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Subgroup $(SU(2)\times Spin(12))/Z_2$ of compact simple Lie group E_7 and non-compact simple Lie group $E_{7,\sigma}$ of type $E_{7(-5)}$

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Introduction

It is known that there exist four simple Lie groups of type E_7 up to local isomorphism, one of them is compact and the others are non-compact. We have shown that in [3], [5] the group

$$\begin{aligned}E_7 &= \{\alpha\in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{P}^{\boldsymbol{C}}, \mathfrak{P}^{\boldsymbol{C}})\,|\,\alpha(P\times Q)\alpha^{-1} = \alpha P\times\alpha Q,\ \langle\alpha P, \alpha Q\rangle = \langle P, Q\rangle\}\\ &= \{\alpha\in \mathrm{Iso}_{\boldsymbol{C}}(P^{\boldsymbol{C}}, \mathfrak{P}^{\boldsymbol{C}})\,|\,\alpha\mathfrak{M}^{\boldsymbol{C}} = \mathfrak{M}^{\boldsymbol{C}},\ \{\alpha P, \alpha Q\} = \{P, Q\},\ \langle\alpha P, \alpha Q\rangle = \langle P, Q\rangle\}\end{aligned}$$

is a simply connected compact simple Lie group of type E_7 and in [4], [5] the group

$$\begin{aligned}E_{7,\iota} &= \{\alpha\in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{P}^{\boldsymbol{C}}, \mathfrak{P}^{\boldsymbol{C}})\,|\,\alpha(P\times Q)\alpha^{-1} = \alpha P\times\alpha Q,\ \langle\alpha P, \alpha Q\rangle_\iota = \langle P, Q\rangle_\iota\}\\ &= \{\alpha\in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{P}^{\boldsymbol{C}}, \mathfrak{P}^{\boldsymbol{C}})\,|\,\alpha\mathfrak{M}^{\boldsymbol{C}} = \mathfrak{M}^{\boldsymbol{C}},\ \{\alpha P, \alpha Q\} = \{P, Q\},\ \langle\alpha P, \alpha Q\rangle_\iota = \langle P, Q\rangle_\iota\}\end{aligned}$$

is a connected non-compact simple Lie group of type $E_{7(-25)}$ and its polar decomposition is given by

$$E_{7,\iota} \simeq (U(1)\times E_6)/\boldsymbol{Z}_3 \times \boldsymbol{R}^{54}.$$

In this paper, we show that the group

$$\begin{aligned}E_{7,\sigma} &= \{\alpha\in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{P}^{\boldsymbol{C}}, \mathfrak{P}^{\boldsymbol{C}})\,|\,\alpha(P\times Q)\alpha^{-1} = \alpha P\times\alpha Q,\ \langle\alpha P, \alpha Q\rangle_\sigma = \langle P, Q\rangle_\sigma\}\\ &= \{\alpha\in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{P}^{\boldsymbol{C}}, \mathfrak{P}^{\boldsymbol{C}})\,|\,\alpha\mathfrak{M}^{\boldsymbol{C}} = \mathfrak{M}^{\boldsymbol{C}},\ \{\alpha P, \alpha Q\} = \{P, Q\},\ \langle\alpha P, \alpha Q\rangle_\sigma = \langle P, Q\rangle_\sigma\}\end{aligned}$$

is a connected non-compact simple Lie group of type $E_{7(-5)}$ with the center $z(E_{7,\sigma})=\{1, -1\}$. The polar decomposition of the group $E_{7,\sigma}$ is given by

$$E_{7,\sigma} \simeq (SU(2)\times Spin\,(12))/\boldsymbol{Z}_2 \times \boldsymbol{R}^{64}.$$

To give this decomposition, we find subgroups

$$SU(2), \qquad Spin(12), \qquad (SU(2)\times Spin(12))/\boldsymbol{Z}_2$$

in the group E_7 and the group $E_{7,\sigma}$ explicitly.

1. Preliminaries

1.1. Cayley algebra $\mathfrak{C}$ and exceptional Jordan algebra $\mathfrak{J}^C$

Let $\mathfrak{C}$ denote the Cayley algebra over the field $\boldsymbol{R}$ of real numbers and $\mathfrak{C}^C$ its complexification with respect to the field $\boldsymbol{C}$ of complex numbers. Let $\mathfrak{J}^C$ denote the Jordan algebra consisting of all 3×3 Hermitian matrices X with entries in $\mathfrak{C}^C$:

$$X = \begin{bmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{bmatrix}, \qquad \xi_i \in \boldsymbol{C},\ x_i \in \mathfrak{C}^C,$$

with repect to the multiplication $X\circ Y=(XY+YX)/2$. In $\mathfrak{J}^C$, the symmetric inner product (X, Y), the positive definite Hermitian inner product $\langle X, Y\rangle$ and the crossed product $X\times Y$ are defined respectively by

$$(X, Y) = \mathrm{tr}(X\circ Y), \qquad \langle X, Y\rangle = (\tau X, Y) = (\overline{X}, Y),$$

$$X \times Y = (2X\circ Y - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + (\mathrm{tr}(X)\mathrm{tr}(Y) - (X, Y))E)/2$$

where $\tau\colon \mathfrak{J}^C \to \mathfrak{J}^C$ is the conjugation relative to the field $\boldsymbol{C}$ (τX is also denoted by $\overline{X}$) and E the 3×3 unit matrix. We use the following notations in $\mathfrak{J}^C$:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1.2. Compact Lie group E_6 and its Lie algebra $\mathfrak{e}_6$ ([1], [7])

A simply connected compact simple Lie group of type E_6 is given by

$$\begin{aligned} E_6 &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{J}^C, \mathfrak{J}^C) \mid (\alpha X, \alpha X \times \alpha X) = (X, X\times X),\ \langle \alpha X, \alpha Y\rangle = \langle X, Y\rangle\} \\ &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{J}^C, \mathfrak{J}^C) \mid \tau\alpha\tau(X\times Y) = \alpha X \times \alpha Y,\ \langle \alpha X, \alpha Y\rangle = \langle X, Y\rangle\} \end{aligned}$$

and its Lie algebra is

$$\mathfrak{e}_6 = \{\phi \in \mathrm{Hom}_{\boldsymbol{C}}(\mathfrak{J}^C, \mathfrak{J}^C) \mid (\phi X, X\times X) = 0,\ \langle \phi X, Y\rangle = -\langle X, \phi Y\rangle\}.$$

The complexification $\mathfrak{e}_6^C$ of the Lie algebra $\mathfrak{e}_6$:

$$\mathfrak{e}_6^C = \{\phi \in \mathrm{Hom}_{\boldsymbol{C}}(\mathfrak{J}^C, \mathfrak{J}^C) \mid (\phi X, X\times X) = 0\}$$

is a simple Lie algebra over $\boldsymbol{C}$ of type E_6. For $A, B \in \mathfrak{J}^C$, $A \vee B \in \mathfrak{e}_6^C$ is defined by

$$(A \vee B)X = ((B, X)/2)A + ((A, B)/6)X - 2B \times (A \times X), \qquad X \in \mathfrak{J}^C.$$

1.3. Compact Lie group E_7 and its Lie algebra $\mathfrak{e}_7$ ([1], [3])

Let $\mathfrak{P}^C$ be a 56 dimensional vector space over $\boldsymbol{C}$ defined by

$$\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus \boldsymbol{C} \oplus \boldsymbol{C}.$$

An element $P = \begin{bmatrix} X \\ Y \\ \xi \\ \eta \end{bmatrix}$ of $\mathfrak{P}^C$ is often denoted by $P=(X, Y, \xi, \eta)$. In $\mathfrak{P}^C$, the positive definite Hermitian inner product $\langle P, Q\rangle$ and the skew-symmetric inner product $\{P, Q\}$ are defined respectively by

$$\langle P, Q\rangle = \langle X, Z\rangle + \langle Y, W\rangle + \bar{\xi}\zeta + \bar{\eta}\omega,$$
$$\{P, Q\} = (X, W) - (Z, Y) + \xi\omega - \zeta\eta$$

where $P=(X, Y, \xi, \eta)$, $Q=(Z, W, \zeta, \omega) \in \mathfrak{P}^C$.

For $\phi \in \mathfrak{e}_6^C$, $A, B \in \mathfrak{J}^C$ and $\nu \in \boldsymbol{C}$, we define a linear transformation $\Phi(\phi, A, B, \nu)$ of $\mathfrak{P}^C$ by

$$\Phi(\phi, A, B, \nu)\begin{bmatrix} X \\ Y \\ \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \phi-(\nu/3)1 & 2B & 0 & A \\ 2A & \phi'+(\nu/3)1 & B & 0 \\ 0 & A & \nu & 0 \\ B & 0 & 0 & -\nu \end{bmatrix}\begin{bmatrix} X \\ Y \\ \xi \\ \eta \end{bmatrix}$$
$$= \begin{bmatrix} \phi X-(\nu/3)X+2B\times Y+\eta A \\ 2A\times X+\phi' Y+(\nu/3)Y+\xi B \\ (A, Y)+\nu\xi \\ (B, X)-\nu\eta \end{bmatrix}$$

where $\phi' \in \mathfrak{e}_6^C$ denotes the skew-transpose of ϕ with respect to the inner product (X, Y): $(\phi X, Y)+(X, \phi' Y)=0$. Now, for $P=(X, Y, \xi, \eta)$, $Q=(Z, W, \zeta, \omega) \in \mathfrak{P}^C$, we define a linear transformation $P \times Q$ of $\mathfrak{P}^C$ by

$$P \times Q = \Phi(\phi, A, B, \nu), \quad \begin{cases} \phi = -(X \vee W + Z \vee Y)/2, \\ A = -(2Y \times W - \xi Z - \zeta X)/4, \\ B = (2X \times Z - \eta W - \omega Y)/4, \\ \nu = ((X, W) + (Z, Y) - 3(\xi\omega+\zeta\eta))/8. \end{cases}$$

And we define a submanifold $\mathfrak{M}^C$ of $\mathfrak{P}^C$, called a Freudenthal manifold, by

$$\mathfrak{M}^C = \{P \in \mathfrak{P}^C \mid P \times P = 0,\ P \neq 0\}$$
$$= \left\{ P = (X, Y, \xi, \eta) \in \mathfrak{P}^C \,\middle|\, \begin{array}{l} X \vee Y = 0,\ X \times X = \eta Y,\ Y \times Y = \xi X, \\ (X, Y) = 3\xi\eta,\ P \neq 0 \end{array} \right\}.$$

Now, as stated in the introduction, a simply connected compact simple Lie group of type E_7 is given by

$$E_7 = \{\alpha \in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{P}^C, \mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q,\ \langle \alpha P, \alpha Q\rangle = \langle P, Q\rangle\}$$
$$= \{\alpha \in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{P}^C, \mathfrak{P}^C) \mid \alpha \mathfrak{M}^C = \mathfrak{M}^C,\ \{\alpha P, \alpha Q\} = \{P, Q\},\ \langle \alpha P, \alpha Q\rangle = \langle P, Q\rangle\}$$

and its Lie algebra is

$$\mathfrak{e}_7 = \{\Phi(\phi, A, \nu) \in \mathrm{Hom}_{\boldsymbol{C}}(\mathfrak{P}^C, \mathfrak{P}^C) \mid \phi \in \mathfrak{e}_6,\ A \in \mathfrak{J}^C,\ \nu \in \boldsymbol{C},\ \bar{\nu} = -\nu\}$$

where $\Phi(\phi, A, \nu) = \Phi(\phi, A, -\bar{A}, \nu)$, so the action $\Phi(\phi, A, \nu)$ on $\mathfrak{P}^C$ is defined by

$$\Phi(\phi, A, \nu)\begin{bmatrix} X \\ Y \\ \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \phi X - (\nu/3)X - 2\bar{A} \times Y + \eta A \\ 2A \times X + \overline{\phi}\,\overline{Y} + (\nu/3)Y - \xi \bar{A} \\ (A, Y) + \nu\xi \\ -\langle A, X\rangle - \nu\eta \end{bmatrix}.$$

The Lie bracket $[\Phi_1, \Phi_2]$ in $\mathfrak{e}_7$ is given by

$$[\Phi(\phi_1, A_1, \nu_1), \Phi(\phi_2, A_2, \nu_2)] = \Phi(\phi, A, \nu),$$
$$\begin{cases} \phi = [\phi_1, \phi_2] - 2A_1 \vee \bar{A}_2 + 2A_2 \vee \bar{A}_1, \\ A = (\phi_1 + (2\nu_1/3)1)A_2 - (\phi_2 + (2\nu_2/3)1)A_1, \\ \nu = \langle A_1, A_2\rangle - \langle A_2, A_1\rangle. \end{cases}$$

2. Subgroups $(E_7)_\sigma$, $E_{\sigma,\kappa,\lambda}$, $E_{\sigma,\kappa,\lambda,1}$ of E_7 and their Lie algebras

We define linear transformations σ, κ_1 of $\mathfrak{J}^C$ respectively by

$$\sigma\begin{bmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{bmatrix} = \begin{bmatrix} \xi_1 & -x_3 & -\bar{x}_2 \\ -\bar{x}_3 & \xi_2 & x_1 \\ -x_2 & \bar{x}_1 & \xi_3 \end{bmatrix},$$
$$\kappa_1\begin{bmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{bmatrix} = \begin{bmatrix} \xi_1 & 0 & 0 \\ 0 & -\xi_2 & -x_1 \\ 0 & -\bar{x}_1 & -\xi_3 \end{bmatrix}$$

and then define linear transformations σ (denoted by the same notation as the above), κ, λ of $\mathfrak{P}^C$ respectively by

$$\sigma\begin{bmatrix} X \\ Y \\ \xi \\ \eta \end{bmatrix}=\begin{bmatrix} \sigma X \\ \sigma Y \\ \xi \\ \eta \end{bmatrix},\quad \kappa\begin{bmatrix} X \\ Y \\ \xi \\ \eta \end{bmatrix}=\begin{bmatrix} \overline{\kappa_1 Y} \\ \overline{\kappa_1 X} \\ \bar{\xi} \\ \bar{\eta} \end{bmatrix},\quad \lambda\begin{bmatrix} X \\ Y \\ \xi \\ \eta \end{bmatrix}=\begin{bmatrix} \eta E_1+2E_1\times Y \\ \xi E_1+2E_1\times X \\ (E_1, Y) \\ (E_1, X) \end{bmatrix}.$$

Now, we define subgroups $(E_7)_\sigma$, $E_{\sigma,\kappa,\lambda}$, $E_{\sigma,\kappa,\lambda,1}$ of the group E_7 by

$$\begin{aligned} (E_7)_\sigma &= \{\alpha\in E_7 \mid \sigma\alpha=\alpha\sigma\}, \\ E_{\sigma,\kappa,\lambda} &= \{\alpha\in (E_7)_\sigma \mid \kappa\alpha=\alpha\kappa,\ \lambda\alpha=\alpha\lambda\}, \\ E_{\sigma,\kappa,\lambda,1} &= \{\alpha\in E_{\sigma,\kappa,\lambda} \mid \alpha(E_1, E_1, 1, 1)=(E_1, E_1, 1, 1)\}. \end{aligned}$$

Our first aim is to show that these groups are isomorphic to the following groups respectively :

$$(SU(2)\times Spin(12))/\boldsymbol{Z}_2, \qquad Spin(12), \qquad Spin(11).$$

The Lie algebras of these groups are easily calculated as follows.

PROPOSITION 1. (1) *The Lie algebra $(\mathfrak{e}_7)_\sigma$ of the group $(E_7)_\sigma$ is*

$$(\mathfrak{e}_7)_\sigma=\{\Phi\in\mathfrak{e}_7 \mid \sigma\Phi=\Phi\sigma\}=\left\{ \Phi(\phi, A, \nu)\in\mathfrak{e}_7 \;\middle|\; \begin{array}{l} \phi\in\mathfrak{e}_6,\ \sigma\phi=\phi\sigma, \\ A\in\mathfrak{J}^C,\ \sigma A=A, \\ \nu\in\boldsymbol{C},\ \bar{\nu}=-\nu \end{array} \right\}.$$

(2) *The Lie algebra $\mathfrak{e}_{\sigma,\kappa,\lambda}$ of the group $E_{\sigma,\kappa,\lambda}$ is*

$$\begin{aligned} \mathfrak{e}_{\sigma,\kappa,\lambda} &= \{\Phi\in(\mathfrak{e}_7)_\sigma \mid \kappa\Phi=\Phi\kappa,\ \lambda\Phi=\Phi\lambda\} \\ &= \left\{ \Phi(\phi, A, \nu)\in\mathfrak{e}_7 \;\middle|\; \begin{array}{l} \phi\in\mathfrak{e}_6,\ \sigma\phi=\phi\sigma, \\ A\in\mathfrak{J}^C,\ \sigma A=A,\ (E_1, A)=0, \\ \nu=-3(\phi E_1, E_1)/2 \end{array} \right\}. \end{aligned}$$

(3) *The Lie algebra $\mathfrak{e}_{\sigma,\kappa,\lambda,1}$ of the group $E_{\sigma,\kappa,\lambda,1}$ is*

$$\begin{aligned} \mathfrak{e}_{\sigma,\kappa,\lambda,1} &= \{\Phi\in\mathfrak{e}_{\sigma,\kappa,\lambda} \mid \Phi(E_1, E_1, 1, 1)=0\} \\ &= \left\{ \Phi(\phi, A, 0)\in\mathfrak{e}_7 \;\middle|\; \begin{array}{l} \phi\in\mathfrak{e}_6,\ \phi E_1=0, \\ A\in\mathfrak{J}^C,\ 2E_1\times A=\bar{A} \end{array} \right\} \end{aligned}$$

$$(=\{\Phi\in\mathfrak{e}_7 \mid \Phi(E_1, E_1, 1, 1)=0\}=(\mathfrak{e}_7)_1 \qquad \text{(see § 10))}.$$

For $\nu\in\boldsymbol{C}$, $\bar{\nu}=-\nu$, we define a linear transformation $\phi(\nu)$ of $\mathfrak{J}^C$ by

$$\phi(v)\begin{bmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{bmatrix} = (v/3)\begin{bmatrix} 4\xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & -2\xi_2 & -2x_1 \\ x_2 & -2\bar{x}_1 & -2\xi_3 \end{bmatrix}.$$

Then $\phi(v)=2vE_1 \vee E_1 \in \mathfrak{e}_6$ and $\sigma\phi(v)=\phi(v)\sigma$. Further we define a Lie algebra $\mathfrak{a}_1$ by

$$\mathfrak{a}_1 = \{\Phi(\phi(v),\, aE_1,\, v) \in \mathfrak{e}_7 \mid a,\, v \in \boldsymbol{C},\, \bar{v} = -v\}.$$

This $\mathfrak{a}_1$ is a Lie subalgebra of $(\mathfrak{e}_7)_\sigma$ and isomorphic to the special unitary Lie algebra $\mathfrak{su}(2)=\{A \in M(2, \boldsymbol{C}) \mid A^* = -A,\ \mathrm{tr}(A)=0\}$ by the correspondence

$$\mathfrak{a}_1 \ni \Phi(\phi(v),\, aE_1,\, v) \longleftrightarrow \begin{bmatrix} v & a \\ -\bar{a} & -v \end{bmatrix} \in \mathfrak{su}(2).$$

PROPOSITION 2. *The Lie algebra $(\mathfrak{e}_7)_\sigma$ is the direct sum of the Lie subalgebras $\mathfrak{a}_1$ and $\mathfrak{e}_{\sigma,\kappa,\lambda}$ in $\mathfrak{e}_7$:*

$$(\mathfrak{e}_7)_\sigma = \mathfrak{a}_1 + \mathfrak{e}_{\sigma,\kappa,\lambda}.$$

PROOF. The correspondence

$$(\mathfrak{e}_7)_\sigma \in \Phi(\phi,\, A,\, v) \longleftrightarrow \Phi(\phi(v'),\, aE_1,\, v') + \Phi(\phi-\phi(v'),\, A-aE_1,\, v-v') \in \mathfrak{a}_1 + \mathfrak{e}_{\sigma,\kappa,\lambda},$$

where $v'=v/3+(E_1,\, \phi E_1)/2$, $a=(E_1,\, A)$, gives an isomorphism between them.

3. Spinor subgroup *Spin* (11) of E_7

We shall show that the group $E_{\sigma,\kappa,\lambda,1}$ is isomorphic to the spinor group *Spin* (11) (cf. Theorem 20). To show this, consider an 11 dimensional vector space W over $\boldsymbol{R}$ defined by

$$\begin{aligned} W &= \{P \in \mathfrak{P}^C \mid \sigma P = P,\ \kappa P = P,\ \lambda P = P,\ P \times (E_1, E_1, 1, 1) = 0\} \\ &= \left\{ (\eta E_1 + X,\ -\eta E_1 - \bar{X},\ -\eta,\ \eta) \;\middle|\; \begin{array}{l} \eta \in \boldsymbol{C},\ \bar{\eta} = -\eta, \\ X \in \mathfrak{J}^C,\ 2E_1 \times X = -\bar{X} \end{array} \right\} \\ &= \left\{ \left(\begin{bmatrix} \eta & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\bar{\xi} \end{bmatrix},\ \begin{bmatrix} -\eta & 0 & 0 \\ 0 & -\bar{\xi} & -x \\ 0 & -\bar{x} & \xi \end{bmatrix},\ -\eta,\ \eta \right) \;\middle|\; \begin{array}{l} \eta,\ \xi \in \boldsymbol{C},\ \bar{\eta} = -\eta, \\ x \in \mathfrak{C} \end{array} \right\} \end{aligned}$$

and let S^{10} be the unit sphere in W:

$$S^{10}=\{P\in W\,|\,\langle P,P\rangle=4\}$$

$$=\left\{\left(\begin{bmatrix}\eta & 0 & 0\\ 0 & \xi & x\\ 0 & \bar{x} & -\bar{\xi}\end{bmatrix}, \begin{bmatrix}-\eta & 0 & 0\\ 0 & -\bar{\xi} & -x\\ 0 & -\bar{x} & \xi\end{bmatrix}, -\eta, \eta\right) \,\middle|\, \begin{array}{l}\eta, \xi\in \boldsymbol{C}, \bar{\eta}=-\eta, x\in\mathfrak{C},\\ |\eta|^2+|\xi|^2+|x|^2=1\end{array}\right\}.$$

Remember that the spinor group ([7, Proposition 11])

$$Spin\,(10)=\{\alpha\in E_6\,|\,\sigma\alpha=\alpha\sigma,\ \alpha E_1=E_1\}=\{\alpha\in E_6\,|\,\alpha E_1=E_1\}$$

acts transitively on the 9 dimensional sphere S^9 ([7, Lemma 10])

$$S^9=\left\{\left(\begin{bmatrix}0 & 0 & 0\\ 0 & \xi & x\\ 0 & \bar{x} & -\bar{\xi}\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\ 0 & -\bar{\xi} & -x\\ 0 & -\bar{x} & \xi\end{bmatrix}, 0, 0\right) \,\middle|\, \begin{array}{l}\xi\in \boldsymbol{C}, x\in\mathfrak{C},\\ |\xi|^2+|x|^2=1\end{array}\right\}.$$

LEMMA 3. *For $a\in\boldsymbol{C}$, the linear transformation $\alpha_i(a)$ $(i=1,2,3)$ of $\mathfrak{P}^C$ defined by*

$$\alpha_i(a)=\begin{bmatrix}1+(\cos|a|-1)p_i & (2a/|a|)\sin|a|E_i & 0 & -(\bar{a}/|a|)\sin|a|E_i\\ -(2\bar{a}/|a|)\sin|a|E_i & 1+(\cos|a|-1)p_i & (a/|a|)\sin|a|E_i & 0\\ 0 & -(\bar{a}/|a|)\sin|a|E_i & \cos|a| & 0\\ (a/|a|)\sin|a|E_i & 0 & 0 & \cos|a|\end{bmatrix}$$

(if $a=0$, then $(a/|a|)\sin|a|$ means 0) belongs to the group $(E_7)_\sigma$, where the mapping p_i: $\mathfrak{J}^C\to\mathfrak{J}^C$ is

$$p_i\begin{bmatrix}\xi_1 & x_3 & \bar{x}_2\\ \bar{x}_3 & \xi_2 & x_1\\ x_2 & \bar{x}_1 & \xi_3\end{bmatrix}=\begin{bmatrix}\xi_1 & \delta_{3i}x_3 & \delta_{2i}\bar{x}_2\\ \delta_{3i}\bar{x}_3 & \xi_2 & \delta_{1i}x_1\\ \delta_{2i}x_2 & \delta_{1i}\bar{x}_1 & \xi_3\end{bmatrix}\quad (i=1,2,3),$$

and the action of $\alpha_i(a)$ on $\mathfrak{P}^C$ is defined as similar to that of $\Phi(\phi, A, B, \nu)$ in § 1.3. Furthermore, for $a\in\boldsymbol{C}$, $\alpha_{23}(a)=\alpha_2(a)\alpha_3(\bar{a})$ belongs to the group $E_{\sigma,\kappa,\lambda,1}$.

PROOF. For $\Phi(0,-\bar{a}E_i,0)\in(\mathfrak{e}_7)_\sigma$, we have $\alpha_i(a)=\exp\Phi(0,-\bar{a}E_i,0)\in(\mathfrak{e}_7)_\sigma$, $i=1,2,3$. For $\Phi(0,-\bar{a}E_2-aE_3,0)\in\mathfrak{e}_{\sigma,\kappa,\lambda,1}$, we have

$$\begin{aligned}\alpha_{23}(a)&=\alpha_2(a)\alpha_3(\bar{a})=\exp\Phi(0,-\bar{a}E_2,0)\exp\Phi(0,-aE_3,0)\qquad\text{(cf. [3, Lemma 7])}\\ &=\exp(\Phi(0,-\bar{a}E_2,0)+\Phi(0,-aE_3,0))\\ &\qquad\qquad\text{(because } \Phi(0,-\bar{a}E_2,0)\text{ and }\Phi(0,-aE_3,0)\text{ are commutative)}\\ &=\exp\Phi(0,-\bar{a}E_2-aE_3,0)\in E_{\sigma,\kappa,\lambda,1}.\end{aligned}$$

LEMMA 4. *$\alpha \in E_{\sigma,\kappa,\lambda,1}$ satisfies $\alpha(E_1, -E_1, -1, 1)=(E_1, -E_1, -1, 1)$ if and only if $\alpha(0, 0, 1, 0)=(0, 0, 1, 0)$. In particular, we have the following isomorphism:*

$$\{\alpha \in E_{\sigma,\kappa,\lambda,1} \mid \alpha(E_1, -E_1, -1, 1) = (E_1, -E_1, -1, 1)\} = Spin\,(10)\,.$$

PROOF. Suppose that $\alpha \in E_7$ satisfies $\alpha(E_1, E_1, 1, 1)=(E_1, E_1, 1, 1)$ and $\alpha(E_1, -E_1, -1, 1)=(E_1, -E_1, -1, 1)$. Put $\alpha(0, 0, 1, 0)=(X, Y, \xi, \eta)$. Then $\langle\alpha(E_1, E_1, 1, 1), \alpha(0, 0, 1, 0)\rangle=1$, $\langle\alpha(E_1, -E_1, -1, 1), \alpha(0, 0, 1, 0)\rangle=-1$ imply $(E_1, X)+(E_1, Y)+\xi+\eta=1$, $-(E_1, X)+(E_1, Y)+\xi-\eta=1$ respectively. Furthermore $\{\alpha(E_1, E_1, 1, 1), \alpha(0, 0, 1, 0)\}=-1$, $\{\alpha(E_1, -E_1, -1, 1), \alpha(0, 0, 1, 0)\}=-1$ imply $(E_1, Y)-(E_1, X)+\eta-\xi=-1$, $(E_1, Y)+(E_1, X)-\eta-\xi=-1$ respectively. Therefore we have

$$\xi = 1, \qquad (E_1, X) = (E_1, Y) = \eta = 0.$$

Finally $\langle\alpha(0, 0, 1, 0), \alpha(0, 0, 1, 0)\rangle=1$ implies $\langle X, X\rangle+\langle Y, Y\rangle+1+0=1$, hence $X=Y=0$. Thus we have $\alpha(0, 0, 1, 0)=(0, 0, 1, 0)$. The proof of the converse is similar. Since we have the identification

$$E_6 = \{\alpha \in E_7 \mid \alpha(0, 0, 1, 0) = (0, 0, 1, 0)\} \quad ([13, \text{Proposition } 2])$$

and $(E_1, 0, 0, 0)=((E_1, E_1, 1, 1)+(E_1, -E_1, -1, 1)-2(0, 0, 0, 1))/2$ (see [3, Lemma 1]), we have

$$\begin{aligned}
&\{\alpha \in E_{\sigma,\kappa,\lambda,1} \mid \alpha(E_1, -E_1, -1, 1) = (E_1, -E_1, -1, 1)\} \\
&\quad = \{\alpha \in E_7 \mid \alpha(0, 0, 1, 0) = (0, 0, 1, 0),\ \alpha(E_1, 0, 0, 0) = (E_1, 0, 0, 0)\} \\
&\quad = \{\alpha \in E_6 \mid \alpha E_1 = E_1\} = Spin\,(10).
\end{aligned}$$

LEMMA 5. *The group $E_{\sigma,\kappa,\lambda,1}$ acts transitively on S^{10} and the isotropy subgroup of $E_{\sigma,\kappa,\lambda,1}$ at $i(E_1, -E_1, -1, 1)$ is $Spin\,(10)$. Therefore the homogeneous space $E_{\sigma,\kappa,\lambda,1}/Spin\,(10)$ is homeomorphic to S^{10}:*

$$E_{\sigma,\kappa,\lambda,1}/Spin\,(10) \simeq S^{10}.$$

In particular, the group $E_{\sigma,\kappa,\lambda,1}$ is simply connected.

PROOF. Obviously the group $E_{\sigma,\kappa,\lambda,1}$ acts on S^{10}. In order to prove that $E_{\sigma,\kappa,\lambda,1}$ acts transitively on S^{10}, it suffices to show that any element P of S^{10} can be transformed in $i(E_1, -E_1, -1, 1) \in S^{10}$ by a certain element α of $E_{\sigma,\kappa,\lambda,1}$. Now, for a given element $P=\left(\begin{bmatrix} \eta & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\bar{\xi} \end{bmatrix}, \begin{bmatrix} -\eta & 0 & 0 \\ 0 & -\bar{\xi} & -x \\ 0 & -\bar{x} & \xi \end{bmatrix}, -\eta, \eta\right) \in S^{10}$, choose $a \in \boldsymbol{R}$, $\pi/4 \geqq a \geqq 0$, such that

$$\tan 2a = 2\eta/(\bar{\xi}-\xi)$$

and operate $\alpha_{23}(a) \in E_{\sigma,\kappa,\lambda,1}$ of Lemma 3 on P. Then the η-part of $\alpha_{23}(a)P$ becomes $((\xi-\bar{\xi})/2)\sin 2a+\eta\cos 2a=0$. Hence

$$\alpha_{23}(a)P \in S^9.$$

Since the group $Spin\,(10) \subset E_{\sigma,\kappa,\lambda,1}$ acts transitively on S^9, there exists $\beta \in Spin\,(10)$ such that

$$\beta\alpha_{23}(a)P = i(E_2+E_3,\ E_2+E_3,\ 0,\ 0).$$

Again operate $\alpha_{23}(\pi/4)$ of Lemma 3 on it. Then we have

$$\alpha_{23}(\pi/4)\beta\alpha_{23}(a)P = i(E_1,\ -E_1,\ -1,\ 1).$$

This proves the transitivity of $E_{\sigma,\lambda,\lambda,1}$. On the other hand, Lemma 4 shows that the isotropy subgroup of $E_{\sigma,\kappa,\lambda,1}$ at $i(E_1,\ -E_1,\ -1,\ 1)$ is $Spin\,(10)$. Thus we have the required homeomorphism $E_{\sigma,\kappa,\lambda,1}/Spin\,(10) \simeq S^{10}$.

REMARK. The transitivities in Lemma 5 and the following Lemma 8 are easily obtained by another way. In fact, since the compact Lie group $E_{\sigma,\kappa,\lambda,1}$ acts on S^{10}, an orbit $E_{\sigma,\kappa,\lambda,1}i(E_1,\ -E_1,\ -1,\ 1)$ $(\simeq E_{\sigma,\kappa,\lambda,1}/Spin\,(10))$ is $55-45=10$ dimensional compact submanifold of S^{10}, hence it must coincide with S^{10}: $E_{\sigma,\kappa,\lambda,1}/Spin\,(10) \simeq S^{10}$. However, here, we gave their elementary concrete proofs.

THEOREM 6 (cf. Theorem 20). *The subgroup $E_{\sigma,\kappa,\lambda,1}$ of E_7 is isomorphic to the spinor group Spin* (11):

$$E_{\sigma,\kappa,\lambda,1} \cong Spin\,(11).$$

PROOF. Let $SO\,(11) = SO\,(W) = \{\alpha' \in \mathrm{Iso}_{\boldsymbol{R}}(W,\ W) \mid \langle\alpha'P,\ \alpha'Q\rangle = \langle P,\ Q\rangle,\ \det\alpha' = 1\}$ be the rotation group in W. For each $\alpha \in E_{\sigma,\kappa,\lambda,1}$, the restriction $\alpha' = \alpha \mid W$ obviously belongs to $O\,(11) = O\,(W) = \{\alpha' \in \mathrm{Iso}_{\boldsymbol{R}}(W,\ W) \mid \langle\alpha'P,\ \alpha'Q\rangle = \langle P,\ Q\rangle\}$. Hence we can define a homomorphism $p\colon E_{\sigma,\kappa,\lambda,1} \to O\,(11)$ by $p(\alpha) = \alpha'$. Since $E_{\sigma,\kappa,\lambda,1}$ is connected (Lemma 5), p induces a homomorphism

$$p\colon E_{\sigma,\kappa,\lambda,1} \longrightarrow SO\,(11).$$

We shall show that p is onto. Recall that $p' = p \mid Spin\,(10)\colon Spin\,(10) \to SO\,(10) = SO\,(W')$ (where $W' = \{P \in W \mid P = (X,\ -\bar{X},\ 0,\ 0)\}$ is onto ([7, Proposition 11]). By using the five lemma, from the commutative diagram

$$\begin{array}{ccccccccc}
1 & \text{---} & Spin(10) & \longrightarrow & E_{\sigma,\kappa,\lambda,1} & \longrightarrow & S^{10} & \longrightarrow & * \\
\| & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \| & & \| \\
1 & \text{---} & SO(10) & \longrightarrow & SO(11) & \longrightarrow & S^{10} & \longrightarrow & *
\end{array}$$

we see that p is onto. Finally it is easy to see that $\operatorname{Ker} p=\{1, \sigma\}$. Therefore $E_{\sigma,\kappa,\lambda,1}$ is a universal covering group of $SO(11)$. Thus we have proved that $E_{\sigma,\kappa,\lambda,1}$ is isomorphic to the spinor group $Spin(11)$.

From now on, we identify the group $E_{\sigma,\kappa,\lambda,1}$ with the group $Spin(11)$.

4. Spinor subgroup $Spin(12)$ of E_7

We shall show the group $E_{\sigma,\kappa,\lambda}$ is isomorphic to the spinor group $Spin(12)$. To show this, consider a 12 dimensional vector space V over $\boldsymbol{R}$ defined by

$$\begin{aligned} V &= \{P \in \mathfrak{P}^C \mid \sigma P = P, \kappa P = P, \lambda P = P\} \\ &= \{(\eta E_1 + X, \bar{\eta} E_1 - \bar{X}, \bar{\eta}, \eta) \mid \eta \in \boldsymbol{C}, X \in \mathfrak{J}^C, 2E_1 \times X = -\bar{X}\} \\ &= \left\{\left(\begin{bmatrix} \eta & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\bar{\xi} \end{bmatrix}, \begin{bmatrix} \bar{\eta} & 0 & 0 \\ 0 & -\bar{\xi} & -x \\ 0 & -\bar{x} & \xi \end{bmatrix}, \bar{\eta}, \eta\right) \;\middle|\; \begin{matrix} \xi, \eta \in \boldsymbol{C}, \\ x \in \mathfrak{C} \end{matrix} \right\} \end{aligned}$$

and let S^{11} be the unit sphere in V:

$$S^{11} = \{P \in V \mid \langle P, P \rangle = 4\}.$$

LEMMA 7. *For $v \in \boldsymbol{C}$, $\bar{v} = -v$, a linear transformation $\alpha(v)$ of $\mathfrak{P}^C$ defined by*

$$\alpha(v)\left(\begin{bmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{bmatrix}, \begin{bmatrix} \eta_1 & y_3 & \bar{y}_2 \\ \bar{y}_3 & \eta_2 & y_1 \\ y_2 & \bar{y}_1 & \eta_3 \end{bmatrix}, \xi, \eta\right)$$
$$= \left(\begin{bmatrix} e^{2v}\xi_1 & e^{v}x_3 & e^{v}\bar{x}_2 \\ e^{v}\bar{x}_3 & \xi_2 & x_1 \\ e^{v}x_2 & \bar{x}_1 & \xi_3 \end{bmatrix}, \begin{bmatrix} e^{-2v}\eta_1 & e^{-v}y_3 & e^{-v}\bar{y}_2 \\ e^{-v}\bar{y}_3 & \eta_2 & y_1 \\ e^{-v}y_2 & \bar{y}_1 & \eta_3 \end{bmatrix}, e^{-2v}\xi, e^{2v}\eta\right)$$

belongs to the group $E_{\sigma,\kappa,\lambda}$.

PROOF. For $\phi(v) \in \mathfrak{e}_6$ defined in § 2, we have $\Phi(\phi(v), 0, -2v) \in \mathfrak{e}_{\sigma,\kappa,\lambda}$ and $\alpha(v) = \exp \Phi(\phi(v), 0, -2v)$, hence $\alpha(v) \in E_{\sigma,\kappa,\lambda}$.

LEMMA 8. *The group $E_{\sigma,\kappa,\lambda}$ acts transitively on S^{11} and the isotropy subgroup of $E_{\sigma,\kappa,\lambda}$ at $(E_1, E_1, 1, 1)$ is $Spin(11)$. Therefore the homogeneous space $E_{\sigma,\kappa,\lambda}/Spin(11)$ is homeomorphic to S^{11}:*

$$E_{\sigma,\kappa,\lambda}/Spin(11) \simeq S^{11}.$$

In particular, the group $E_{\sigma,\kappa,\lambda}$ is simply connected.

PROOF. Obviously the group $E_{\sigma,\kappa,\lambda}$ acts on S^{11}. In order to prove that $E_{\sigma,\kappa,\lambda}$ acts transitively on S^{11}, it suffices to show that any element P of S^{11} can be transformed in $(E_1, E_1, 1, 1) \in S^{11}$ by a certain element α of $E_{\sigma,\kappa,\lambda}$. Now, for a given element $P=\left(\begin{bmatrix} \eta & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\bar{\xi} \end{bmatrix}, \begin{bmatrix} \bar{\eta} & 0 & 0 \\ 0 & -\bar{\xi} & x \\ 0 & -\bar{x} & \xi \end{bmatrix}, \bar{\eta}, \eta\right) \in S^{11}$, choose $\nu \in \boldsymbol{C}$, $\bar{\nu}=-\nu$ such that

$$\nu = i(\pi/4-\theta/2)$$

where θ is the argument of η: $\eta=|\eta|e^{i\theta}$, and operate $\alpha(\nu)$ of Lemma 7 on P. Then the η-part of $\alpha(\nu)P$ becomes $e^{2\nu}\eta=e^{i\pi/2}e^{-i\theta}\eta=i|\eta|$. Hence

$$\alpha(\nu)P \in S^{10}.$$

Since the group $Spin\,(11)=E_{\sigma,\kappa,\lambda,1}$ acts transitively on S^{10}, there exists $\beta \in Spin\,(11)$ such that

$$\beta\alpha(\nu)P = i(E_1,\ -E_1,\ -1,\ 1).$$

Again operate $\alpha(-i\pi/4)$ of Lemma 7 on it. Then we have

$$\alpha(-i\pi/4)\beta\alpha(\nu)P = (E_1,\ E_1,\ 1,\ 1).$$

This shows the transitivity of $E_{\sigma,\kappa,\lambda}$. The isotropy subgroup of $E_{\sigma,\kappa,\lambda}$ at $(E_1, E_1, 1, 1)$ is $Spin\,(11)$ by the definition. Thus the proof of Lemma 8 is completed.

THEOREM 9. *The subgroup $E_{\sigma,\kappa,\lambda}$ of E_7 is isomorphic to the spinor group $Spin\,(12)$:*

$$E_{\sigma,\kappa,\lambda} \cong Spin\,(12).$$

PROOF. The proof is similar to that of Theorem 6 according to Lemma 8.

From now on, we identify the group $E_{\sigma,\kappa,\lambda}$ with the group $Spin\,(12)$.

REMARK. The group $Spin\,(12)$ has the center $z(Spin\,(12))=\{1, -1, \sigma, -\sigma\} \cong \{1, \sigma\} \times \{1, -\sigma\} \cong \boldsymbol{Z}_2 \times \boldsymbol{Z}_2$. And we have
$Spin\,(12)/\{1, \sigma\} \cong SO\,(12)$. Hence $Spin\,(12)/\{1, -\sigma\} \cong Ss\,(12)$.

5. Special unitary subgroup $SU(2)$ of E_7

THEOREM 10. *The group E_7 contains a subgroup*

$$SU(2) = \{\alpha_A \in E_7 \mid A \in \mathrm{SU}(2)\}$$

which is isomorphic to the special unitary group $\mathrm{SU}(2)=\{A \in M(2, \boldsymbol{C}) \mid A^*A= E, \det A=1\}$. *Here, for* $A \in \mathrm{SU}(2)$, α_A *is defined by*

$$\alpha_A\left(\begin{bmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{bmatrix}, \begin{bmatrix} \eta_1 & y_3 & \bar{y}_2 \\ \bar{y}_3 & \eta_2 & y_1 \\ y_2 & \bar{y}_1 & \eta_3 \end{bmatrix}, \xi, \eta\right)$$
$$= \left(\begin{bmatrix} \xi'_1 & x'_3 & \bar{x}'_2 \\ \bar{x}'_3 & \xi'_2 & x'_1 \\ x'_2 & \bar{x}'_1 & \xi'_3 \end{bmatrix}, \begin{bmatrix} \eta'_1 & y'_3 & \bar{y}'_2 \\ \bar{y}'_3 & \eta'_2 & y'_1 \\ y'_2 & \bar{y}'_1 & \eta_3 \end{bmatrix}, \xi', \eta'\right).$$

where

$$\begin{bmatrix} \xi'_1 \\ \eta' \end{bmatrix} = A\begin{bmatrix} \xi_1 \\ \eta \end{bmatrix}, \quad \begin{bmatrix} \xi' \\ \eta'_1 \end{bmatrix} = A\begin{bmatrix} \xi \\ \eta_1 \end{bmatrix}, \quad \begin{bmatrix} \eta'_2 \\ \xi'_3 \end{bmatrix} = A\begin{bmatrix} \eta_2 \\ \xi_3 \end{bmatrix}, \quad \begin{bmatrix} \eta'_3 \\ \xi'_2 \end{bmatrix} = A\begin{bmatrix} \eta_3 \\ \xi_2 \end{bmatrix},$$

$$\begin{bmatrix} x'_1 \\ y'_1 \end{bmatrix} = \bar{A}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \quad \begin{bmatrix} x'_2 \\ y'_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}, \quad \begin{bmatrix} x'_3 \\ y'_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}.$$

PROOF. For $A=\exp\begin{bmatrix} v & a \\ -\bar{a} & -v \end{bmatrix} \in \mathrm{SU}(2)$, $(a, v \in \boldsymbol{C}, \bar{v}=-v)$, we have $\alpha_A= \exp \Phi(\phi(v), aE_1, v) \in SU(2)$.

6. Connectedness of $(E_7)_\sigma$

We shall prove that the group $(E_7)_\sigma$ is connected. We denote, for a while, the connected component of $(E_7)_\sigma$ containing the identity 1 by $((E_7)_\sigma)_0$.

LEMMA 11. *Any element* $X \in (\mathfrak{J}^{\boldsymbol{C}})_\sigma=\{X \in \mathfrak{J}^{\boldsymbol{C}} \mid \sigma X=X\}$ *can be transformed in a diagonal form by a certain element* α *of the group* $(E_6)_\sigma=\{\alpha \in E_6 \mid \sigma\alpha=\alpha\sigma\}$:

$$\alpha X = \begin{bmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{bmatrix}, \qquad \xi_i \in \boldsymbol{C}.$$

PROOF. In the proof of [7, Proposition 5], if we remember that $i(\tilde{E}_1-\tilde{E}_2)$, $i(\tilde{E}_1-\tilde{E}_3)$, $i\tilde{F}_1(a)$, $\tilde{A}_1(a) \in (\mathfrak{e}_6)_\sigma=\{\phi \in \mathfrak{e}_6 \mid \sigma\phi=\phi\sigma\}$ (which is the Lie algebra of the group $(E_6)_\sigma$), then we can prove this lemma by the same way as [7, Proposition 5].

We define the spaces $(\mathfrak{M}^{\boldsymbol{C}})_\sigma$ and $(\mathfrak{M}_1)_\sigma$ respectively by

$$(\mathfrak{M}^C)_\sigma = \{P \in \mathfrak{M}^C \mid \sigma P = P\}, \qquad (\mathfrak{M}_1)_\sigma = \{P \in (\mathfrak{M}^C)_\sigma \mid \langle P, P\rangle = 1\}.$$

LEMMA 12. *Any element P of $(\mathfrak{M}^C)_\sigma$ can be transformed in a diagoanl form by a certain element α of $((E_7)_\sigma)_0$:*

$$\alpha P = (X, Y, \xi, \eta), \qquad X, Y \text{ are diagonal forms.}$$

Moreover we can choose $\alpha \in ((E_7)_\sigma)_0$ so that ξ is a positive real number.

PROOF. By making use of Lemma 11, we can prove this lemma by the same way as [3, Proposition 8].

PROPOSITION 13. *The group $(E_7)_\sigma$ acts transitively on $(\mathfrak{M}_1)_\sigma$ (which is connected) and the isotropy subgroup of $(E_7)_\sigma$ at $(0, 0, 1, 0) \in (\mathfrak{M}_1)_\sigma$ is $(E_6)_\sigma$. Therefore the homogeneous space $(E_7)_\sigma/(E_6)_\sigma$ is homeomorphic to $(\mathfrak{M}_1)_\sigma$:*

$$(E_7)_\sigma/(E_6)_\sigma \simeq (\mathfrak{M}_1)_\sigma.$$

In particular, the group $(E_7)_\sigma$ is connected.

PROOF. For $a \in \boldsymbol{C}$, remember $\Phi(0, -\bar{a}E_i, 0) \in (\mathfrak{e}_7)_\sigma$, $i=1, 2, 3$. Then by the use of Lemmas 12 and 3, we can prove the homeomorphism $(E_7)_\sigma/(E_6)_\sigma \simeq (\mathfrak{M}_1)_\sigma$ by the same way as [3, Theorem 9]. Now, since the group $(E_6)_\sigma$ is isomorphic to the group

$$(E_6)_\sigma \cong (U(1) \times Spin\,(10))/\boldsymbol{Z}_4 \qquad ([7, \text{Theorem } 13]),$$

$(E_6)_\sigma$ is connected. Therefore the group $(E_7)_\sigma$ is also connected.

7. Isomorphism $(E_7)_\sigma \cong (SU(2) \times Spin\,(12))/\boldsymbol{Z}_2$

THEOREM 14. *The subgroup $(E_7)_\sigma = \{\alpha \in E_7 \mid \sigma\alpha = \alpha\sigma\}$ of E_7 is isomorphic to the group $(SU(2) \times Spin\,(12))/\boldsymbol{Z}_2$:*

$$(E_7)_\sigma \cong (SU(2) \times Spin\,(12))/\boldsymbol{Z}_2 \qquad \text{where } \boldsymbol{Z}_2 = \{(1, 1), (-1, -\sigma)\}.$$

PROOF. We define a mapping

$$\psi\colon SU(2) \times Spin\,(12) \longrightarrow (E_7)_\sigma, \quad \psi(\alpha, \beta) = \alpha\beta.$$

Since the Lie algebra $(\mathfrak{e}_7)_\sigma$ is the direct sum of Lie algebras $\mathfrak{a}_1$ and $\mathfrak{e}_{\sigma,\kappa,\lambda}$ as ideals (Proposition 2), $\alpha \in SU(2)$ and $\beta \in Spin\,(12)$ are commutative. Hence we see that ψ is a homomorphism. Moreover ψ is onto, because the group $(E_7)_\sigma$ is connected (Proposition 13). $\operatorname{Ker}\psi = \boldsymbol{Z}_2 = \{(1, 1), (-1, -\sigma)\}$ is easily obtained. Thus the proof of Theorem 14 is completed.

8. Lie group $E_{7,\sigma}$ and its polar decomposition

We define an inner product $\langle P, Q\rangle_\sigma$ in $\mathfrak{P}^C$ by

$$\langle P, Q\rangle_\sigma = \langle \sigma P, Q\rangle = \langle P, \sigma Q\rangle$$

and a group $E_{7,\sigma}$ by (cf. [3], [5])

$$E_{7,\sigma} = \{\alpha \in \mathrm{Iso}^C(\mathfrak{P}^C, \mathfrak{P})^C \mid \alpha\mathfrak{M}^C = \mathfrak{M}^C, \{\alpha P, \alpha Q\} = \{P, Q\}, \langle \alpha P, \alpha Q\rangle_\sigma = \langle P, Q\rangle_\sigma\}.$$

(Later, we see that this group $E_{7,\sigma}$ is connected (Theorem 17), therefore it may also defined by (see [5])

$$E_{7,\sigma} = \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C, \mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q\rangle_\sigma = \langle P, Q\rangle_\sigma\}.)$$

In order to give a polar decomposition of the group $E_{7,\sigma}$, we use

LEMMA 15 ([2, p. 345]). *Let G be a pseudoalgebraic subgroup of the general linear group $GL(n, \boldsymbol{C})$ such that the condition $A \in G$ implies $A^* \in G$. Then G is homeomorphic to the topological product of the group $G \cap U(n)$ and a Euclidean space $\boldsymbol{R}^d$:*

$$G \simeq (G \cap U(n)) \times \boldsymbol{R}^d$$

where $U(n)$ is the unitary subgroup of $GL(n, \boldsymbol{C})$.

LEMMA 16. *The group $E_{7,\sigma}$ is a pseudoalgebraic subgroup of the general linear group $GL(56, \boldsymbol{C}) = \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{P}^C, \mathfrak{P}^C)$ and satisfies the condition that $\alpha \in E_{7,\sigma}$ implies $\alpha^* \in E_{7,\sigma}$, where α^* is the transpose of α with respect to the inner product $\langle P, Q\rangle$: $\langle \alpha P, Q\rangle = \langle P, \alpha^* Q\rangle$.*

PROOF. Since $\langle \alpha^* P, Q\rangle = \langle P, \alpha Q\rangle = \langle \sigma P, \alpha Q\rangle_\sigma = \langle \alpha^{-1}\sigma P, Q\rangle_\sigma = \langle \sigma\alpha^{-1}\sigma P, Q\rangle$ for $\alpha \in E_{7,\sigma}$, we have

$$\alpha^* = \sigma\alpha^{-1}\sigma \in E_{7,\sigma}.$$

It is obvious that $E_{7,\sigma}$ is pseudoalgebraic, because $E_{7,\sigma}$ is defined by pseudoalgebraic relations $\alpha\mathfrak{M}^C = \mathfrak{M}^C$, $\{\alpha P, \alpha Q\} = \{P, Q\}$, $\langle \alpha P, \alpha Q\rangle_\sigma = \langle P, Q\rangle_\sigma$.

Let $U(56) = U(\mathfrak{P}^C) = \{\alpha \in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{P}^C, \mathfrak{P}^C) \mid \langle \alpha P, \alpha Q\rangle = \langle P, Q\rangle\}$ denote the unitary subgroup of the general linear group $GL(56, \boldsymbol{C}) = \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{P}^C, \mathfrak{P}^C)$. Then

$$E_{7,\sigma} \cap U(56) = \{\alpha \in E_{7,\sigma} \mid \sigma\alpha = \alpha\sigma\} = \{\alpha \in E_7 \mid \sigma\alpha = \alpha\sigma\}$$
$$\cong (SU(2) \times Spin(12))/\boldsymbol{Z}_2 \qquad \text{(Theorem 14)}.$$

Since it is easy to see that $E_{7,\sigma}$ is a simple Lie group of type E_7 (see [3], [4]), the dimension of $E_{7,\sigma}$ is 133. Hence the dimension d of the Euclidean part of $E_{7,\sigma}$ and the Cartan index i are calculated as follows:

$$d = \dim E_{7,\sigma} - \dim(SU(2)\times Spin(12)) = 133 - (3+66) = 64,$$
$$i = \dim E_{7,\sigma} - 2\dim(SU(2)\times Spin(12)) = 133 - 2(3+66) = -5.$$

Thus we have the following

THEOREM 17. *The group $E_{7,\sigma}$ is homeomorphic to the topological product of the group $(SU(2)\times Spin(12))/\boldsymbol{Z}_2$ and the Euclidean space $\boldsymbol{R}^{64}$:*

$$E_{7,\sigma} \simeq (SU(2)\times Spin(12))/\boldsymbol{Z}_2 \times \boldsymbol{R}^{64}.$$

In particular, the group $E_{7,\sigma}$ is a connected non-compact simple Lie group of type $E_{7(-5)}$.

9. Center $z(E_{7,\sigma})$ of $E_{7,\sigma}$

THEOREM 18. *The center $z(E_{7,\sigma})$ of the group $E_{7,\sigma}$ is the cyclic group of order 2:*

$$z(E_{7,\sigma}) = \{1, -1\}.$$

PROOF. Let $\alpha\in z(E_{7,\sigma})$. From the commutativity with $\sigma\in E_{7,\sigma}$, α is contained in the center $z((E_7)_\sigma)$ of the group $(E_7)_\sigma$: $\alpha\in z((E_7)_\sigma)=\{1, -1, \sigma, -\sigma\}$ (cf. Theorem 14). Obviously, $\sigma, -\sigma\notin z(E_{7,\sigma})$, so we have $z(E_{7,\sigma})=\{1, -1\}$.

10. Remark on the definition of *Spin* (11) in E_7

We shall show that

$$(E_7)_1 = \{\alpha\in E_7 \mid \alpha(E_1, E_1, 1, 1) = (E_1, E_1, 1, 1)\} = Spin(11),$$

that is, in the definition of the group $E_{\sigma,\kappa,\lambda,1}$, the conditions $\sigma\alpha=\alpha\sigma$, $\kappa\alpha=\alpha\kappa$, $\lambda\alpha=\alpha\lambda$ are of no use.

We see that the Lie algebra $(\mathfrak{e}_7)_1$ of the group $(E_7)_1$ coincides with the Lie algebra $\mathfrak{e}_{\sigma,\kappa,\lambda,1}$ of the group $E_{\sigma,\kappa,\lambda,1}$ (Proposition 1, (3)). So, if we prove that the group $(E_7)_1$ is connected, then we can conclude $(E_7)_1=E_{\sigma,\kappa,\lambda,1}$.

We consider a vector space W^C which is invariant by the group $(E_7)_1$:

$$W^C=\{P\in\mathfrak{P}^C \mid P\times(E_1, E_1, 1, 1) = 0\}$$
$$=\left\{\left(\begin{bmatrix}-\xi & 0 & 0\\ 0 & \xi_2 & x\\ 0 & \bar{x} & \xi_3\end{bmatrix}, \begin{bmatrix}\xi & 0 & 0\\ 0 & \xi_3 & -x\\ 0 & -\bar{x} & \xi_2\end{bmatrix}, \xi, -\xi\right) \middle| \begin{array}{l}\xi, \xi_2, \xi_3\in\boldsymbol{C},\\ x\in\mathfrak{C}^C\end{array}\right\}.$$

This W^{C} is the complexification of W in § 3 and of course W^{C} has the positive definite Hermitian inner product $\langle P, Q\rangle$ which is invariant by the group $(E_7)_1$. We shall define one more inner product (P, Q) in W^{C} which is also invariant by the group $(E_7)_1$.

LEMMA 19. *If* $\alpha\in E_7$ *satisfies* $\alpha(E_1, E_1, 1, 1)=(E_1, E_1, 1, 1)$, *then* $\alpha(E_1, 0, 1, 0)=(E_1, 0, 1, 0)$. *Therefore this* α *also satisfies*

$$\alpha(E_1, iE_1, 1, i)=(E_1, iE_1, 1, i) \text{ and } \alpha(E_1, -iE_1, 1, -i)=(E_1, -iE_1, 1, -i).$$

PROOF. The proof is similar to that of Lemma 4.

We define vector spaces $U^{C}_{\varepsilon}(\varepsilon=1, -1)$ and U^{C} which are invariant by the group $(E_7)_1$ respectively by

$$U^{C}_{\varepsilon}=\{P\in \mathfrak{P}^{C} \mid P\times(E_1, \varepsilon iE_1, 1, \varepsilon i)=0\}$$

$$=\left\{\left(\begin{bmatrix}-\xi & 0 & 0\\ 0 & \xi_2 & x\\ 0 & \bar{x} & \xi_3\end{bmatrix}, \begin{bmatrix}\varepsilon i\xi & 0 & 0\\ 0 & -\varepsilon i\xi_3 & \varepsilon ix\\ 0 & \varepsilon i\bar{x} & -\varepsilon i\xi_2\end{bmatrix}, \xi, -\varepsilon i\xi\right) \,\middle|\, \begin{matrix}\xi, \xi_2, \xi_3\in \boldsymbol{C},\\ x\in\mathfrak{C}^{C}\end{matrix}\right\},$$

$U^{C}=U^{C}_{1}+U^{C}_{1}$ (which is the direct sum)

$$=\left\{\left(\begin{bmatrix}\xi_1 & 0 & 0\\ 0 & \xi_2 & x\\ 0 & \bar{x} & \xi_3\end{bmatrix}, \begin{bmatrix}\eta_1 & 0 & 0\\ 0 & \eta_2 & y\\ 0 & \bar{y} & \eta_3\end{bmatrix}, -\xi_1, -\eta_1\right) \,\middle|\, \begin{matrix}\xi_i, \eta_i\in \boldsymbol{C},\\ x, y\in\mathfrak{C}^{C}\end{matrix}\right\}.$$

We define a linear involutive transformation κ' of U^{C} by

$$\kappa'\left(\begin{bmatrix}\xi_1 & 0 & 0\\ 0 & \xi_2 & x\\ 0 & \bar{x} & \xi_3\end{bmatrix}, \begin{bmatrix}\eta_1 & 0 & 0\\ 0 & \eta_2 & y\\ 0 & \bar{y} & \eta_3\end{bmatrix}, -\xi_1, -\eta_1\right)$$

$$=\left(\begin{bmatrix}i\eta_1 & 0 & 0\\ 0 & i\eta_3 & -iy\\ 0 & -i\bar{y} & i\eta_2\end{bmatrix}, \begin{bmatrix}-i\xi_1 & 0 & 0\\ 0 & -i\xi_3 & ix\\ 0 & i\bar{x} & -i\xi_2\end{bmatrix}, -i\eta_1, i\xi_1\right).$$

Then $U^{C}=U^{C}_{1}+U^{C}_{-1}$ is the decomposition into the eigen spaces of κ'. Therefore we have, for any $\alpha\in(E_7)_1$,

$$\kappa'\alpha=\alpha\kappa' \qquad \text{on} \quad U^{C}.$$

Now, we define an inner product (P, Q) in U^{C} by

$$(P, Q) = i\{\kappa' P, Q\}.$$

Then (P, Q) is a symmetric non-degenerate inner product in U^C (of course so is in $W^C(\subset U^C)$) which is invariant by the group $(E_7)_1$: $(\alpha P, \alpha Q)=(P, Q)$ for $\alpha \in (E_7)_1$. Furthermore the two inner products $\langle P, Q\rangle$, (P, Q) coincide on W:

$$\langle P, Q\rangle = (P, Q) \qquad \text{for} \quad P, Q \in W.$$

Let p' be the natural homomorphism

$$p' : (E_7)_1 \longrightarrow O(W^C) = \{\alpha \in \mathrm{Iso}_C(W^C, W^C) \,|\, (\alpha P, \alpha Q) = (P, Q)\}.$$

Since $p'((E_7)_1)$ is a compact subgroup of $O(W^C)$, it is contained in a maximal compact subgroup of $O(W^C)$. On the other hand, maximal compact subgroups of $O(W^C)$ are conjugate to each other ([6, Theorem 3.1]), so there exists $\alpha \in O(W^C)$ such that

$$p'((E_7)_1) \subset \alpha O(W)\alpha^{-1}.$$

Let $e_1,\dots, e_{11}$ be an orthogonal basis in W and put $w_1=\alpha(e_1),\dots, w_{11}=\alpha(e_{11}) \in W^C$.

Case 1. $\langle w_k, w_l\rangle=0$ for all k, l $(k\neq l)$. In this case, $\langle \bar{w}_k, w_l\rangle=(w_k, w_l)=\delta_{kl}=\langle w_k, w_l\rangle/\langle w_k, w_k\rangle$ for all l, so we have $\bar{w}_k=w_k/\langle w_k, w_k\rangle$ (for $w=u+iv \in W^C$ $(u, v\in W)$, $\bar{w}$ means $u-iv$). Hence $w_k \in W$, $k=1,\dots, 11$, so $\alpha \in O(W)$, that is, $\alpha W=W$. Therefore the group $(E_7)_1$ acts on W. Then by the same arguments as those in § 3, we can conclude that the group $(E_7)_1$ is connected.

Case 2. There exist w_k, w_l $(k\neq l)$ such that $\langle w_k, w_l\rangle\neq 0$ and $(E_7)_1$ is not connected. Since $\mathrm{Ker}\, p'=\{1, \sigma\}\subset((E_7)_1)_0$ (which denotes the connected component of $(E_7)_1$ containing the identity 1), $p'((E_7)_1)$ is not also connected, so $p'((E_7)_1)=\alpha SO(W)\alpha^{-1}$ does not occur. Hence we have

$$p'((E_7)_1) = \alpha O(W)\alpha^{-1} = O(\alpha W).$$

Let $\beta \in O(\alpha W)$ be the reflection in W^C satisfying

$$\beta(w_k) = -w_k,\ \beta(w_j) = w_j \qquad (j\neq k).$$

Then we have $\langle w_k, w_l\rangle=\langle \beta w_k, \beta w_l\rangle=\langle -w_k, w_l\rangle$, hence $\langle w_k, w_l\rangle=0$. This contradicts the hypothesis.

Thus we have

THEOREM 20. *The subgroup* $(E_7)_1=\{\alpha \in E_7 \,|\, \alpha(E_1, E_1, 1, 1)=(E_1, E_1, 1, 1)\}$ *of* E_7 *is isomorphic to the spinor group* $Spin(11)$:

$$(E_7)_1 = Spin(11).$$

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