_1, x_1^s)$, $(2, v_1^2, x_1^s)$ or $(2^2, v_1^2, x_2^s)$. These are invariant by Lemmas 2.6 and 2.10 (iii).

Since $2^i\mid j$, $(2^{i+1}, v_1^j)$ is invariant. If $n-i\geqq 2$, then x_n^s is invariant mod$(2^{i+1}, v_1^j)$ for $j\leqq a_{n-i}$ with $2^i\mid j$ by Lemma 2.10 (iv).

(2) for (b) We use the following which is seen easily:

(3.9) *If* $x\equiv y$ mod$(2, u)$, *then* $x^{2^i s}\equiv y^{2^i s}$ mod$(2^{i+1}, 2u, u^{2^i})$ *for* $i\geqq 1$.

Applying this to Lemma 2.6, we obtain that $x_n^s=x_{n-i-1}^{2^{i+1}s}$ in (1.1) is invariant mod$(2^{i+2}, 2v_1^q, v_1^{2^{i+1}a})$, where $a=a_{n-i-1}$. Since $v_1^{2^i m}\equiv 2^{i+1}mv_1^{2^i m-3}v_2$ mod$(2^{i+2}, y_i^m)$ by (1.1), both $2v_1^q$ and $v_1^{2^{i+1}a}$ belong to the ideal $(2^{i+2}, y_i^m)$. Hence x_n^s is invariant mod$(2^{i+2}, y_i^m)$.

(2) for (a) By (2.4) and (3.7), we have

$$d(x_n^s/2^3y_1) = d(x_n^s)/2^3v_1^2 + d(x_n^s)v_2/2v_1^5.$$

Further $d(x_n^s)\equiv 0$ mod$(2^3, v_1^2)$ by Lemma 2.10 (ii) and (iii), and $d(x_n^s)\equiv 0$ mod$(2, v_1^5)$ by Lemma 2.6. Therefore $d(x_n^s/2^3y_1)=0$. q.e.d.

LEMMA 3.10. *For any cycle $\alpha_2/\alpha_0\alpha_1=x_n^s/2^{i+2}y_i^m$ of type* (2) *in Lemma* 3.8,

(3.11)
$$\alpha_2/\alpha_0\alpha_1 = x_n^s/2^{i+2}v_1^{2^i m} \quad \text{if } m \text{ is even},$$
$$2(\alpha_2/\alpha_0\alpha_1) = x_n^s/2^{i+1}v_1^{2^i m}.$$

Furthermore, the cycles given in Theorem 3.3 *are linearly independent.*

PROOF. (3.11) follows immediately from (3.7). Moreover, we see easily that (3.11) is the only relations between the cycles in Lemma 3.8 for $s=0$ or odd s, by comparing their degrees. Thus the latter half follows. q.e.d.

Now consider the exact sequence

$$(3.12)\qquad \mathrm{Ext}^0 M_1^1 \xrightarrow{\delta} \mathrm{Ext}^1 M_2^0 \xrightarrow{i} \mathrm{Ext}^1 M_1^1 \xrightarrow{v_1} \mathrm{Ext}^1 M_1^1$$

associated to the exact sequence (3.2) for $s=1$ and $n=1$, where the first two modules are given as follows:

(3.13) ([6; Prop. 3.18]) $\mathrm{Ext}^1 M_2^0$ *is the* $F_2[v_2, v_2^{-1}]$*-vector space generated by the four elements* $h_0=\{t_1\}$, $h_1=\{t_1^2\}$, ζ_2 *of* (2.8) *and* ρ_2.

(3.14) ([6; Th. 5.10]) $\mathrm{Ext}^0 M_1^1$ *is the direct sum of* $F_2[v_1, v_1^{-1}]/F_2[v_1]$ *and the cyclic* $F_2[v_1]$*-submodules generated by* $x_i^s/v_1^{q_i}$ *for* $i\geqq 0$ *and odd* s.

LEMMA 3.15. *The kernel of* v_1: $\mathrm{Ext}^1 M_1^1 \to \mathrm{Ext}^1 M_1^1$ *in* (3.12) *is the* F_2*-vector space generated by the elements represented by the following cycles*:

$$v_2^s t_1/v_1,\ v_2^{2t-1}t_1^2/v_1,\ v_2^{2^k(4t-1)}\zeta_2/v_1,\ v_2^{4t}\tilde{\rho}_1/v_1^2,\ \textit{and}\ v_2^t\rho_2/v_1$$

for any integers s, t *and* k *with* $s\not\equiv 1 \bmod 4$ *and* $k\geqq 0$.

PROOF. We study the cokernel of δ: $\mathrm{Ext}^0 M_1^1 \to \mathrm{Ext}^1 M_2^0$ in (3.12). We see easily that $d(1/v_1^j)=0$ by (2.1), which implies $\delta(F_2[v_1, v_1^{-1}]/F_2[v_1])=0$. By Lemma 2.6,

$$\delta(x_i^s/v_1^{q_i}) = \begin{cases} sv_2^{s-1}t_1^2 & (i=0), \\ sv_2^{2s-1}t_1 & (i=1), \\ sv_2^{(2s-1)2^{i-1}}\zeta_2^{2^{i-1}} & (i\geqq 2). \end{cases}$$

In the last term, $\zeta_2^{2^{i-1}}$ is homologous to ζ_2 by [6; Lemma 3.19]. Therefore, (3.13), (3.14) and the above equalities imply that $\mathrm{Coker}\,\delta$ is the F_2-vector space spanned by

$$v_2^s t_1,\quad v_2^{2t-1}t_1^2,\quad v_2^{2^k(4t-1)}\zeta_2,\quad v_2^{4t+1}\zeta_2\quad \text{and}\quad v_2^t\rho_2$$

for any integers s, t and k with $s\not\equiv 1 \bmod 4$ and $k\geqq 0$. Furthermore, by using Lemma 2.6, we see that

$$dv_2^{4t}x_1/v_1^4 = v_2^{4t}\tilde{\rho}_1/v_1^2 + v_2^{4t+1}\zeta_2/v_1.$$

Therefore $v_2^{4t}\tilde{\rho}_1/v_1^2$ is homologous to $i(v_2^{4t+1}\zeta_2)=v_2^{4t+1}\zeta_2/v_1$. Since $\mathrm{Ker}\,v_1 = i(\mathrm{Coker}\,\delta)$ by the exact sequence (3.12), these imply the lemma. q.e.d.

Next consider the exact sequence

(3.16) $$0 \longrightarrow \mathrm{Ext}^0 M_1^1 \xrightarrow{i} \mathrm{Ext}^0 M_0^2 \xrightarrow{2} \mathrm{Ext}^0 M_0^2 \xrightarrow{\delta} \mathrm{Ext}^1 M_1^1$$

associated to the exact sequence (3.2) for $s=2$ and $n=0$.

On the boundary homomorphism δ in (3.16), we have the following

LEMMA 3.17. *The δ-image of the cycle $\alpha_2/\alpha_0\alpha_1 \in \mathrm{Ext}^0 M_0^2$ in* (3.3.1–3) *is given as follows*:

(3.17.1) *For the cycles $x_n^s/2^{i+1}v_1^j$ in* (3.3.1),

(a) $\delta(v_2^s/2v_1) = \{v_2^{s-1}\tilde{\rho}_1/v_1^2\}$, (b) $\delta(x_1^s/2v_1^2) = \{v_2^{2s-1}t_1^2/v_1\}$,

(c) $\delta(x_n^s/2v_1^j) = \{v_2^k t_1/v_1^{j+1} + \cdots\}$ $(k=2^n s)$ *for* $n \geqq 1$, *odd* $j \leqq a_n$,

(d) $\delta(x_n^s/2^{i+1}v_1^j) = \{v_2^{k-l}\zeta_2^l/v_1^{j-3l} + \cdots\}$ $(k=2^n s,\ l=2^{n-i-2})$ *for the other cycles.*

(3.17.2) $\delta(x_n^s/2^{i+2}y_i^m) = \{v_2^k\tilde{\rho}_1/v_1^{j+4} + \cdots\}$ $(k=2^n s,\ j=2^i m)$ *for the cycles in* (3.3.2).

(3.17.3) $\delta(1/2v_1^m) = \{t_1/v_1^{m+1}\}$, $\delta(1/2^{i+2}y_i^m) = \{\tilde{\rho}_1/v_1^{j+4}\}$ $(j=2^i m)$ *for the cycles in* (3.3.3).

Here $\{x\}$ means the cohomology class of x, and $\cdots$ denotes an element of $\Omega^1 M_1^1$ killed by a lower power of v_1 than those shown.

PROOF. By the definition of δ, we see that

(3.18) $$\delta(\alpha_2/\alpha_0\alpha_1) = \{i^{-1}\{d(\alpha_2)/2\alpha_0\alpha_1 + \eta_R(\alpha_2)d(1/2\alpha_0\alpha_1)\}\}.$$

On the $d(1/2\alpha_0\alpha_1)$ in the second term of the right hand side of (3.18), we have the following by (2.1) and (2.4):

(3.19) (i) $d(1/2^2v_1^j) = jt_1/2v_1^{j+1}$ for $j \geqq 1$;

(ii) $d(1/2^{i+2}v_1^j) = j(t_1^2+v_1t_1)/2^{i+1}v_1^{j+2}$ for $i \geqq 1$, $2^i \mid j \geqq 1$;

(iii) $d(1/2^{i+3}y_i^m) = m\tilde{\rho}_1/2v_1^{j+4}$ $(j=2^i m)$ for $i \geqq 1$, $m \geqq 1$.

By Lemmas 2.6, 2.10, (3.19) and (3.18), we see the lemma as follows.

(3.17.1) By (2.2) and (3.19) (i), we see that

$$d(v_2^s)/2^2v_1 = v_2^{s-1}t_2/2v_1,\quad \eta_R(v_2^s)d(1/2^2v_1) = (v_2^s t_1 + v_1v_2^{s-1}t_1^3)/2v_1^2.$$

These imply (a). By Lemma 2.10,

$$d(x_1^s)/2^2v_1 = 0,\quad d(x_1^s)/2^2v_1^2 = v_2^{2s-1}t_1^2/2v_1,$$

$$d(x_n^s)/2^{i+2}v_1^j = v_2^{k-l}\zeta_2^l/2v_1^{j-3l} \text{ for } n-2 \geqq i \geqq 0,\ 0<j \leqq a_{n-i} \text{ with } 2^i \mid j,$$

$$(k=2^n s,\ l=2^{n-i-2}).$$

Also by (1.1), Lemma 2.6 and (3.19),

$$\eta_R(x_n^s)d(1/2^2v_1^j) = j((v_2^k t_1/2v_1^{j+1}) + \cdots) \ (k=2^n s) \quad \text{for} \quad n \geqq 1,$$

$$\{\eta_R(x_n^s)d(1/2^{i+2}v_1^j)\} = \{jv_2^{k-l'}(t_1^2+v_1t_1)/2^{i+1}v_1^{j+2-3l'}\} \quad (l'=2^{n-2})$$

for $n-2 \geqq i > 0$, $0 < j \leqq a_{n-i}$ with $2^i \mid j$, because $d(v_2^{k+1}/2v_1^{j+3}) = v_2^k(t_1^2+v_1t_1)/2v_1^{j+2} + \delta_{n,3}v_2^{8s-7}t_1^{16}/2v_1^{j-5}$ by (2.2) ($\delta_{n,3}$ is the Kronecker δ). These imply (b)–(d).

(3.17.2) In the right hand side of (3.18) for the cycle $\alpha_2/\alpha_0\alpha_1$ in (3.3.2), we see that the power of v_1 in the denominator of the first term is less than that of the second term by using (1.1), Lemma 2.10 and (3.19) (iii). Now (3.17.2) follows from (1.1), (3.18) and (3.19) (iii).

(3.17.3) The equality follows immediately from (3.19) (i), (iii). q. e. d.

COROLLARY 3.20. *For every cycle $g=\alpha_2/\alpha_0\alpha_1$ in (3.3.1–3), there is a non-negative integer $l(g)$ such that $v_1^{l(g)}\delta(g)$ is a generator of the kernel of v_1 given in Lemma 3.15. More precisely, we have the following equalities:*

(3.20.1) (a) $v_1\delta(v_2^s/2v_1) = \{v_2^s t_1/v_1\}$ *for* $s \equiv -1 \mod 4$,

$\delta(v_2^s/2v_1) = \{v_2^{s-1}\tilde{\rho}_1/v_1^2\}$ *for* $s \equiv 1 \mod 4$,

(b) $\delta(x_1^s/2v_1^2) = \{v_2^{2s-1}t_1^2/v_1\}$,

(c) $v_1^j\delta(x_n^s/2v_1^j) = \{v_2^k t_1/v_1\}$ $(k=2^n s)$ *for* $n \geqq 1$, *odd* $j \leqq a_n$,

(d) $v_1^{j-3l-1}\delta(x_n^s/2^{i+1}v_1^j) = \{v_2^{k-l}\zeta_2/v_1\}$ $(k=2^n s,\ l=2^{n-i-2})$

for the other cycles in (3.3.1).

(3.20.2) $v_1^{j+2}\delta(x_n^s/2^{i+2}y_i^m) = \{v_2^k\tilde{\rho}_1/v_1^2\}$ $(k=2^n s,\ j=2^i m)$ *for the cycles in* (3.3.2).

(3.20.3) $v_1^m\delta(1/2v_1^m) = \{t_1/v_1\}$, $v_1^{j+2}\delta(1/2^{i+2}y_i^m) = \{\tilde{\rho}_1/v_1^2\}$ $(j=2^i m)$

for the cycles in (3.3.3).

Now we are ready to prove Theorem 3.3, by using the following fact which is an immediate consequence of [6; Remark 3.11]:

(3.21) *In the exact sequence* (3.16), *let B be a $Z_{(2)}$-submodule of* $\mathrm{Ext}^0 M_0^2$ *such that B contains both $2B$ and the image of i:* $\mathrm{Ext}^0 M_1^1 \to \mathrm{Ext}^0 M_0^2$, *and that the sequence*

$$B \xrightarrow{2} B \xrightarrow{\delta} \mathrm{Ext}^1 M_1^1 \tag{3.22}$$

is exact. Then $B = \mathrm{Ext}^0 M_0^2$.

PROOF OF THEOREM 3.3. Let B be the submodule generated by the cycles in (3.3.1–3). Then by Lemma 3.10 and (3.14), we see that $B \supset 2B$ and $B \supset \mathrm{Im}\, i$.

Thus it suffices to show that the sequence (3.22) is exact. By Lemma 3.17, it is clear that the composition of 2 and δ in (3.22) is zero.

Now, take a homogeneous element $\beta \in B$ and write $\beta = \sum \lambda_g g$ as a linear combination of the generators of B. Assume $\beta \notin 2B$, and consider $l(\beta) = \max\{l(g): \lambda_g \equiv 1 \bmod 2\}$, where $l(g)$ is the integer given in Corollary 3.20. Then we see easily that $v_1^{l(\beta)}\delta(\beta) = \sum_{l(g)=l(\beta)} v_1^{l(g)}\delta(g)$ is not zero by the definition of $l(\beta)$, Lemma 3.15 and Corollary 3.20. Hence $\delta(\beta) \neq 0$. Therefore $\delta(\beta) = 0$ implies $\beta \in 2B$, as desired. q.e.d.

§4. $\mathrm{Ext}^2 BP_*$

In the first place, we notice the following

PROPOSITION 4.1. *The boundary homomorphism*

$$\delta_0 \colon \ \mathrm{Ext}^t N_0^1 \longrightarrow \mathrm{Ext}^{t+1} N_0^0 = \mathrm{Ext}^{t+1} BP_*\,,$$

associated to the exact sequence $0 \to N_0^0 \to M_0^0 \to N_0^1 \to 0$ *of* (3.1), *is an isomorphism for* $t > 0$.

This follows immediately from the fact that $\mathrm{Ext}^t M_0^0 = 0$ for $t > 0$ ([5; Th. 2.10], [6; Th. 3.15]).

To study $\mathrm{Ext}^1 N_0^1 \approx \mathrm{Ext}^2 BP_*$, we consider the exact sequence

$$(4.2)\quad \mathrm{Ext}^0 M_0^1 \xrightarrow{k_*} \mathrm{Ext}^0 N_0^2 \xrightarrow{\delta'} \mathrm{Ext}^1 N_0^1 \xrightarrow{j_*} \mathrm{Ext}^1 M_0^1 \xrightarrow{k_*} \mathrm{Ext}^1 N_0^2 \xrightarrow{\delta'} \mathrm{Ext}^2 N_0^1$$

associated to the exact sequence (3.1) for $s = 1$ and $n = 0$. Here, the module $\mathrm{Ext}^0 N_0^2$ is given in Theorem 3.4 and the modules $\mathrm{Ext}^0 M_0^1$ and $\mathrm{Ext}^1 M_0^1$ are given in [6; Th. 4.16] as follows:

(4.3.a) $\mathrm{Ext}^0 M_0^1$ *is the direct sum of* (1) *the cyclic* $Z_{(2)}$*-modules generated by* $v_1^s/2$, $y_i^s/2^{i+2}$ *for* $i \geqq 1$ *and odd* s, *and* (2) $Q/Z_{(2)}$ *generated by* $1/2^j$ *for* $j \geqq 1$.

(4.3.b) $\mathrm{Ext}^1 M_0^1$ *is the direct sum of* (1) $Z/2$ *generated by* $\rho_1/2$, $v_1^s t_1/2$, $v_1^s \rho_1/2$ *for odd* s, *and* (2) $Q/Z_{(2)}$ *generated by* z_j *such that* $2^j z_j = \{\rho_1/2\}$ *for* $j \geqq 1$.

LEMMA 4.4. *The image* A_0 *of* $k_*\colon \mathrm{Ext}^0 M_0^1 \to \mathrm{Ext}^0 N_0^2$ *in* (4.2) *is the submodule generated by the cycles in* (3.3.3) *and* $v_2/2v_1$.

PROOF. For the generators given in (4.3.a), we see that

$$k_*(1/2^j) = 0, \quad k_*(v_1^s/2) = \begin{cases} 0 & (s > 0), \\ 1/2v_1^{-s} & (s < 0), \end{cases}$$

$$k_*(y_i^s/2^{i+2}) = \begin{cases} 0 & (s \geqq 3 \text{ or } i \geqq 2), \\ v_2/2v_1 & (s=1, i=1), \\ 1/2^{i+2}y_i^{-s} & (s<0, i \geqq 1). \end{cases}$$

These imply the lemma. q.e.d.

LEMMA 4.5. (a) *The image of* $k_*\colon \mathrm{Ext}^1 M_0^1 \to \mathrm{Ext}^1 N_0^2$ *in* (4.2) *is the direct sum of* (1) $Z/2$ *generated by* $\tilde{\rho}_1/2v_1^4$, $v_2t_1/2v_1$, $t_1/2v_1^{2t+1}$ *and* $\rho_1/2v_1^{2t-1}$ *for* $t \geqq 0$, *and* (2) $Q/Z_{(2)}$ *generated by* $k_*(z_j)$ *for* $j \geqq 1$.

(b) *The kernel* A_1 *of* $k_*\colon \mathrm{Ext}^1 M_0^1 \to \mathrm{Ext}^1 N_0^2$ *in* (4.2) *is the direct sum of* $Z/2$ *generated by* $v_1^{2t-1}t_1/2$ *and* $v_1^{2t+3}\rho_1/2$ *for* $t \geqq 1$.

PROOF. For the generators given in (4.3.b), we see that

$$k_*(\rho_1/2) = \tilde{\rho}_1/2v_1^4, \quad k_*(v_1^{2t-1}t_1/2) = \begin{cases} 0 & (t \geqq 1), \\ t_1/2v_1^{-2t+1} & (t \leqq 0), \end{cases} \tag{4.6}$$

$$k_*(v_1^{2t+1}\rho_1/2) = \begin{cases} 0 & (t \geqq 2), \\ v_2t_1/2v_1 & (t=1), \\ \rho_1/2v_1^{-2t-1} & (t \leqq 0). \end{cases}$$

Since $\mathrm{Ext}^1 N_0^1 \approx \mathrm{Ext}^2 BP_*$ by Proposition 4.1 and $\mathrm{Ext}^2 BP_*$ is 3-connected by [6; Lemma 1.16], $\mathrm{Ext}^1 N_0^1$ is also 3-connected. Therefore the elements $\rho_1/2$, $v_1^{2t-1}t_1/2$ $(t \leqq 0)$, $v_1^{2t+1}\rho_1/2$ $(t \leqq 0)$ and z_j $(j \geqq 1)$ of degree less than 3 do not belong to the image of j_* in (4.2). Hence $\tilde{\rho}_1/2v_1^4$, $t_1/2v_1^{-2t+1}$ $(t \leqq 0)$, $\rho_1/2v_1^{-2t-1}$ $(t \leqq 0)$ and $k_*(z_j)$ $(j \geqq 1)$ are not zero by the exact sequence (4.2).

Next consider the exact sequence

$$\mathrm{Ext}^0 M_0^2 \xrightarrow{k} \mathrm{Ext}^0 N_0^3 \xrightarrow{\delta'} \mathrm{Ext}^1 N_0^2 \tag{4.7}$$

associated to the exact sequence (3.1) for $s=2$ and $n=0$. We see that $\delta'(v_3/2v_1v_2) = \{v_2t_1/2v_1\}$ by (2.2) and (2.3). Further Theorem 3.3 implies that $v_3/2v_1v_2$ does not belong to the image of k in (4.7). Therefore $v_2t_1/2v_1 \neq 0$. Hence (a) is proved.

(b) follows immediately from (a) and (4.6). q.e.d.

PROOF OF THEOREM 1.1. By the exact sequence (4.2) and the isomorphism $\delta_0\colon \mathrm{Ext}^1 N_0^1 \approx \mathrm{Ext}^2 BP_*$ in Proposition 4.1, we have the short exact sequence

$$0 \longrightarrow \mathrm{Ext}^0 N_0^2/A_0 \xrightarrow{\delta_0\delta'} \mathrm{Ext}^2 BP_* \xrightarrow{j_*\delta_0^{-1}} A_1 \longrightarrow 0 \tag{4.8}$$

where $A_0 = \mathrm{Im}\{k_*\colon \mathrm{Ext}^0 M_0^1 \to \mathrm{Ext}^0 N_0^2\}$ and $A_1 = \mathrm{Ker}\{k_*\colon \mathrm{Ext}^1 M_0^1 \to \mathrm{Ext}^1 N_0^2\}$. By Lemma 4.5(b), we can define naturally a right inverse $h\colon A_1 \to \mathrm{Ext}^2 BP_*$ of $j_*\delta_0^{-1}$ by $h(x) = \delta_0(x)$ for $x \in A_1$ $(j_*^{-1}(x) = x \in \mathrm{Ext}^1 N_0^1)$. Thus (4.8) is a split exact sequence.

By the definition of the universal Greek letter map η (cf. [6; (3.6)]), we see that $\eta(x)=\delta_0\delta'(x)$ if $x\in \mathrm{Ext}^0 N_0^2$. Thus we deduce from (1.3.1–2) that the $\delta_0\delta'$-image of $\mathrm{Ext}^0 N_0^2/A_0$ given by Theorem 3.4 and Lemma 4.4 is the submodule generated by the elements in (1.7.1–2). For $\alpha\in \mathrm{Ext}^1 BP_*$ given in (1.2), we see by (2.4) that

$$\alpha_m \equiv v_1^{m-1}t_1, \quad \alpha_{2^i m/i+2} \equiv v_1^{2^i m}\rho_1 \quad \mathrm{mod}(2) \tag{4.9}$$

since $\eta(x)=\delta_0(x)$ if $x\in \mathrm{Ext}^0 N_0^1$ (cf. [6; (3.6)]). Hence we obtain

$$\alpha_1\alpha_m = \delta_0(v_1^m t_1/2), \quad \alpha_1\alpha_{2^i m/i+2} = \delta_0(v_1^{2^i m+1}\rho_1/2) \quad \text{in} \quad \mathrm{Ext}^2 BP_*.$$

These and Lemma 4.5(b) give the elements in (1.6). Thus we obtain the theorem.
q. e. d.

§ 5. Some relations in $\mathrm{Ext}^3 BP_*$

By the exact sequence (4.2) and Lemma 4.5(a), the sequence

$$0 \longrightarrow Z/2 \xrightarrow{k_*} \mathrm{Ext}^1 N_0^2 \xrightarrow{\delta'} \mathrm{Ext}^2 N_0^1 \tag{5.1}$$

is exact in positive degree, where $Z/2$ is generated by $v_2t_1/2v_1$.

In this section we study some relations in $\mathrm{Ext}^3 BP_*$ by the exact sequence (5.1) whose last term is isomorphic to $\mathrm{Ext}^3 BP_*$ by Proposition 4.1.

THEOREM 5.2 (cf. [9; Th. 7.9]). *Let $n\geq 0$, $s\geq 1$ and $j\geq 1$ be integers such that s is odd, $j\leq a_n$, and especially $j\leq 2^n$ if $s=1$. Then*

(i) $\alpha_1\beta_{2^n s/j,1}\neq 0$ *in* $\mathrm{Ext}^3 BP_*$ *if and only if*

(a) $n=0, j=1$ *and* $s\equiv -1 \mod 4$, (b) $n=1$ *and* $j=1$, *or*
(c) $n\geq 2$ *and either j is odd or* $j-1>3\cdot 2^{n-2}$.

(ii) $\alpha_{4/4}\beta_{2^n s/j,1}\neq 0$ *in* $\mathrm{Ext}^3 BP_*$ *if and only if*

(a) $n=0, 1$ *and* $j=1$, (b) $n\geq 2$ *and either* $j=4$ *or j is odd* ≥ 3, *or*
(c) $n\geq 3$, *and j is even* $>\max\{6, 3\cdot 2^{n-i-3}+4\}$, *where*
$j-4\equiv 2^i \mod 2^{i+1}$.

(iii) *In* $\mathrm{Ext}^3 BP_*$,

$$\alpha_t\beta_{2^n s/j,1} = \alpha_1\beta_{2^n s/j-t+1,1}, \qquad \alpha_{2^i t/i+2}\beta_{2^n s/j,1} = \alpha_{4/4}\beta_{2^n s/j-2^i t+4,1}$$

for odd $t\geq 1$ and $i\geq 0$ with $2^i t\geq 4$.

PROOF. By (4.9) and (1.2.1), we obtain

$$\alpha_t\beta_{2^n s/j,1} = \delta_0\delta'(x_n^s t_1/2v_1^{j-t+1}), \qquad \alpha_{2^i t/i+2}\beta_{2^n s/j,1} = \delta_0\delta'(x_n^s\rho_1/2v_1^{j-2^i t}) \tag{5.3}$$

where δ_0 and δ' are the boundary homomorphisms in Proposition 4.1 for $t=2$ and in (5.1), respectively. These imply (iii) immediately.

For n, s and j in the theorem, (5.3) implies that the elements $\alpha_1\beta_{2^n s/j,1}$ and $\alpha_{4/4}\beta_{2^n s/j,1}$ are the $\delta_0\delta'$-images of the elements

$$b(n, s, j) = x_n^s t_1/2v_1^j \text{ and } c(n, s, j) = x_n^s\rho_1/2v_1^{j-4}\ (= x_n^s\tilde{\rho}_1/2v_1^j \text{ by (2.7)})$$

in $\Omega^1 N_0^2$, respectively. By Lemmas 2.6, 2.10 and (3.19), we have the following in $\Omega^1 M_0^2$:

(5.4) $b(0, s, 1) = d(x_1^{(s+1)/2}/2v_1^3)$ for $s \geqq 3$ with $s \equiv 1 \mod 4$,

$b(1, s, 2) = d(x_1^s/2^2 v_1)$,

$b(n, s, j) = d(x_n^s/2^2 v_1^{j-1}) + b'$, $b' = (v_2^{k-l}\zeta_2^l/2v_1^{j-1-3l} + \cdots)$,

$(k=2^n s,\ l=2^{n-2})$, for $n \geqq 2$ and even j;

(5.5) $c(n, s, 2) = d(v_2^{2^n s+1}/2^2 v_1)$ for $n \geqq 1$,

$c(n, s, 1) = d(x_1^{2^{n-1}s+1}/2v_1^3)$ for $n \geqq 2$,

$c(n, s, 6) = d(x_n^s/2^4 y_1)$ for $n = 2, 3$,

$c(n, s, j) = d(x_n^s/2^{i+3} y_i^m) + c'$, $c' = (v_2^{k-l}\zeta_2^l/2v_1^{j-4-3l} + \cdots)$,

$(m=(j-4)/2^i$ where $j-4=2^i \mod 2^{i+1}$, and $k=2^n s$, $l=2^{n-i-3})$

for $n-i \geqq 3$, and $6 \leqq j \leqq a_{n-i-1}+4$.

In the above (5.4) and (5.5), $\cdots$ denotes an element of $\Omega^1 N_0^2$ killed by a lower power of v_1 than those shown.

Assume that (a)–(c) in (i) do not hold. Then we see that $b'=0$ in (5.4) and hence $b(n, s, j)=0$ except $b(0, 1, 1)$. Further, $\delta_0\delta'(b(0, 1, 1))=0$ by the exact sequence (5.1). Hence the sufficiency in (i) is proved. The sufficiency in (ii) is proved similarly by (5.5) without exception.

To prove the necessity, consider the diagram

$$\begin{tikzcd} & \mathrm{Ext}^1 M_1^1 & \mathrm{Ext}^1 N_0^2 \arrow[d, "j_*"] \\ \mathrm{Ext}^0 M_0^2 \arrow[r, "\delta"] & \mathrm{Ext}^1 M_1^1 \arrow[u, "v_1"] \arrow[r, "i_*"] & \mathrm{Ext}^1 M_0^2 \end{tikzcd}$$

Assume that one of (a)–(c) in (i) holds, and consider $\bar{b}=b(n, s, j)$ (resp. b' in (5.4)) in $\Omega^1 M_0^2$ and $t=j-1$ (resp. $j-2-3\cdot 2^{n-2}$) if either $n=0$, 1 or j is odd (resp. $n\geqq 2$ and j is even). Then we see that $v_1^t i_*^{-1}(\bar{b})$ ($\in \mathrm{Ker}\, v_1$) does not belong to $v_1^t \mathrm{Im}\,\delta$ by (3.20.1). Thus $i_*^{-1}(\bar{b}) \notin \mathrm{Im}\,\delta$ and $\bar{b}$ is not zero in $\mathrm{Ext}^1 M_0^2$. Therefore $b(n, s, j)$ is not zero in $\mathrm{Ext}^1 N_0^2$ by (5.4), and is not also equal to $v_2 t_1/2v_1$ clearly. Thus its $\delta_0\delta'$-image $\alpha_1\beta_{2^n s/j,1}$ is not zero by the exact sequence (5.1) and Proposition 4.1.

The necessity in (ii) can be proved similarly. q.e.d.

§ 6. Some elements in $\pi_*(S)$

In this section, we prove Theorem 1.8 by showing the following

THEOREM 6.1. *The elements $\beta_{8t/3,1} \in \mathrm{Ext}^{2,48t-6}BP_*$ for $t \geqq 1$ in Theorem 1.5 are all nontrivial permanent cycles converging to the homotopy elements in $\pi_*(S)$ of order 2.*

PROOF. Let $M(2)$ be the mod 2 Moore spectrum and $S \xrightarrow{2} S \xrightarrow{\iota} M(2) \xrightarrow{\pi} \Sigma S$ the cofiber sequence. Recall ([9; Th. 5.13]) that $v_1 \in \mathrm{Ext}^{0,2}BP_*/(2)$ is a permanent cycle converging to the element $\alpha \in \pi_*(M(2))$ such that $\iota\alpha_1^2 = 2\alpha$ and $\pi\alpha = \alpha_1$ (α_1 is the generator of $\pi_1(S)$).

Also, recall the following result due to M. Mahowald:

(6.2) (cf. [9; Cor. 7.6]) *The elements $\beta_{8t/4,2} \in \mathrm{Ext}^{2,48t-8}BP_*$ for all $t>0$ are nontrivial permanent cycles and the corresponding homotopy elements have order 4.*

Let $\xi \in \pi_*(S)$ be an element detected by $\beta_{8t/4,2}$. Then $2\xi \in \pi_*(S)$ has order 2 by (6.2), and it extends to a map $(2\xi)^{\sim}: M(2) \to S$; and we see easily that $\xi' = (2\xi)^{\sim}\alpha \in \pi_*(S)$ is detected by $\beta_{8t/3,1}$ from the Geometric Boundary Theorem ([3; Th. 1.7]). Clearly $2\xi' = 2\xi\alpha_1^2 = 0$. On the other hand, $\beta_{8t/3,1}$ ($t \geqq 1$) is nontrivial in the E_2-term by Theorem 1.5, and there is no Novikov differential hitting it since 0-line is trivial in positive dimension. Thus ξ' has order 2. q.e.d.

PROOF OF THEOREM 1.8. By Theorem 5.1, $\alpha_{4/4}\beta_{8t/3,1} \neq 0$ and $\alpha_{4/4}\beta_{8t/4,1} \neq 0$. Also $2\alpha_{4/4}\beta_{8t/4,2} = \alpha_{4/4}\beta_{8t/4,1}$ by (3.11). It is well known that α_1 and $\alpha_{4/4}$ are nontrivial permanent cycles (cf. [9; Th. 5.8]). Therefore by Theorem 6.1 and (6.2), we see that $\alpha_1\beta_{8t/3,1}$, $\alpha_{4/4}\beta_{8t/3,1}$, $\alpha_{4/4}\beta_{8t/4,1}$ and $\alpha_{4/4}\beta_{8t/4,2}$ are also permanent cycles. By sparseness, a Novikov differential hitting any of them would have to originate on 0-line which is trivial in positive dimensions. Hence they are nontrivial permanent cycles, and the theorem is proved since α_1, $\beta_{8t/3,1}$ and $\beta_{8t/4,2}$ have orders 2, 2 and 4, respectively. q.e.d.

Finally, we notice the following remark which follows from [9; Th. 7.9].

REMARK 6.3. *$\beta_{8t/2,1} \in \mathrm{Ext}^{2,48t-4}BP_*$ for $t \geqq 1$ in Theorem 1.5 are also nontrivial permanent cycles and the corresponding homotopy elements have order 4.*

PROOF. By continuing the proof of Theorem 6.1, we see similarly that ξ' extends to a map $(\xi')^{\sim}: M(2) \to S$ and that $\xi'' = (\xi')^{\sim}\alpha \in \pi_*(S)$ is detected by $\beta_{8t/2,1}$. Furthermore, $2\xi'' = \xi'\alpha_1^2$ is detected by $\alpha_1^2\beta_{8t/3,1} \in \mathrm{Ext}^4BP_*$ which is shown to be nontrivial in [9; Th. 7.9]. Thus we see similarly that ξ'' has order 4. q. e. d.

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Department of Mathematics,
Faculty of Science,
Hiroshima University