Orbits on affine symmetric spaces under the action of parabolic subgroups

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Introduction

Let $G$ be a connected Lie group, $\sigma$ an involutive automorphism of $G$ and $H$ a subgroup of $G$ satisfying $(G_\sigma)_0 \subset H \subset G_\sigma$ where $G_\sigma = \{ x \in G \mid \sigma(x) = x \}$ and $(G_\sigma)_0$ is the identity component of $G_\sigma$. Then the triple $(G, H, \sigma)$ is called an affine symmetric space. We assume that $G$ is real semisimple throughout this paper.

Let $P$ be a minimal parabolic subgroup of $G$. Then the double coset decomposition $H \backslash G / P$ is studied in [3] and [4]. Let $P'$ be an arbitrary parabolic subgroup of $G$ containing $P$. Then we have a canonical surjection

$$f: H \backslash G / P \twoheadrightarrow H \backslash G / P'.$$

The purpose of this paper is to determine $f^{-1}(\Theta)$ for an arbitrary double coset $\Theta$ in $H \backslash G / P'$.

When $G$ is a complex semisimple Lie group and $H$ is a real form of $G$, the double coset decomposition $H \backslash G / P$ is studied in [1] and [7] and structures of $H$-orbits on $G / P'$ are studied in [7].

When $G$ is a complex semisimple Lie group, $H$ is a complex subgroup of $G$ and $P'$ is a parabolic subgroup of $G$ corresponding to a simple root, the structure of $f^{-1}(\Theta)$ is determined for an arbitrary double coset $\Theta$ in $H \backslash G / P'$ in [5], p. 29, Lemma 5.2.

The results of this paper are as follows. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of $G$ and $H$ respectively, and the automorphism $\sigma$ of $\mathfrak{g}$ be the one induced from the automorphism $\sigma$ of $G$. Let $\theta$ be a Cartan involution of $\mathfrak{g}$ such that $\sigma \theta = \theta \sigma$. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ (resp. $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$) be the decomposition of $\mathfrak{g}$ into the $+1$ and $-1$ eigenspaces for $\sigma$ (resp. $\theta$).

Let $P^0$ be a minimal parabolic subgroup of $G$. Then the factor space $G / P^0$ is identified with the set of minimal parabolic subalgebras of $\mathfrak{g}$. By Theorem 1 of [3], every $H$-conjugacy class of minimal parabolic subalgebras of $\mathfrak{g}$ contains a minimal parabolic subalgebra of the form $\mathfrak{B} = \mathfrak{B}(\alpha, \Sigma(\alpha)^*)$ where $\alpha$ is a $\sigma$-stable maximal abelian subspace of $\mathfrak{p}$, $\Sigma(\alpha)^*$ is a positive system of the root system $\Sigma(\alpha)$ of the pair $(\mathfrak{g}, \alpha)$ and $\mathfrak{B}(\alpha, \Sigma(\alpha)^*) = \mathfrak{m} + \alpha + \mathfrak{n}$ is the corresponding minimal parabolic subalgebra of $\mathfrak{g}$. 


Thus the problem is reduced to the following. Fix a $\sigma$-stable maximal abelian subspace $a$ of $p$ and a minimal parabolic subalgebra $\mathfrak{P} = \mathfrak{P}(a, \Sigma(a)^\pm)$. Let $\mathfrak{P}'$ be an arbitrary parabolic subalgebra of $g$ containing $\mathfrak{P}$ and $P'$ the corresponding parabolic subgroup of $G$. Then we have only to determine the double coset decomposition

$$H \backslash H P' / P.$$

Since there is a canonical bijection $H \cap P \backslash P' \simeq H \backslash H P' / P$ and since the factor space $P' / P$ is identified with the set of minimal parabolic subalgebras of $g$ contained in $\mathfrak{P}'$, we have only to consider $H \cap P'$-conjugacy classes of minimal parabolic subalgebras of $g$ contained in $\mathfrak{P}'$. Let $\mathfrak{P}' = \mathfrak{m}' + a' + n'$ be the Langlands decomposition of $\mathfrak{P}'$ such that $a' \subseteq a$. A subset $a'_+$ of $a'$ is defined by $a'_+ = \{Y \in a' \mid \alpha(Y) > 0 \text{ for all } \alpha \in \Sigma(a) \text{ satisfying } g(a; \alpha) \subseteq n'\}$ ($g(a; \alpha) = \{X \in g \mid [Y, X] = \alpha(Y)X \text{ for all } Y \in a\}$). Now we can state the main result of this paper as follows.

**Theorem.** Every minimal parabolic subalgebra of $g$ contained in $\mathfrak{P}'$ is $H \cap P'$-conjugate to a minimal parabolic subalgebra $\mathfrak{P}_1$ of $g$ of the form

$$\mathfrak{P}_1 = \mathfrak{P}(a_1, \Sigma(a_1)^+)$$

where $a_1$ is a $\sigma$-stable maximal abelian subspace of $p$ such that $a_1 \supseteq a'$ and $\Sigma(a_1)^+$ satisfies $\langle \Sigma(a_1)^+, a'_+ \rangle \subseteq \mathbb{R}_+$ ($= \{t \in \mathbb{R} \mid t \geq 0\}$).

Let $\mathfrak{Z}_g(a' + \sigma a')$ denote the centralizer of $a' + \sigma a'$ and $\mathfrak{Z}$ the center of $\mathfrak{Z}_g(a' + \sigma a')$. Define a subalgebra $\mathfrak{m}''$ of $\mathfrak{Z}_g(a' + \sigma a')$ by $\mathfrak{m}'' = \{X \in \mathfrak{Z}_g(a' + \sigma a') \mid B(X, \mathfrak{Z} \cap a) = \{0\}\}$ where $B(,)$ is the Killing form of $g$. Then a subspace $a_1$ of $p$ satisfying the condition of Theorem contains $\mathfrak{Z} \cap a$. For such a subspace $a_1$ of $p$, define subsets $\Sigma(a_1)^{m'}_{m''}$ and $\Sigma(a_1)^{m''}_{m''}$ of $\Sigma(a_1)$ by

$$\Sigma(a_1)^{m'}_{m''} = \{x \in \Sigma(a_1) \mid \langle x, a' \rangle = \{0\}\}$$

and

$$\Sigma(a_1)^{m''}_{m''} = \{x \in \Sigma(a_1) \mid \langle x, a' + \sigma a' \rangle = \{0\}\}.$$

We consider closed $H$-orbits and open $H$-orbits on $H P'/P$ with respect to the topology of $H P'/P$.

**Corollary 1.** (a) A minimal parabolic subalgebra $\mathfrak{P}_1 = \mathfrak{P}(a_1, \Sigma(a_1)^+)$ satisfying the conditions of Theorem is contained in a closed $H$-orbit on $H P'/P$ (here we identified $\mathfrak{P}_1$ with a point in $P' / P$) if and only if the following three conditions are satisfied:

1. $\langle \Sigma(a_1)^{m''}_{m''}, \sigma a'_+ \rangle \subseteq \mathbb{R}_+$ where $\Sigma(a_1)^{m''}_{m''} = \Sigma(a_1)^{m''}_{m''} \cap \Sigma(a_1)^+$,
2. $\Sigma(a_1)^{m''}_{m''}$ is $\sigma$-compatible (i.e. $\alpha \in \Sigma(a_1)^{m''}_{m''}, \alpha |_{m'' \cap a_1} \neq 0 \Rightarrow \sigma \alpha \in \Sigma(a_1)^{m''}_{m''}$)

where $\Sigma(a_1)^{m''}_{m''} = \Sigma(a_1)^{m''}_{m''} \cap \Sigma(a_1)^+$,
(iii) $m'' \cap a_1 \cap h$ is maximal abelian in $m'' \cap p \cap h$.

(b) A minimal parabolic subalgebra $\mathfrak{P}_1 = \Psi(a_1, \Sigma(a_1)^+)$ satisfying the conditions of Theorem is contained in an open $H$-orbit on $HP'/P$ if and only if the following three conditions are satisfied:

(i) $\langle \Sigma(a_1)_m^+, \sigma \alpha_i' \rangle \in \mathbb{R}_+$,

(ii) $\Sigma(a_1)^+_m$ is $\sigma \theta$-compatible (i.e. $\sigma \in \Sigma(a_1)^+_m$, $\alpha |_{m'' \cap a_1 \cap h} \neq 0 \Rightarrow \sigma \theta \alpha \in \Sigma(a_1)^+_m$),

(iii) $m'' \cap a_1 \cap q$ is maximal abelian in $m'' \cap p \cap q$.

For an affine symmetric space $(G, H, \sigma)$, the associated affine symmetric space $(G, H', \sigma \theta)$ is defined by $H' = (K \cap H) \exp (p \cap q)$. Then there exists a one-to-one correspondence between the double coset decompositions $H \backslash G/P$ and $H' \backslash G/P$. If $a$ is a $\sigma$-stable maximal abelian subspace of $p$, then the $H$-orbit containing $\Psi(a, \Sigma(a)^+)$ corresponds to the $H'$-orbit containing the same $\Psi(a, \Sigma(a)^+)$ ([3], Corollary 2 of Theorem 1).

**Corollary 2.** (a) In the above correspondence between $H \backslash G/P$ and $H' \backslash G/P$, $H \backslash HP'/P$ corresponds to $H' \backslash H'P'/P$. Moreover closed $H$-orbits on $HP'/P$ correspond to open $H'$-orbits on $H'P'/P$ and open ones to closed ones.

(b) Let $P''$ be a parabolic subgroup of $G$ containing $P'$. Then there is a one-to-one correspondence between $H \backslash HP''/P'$ and $H' \backslash H''P''/P'$. In this correspondence closed $H$-orbits on $HP''/P'$ correspond to open $H'$-orbits on $H''P''/P'$ and open ones to closed ones.

Lastly we state an explicit formula for the decomposition $H \backslash HP'/P$ applying the method used in §2 of [3]. Let $a_0$ be a $\sigma$-stable maximal abelian subspace of $p$ such that $a_0 \subset a'$ and that $m'' \cap a_0 \cap h$ is maximal abelian in $m'' \cap p \cap h$. Fix a positive system $\Sigma(a_0)^+$ of $\Sigma(a_0)$ such that $\langle \Sigma(a_0)^+_m, \alpha_i' \rangle \in \mathbb{R}_+$. Then $\Psi(a_0) = \Psi(a_0, \Sigma(a_0)^+)$ is contained in $\Psi'$. Let $P(0)$ be the corresponding minimal parabolic subgroup of $G$.

Let $\bar{a}$ be a $\sigma$-stable maximal abelian subspace of $p$ such that $\bar{a} \cap h$ is maximal abelian in $p \cap h$, $\bar{a} \cap h \ni a_0 \cap h$ and $\bar{a} \cap q \subset a_0 \cap q$. Put $\tau = \{Y \in \bar{a} \cap h | B(Y, a_0 \cap h) = \{0\}\}$. Put $\Sigma'(\bar{a}_0)_m = \{\alpha \in \Sigma(a_0)_m | H_x \in m'' \cap a_0 \cap h\}$ where $H_x \in a_0$ is defined by $B(H_x, Y) = \alpha(Y)$ for $Y \in a_0$. Then a set of root vectors $Q = \{X_{a_1}, ..., X_{a_k}\}$ is said to be a $q$-orthogonal system of $\Sigma'(\bar{a}_0)_m$ if the following two conditions are satisfied:

(i) $\alpha_i \in \Sigma'(\bar{a}_0)_m$ and $X_{a_i} \in g(a_0; \alpha_i) \cap q - \{0\}$ for $i = 1, ..., k$,

(ii) $[X_{a_i}, X_{a_j}] = [X_{a_i}, \theta X_{a_j}] = 0$ for $i \neq j$.

We normalize $X_{a_i}, i = 1, ..., k$ so that $2\alpha_i(H_{a_i})B(X_{a_i}, \theta X_{a_i}) = -1$. Define an element $c(Q)$ of $M_0^\sigma$ by

$$ c(Q) = \exp (\pi/2)(X_{a_1} + \theta X_{a_1}) \cdots \exp (\pi/2)(X_{a_k} + \theta X_{a_k}). $$
Then \( \alpha^i = \text{Ad}(c(Q))a_0 \) is a \( \sigma \)-stable maximal abelian subspace of \( p \) such that \( \alpha' \subset \alpha^i \).

Let \( \{Q_0, \ldots, Q_n\} (Q_0 = \emptyset) \) be a complete set of representatives of \( q \)-orthogonal systems of \( \Sigma_q(a_0)_{m} \) with respect to the following equivalence relation \( \sim \). For two \( q \)-orthogonal systems \( Q = \{X_{a_1}, \ldots, X_{a_k}\} \) and \( Q' = \{X_{b_1}, \ldots, X_{b_k}\} \) of \( \Sigma_q(a_0)_{m} \), \( Q \sim Q' \) if and only if there exists a \( w \in W_{K \cap H}(\bar{a}) (= N_{K \cap H}(\bar{a})/Z_{K \cap H}(\bar{a})) \) such that

\[
W\left( r + \sum_{j=1}^{k} H_{b_j} \right) = r + \sum_{j=1}^{k} H_{b_j}.
\]

Put \( a_i = \text{Ad}(c(Q_i))a_0, \ i = 1, \ldots, n \). Then we have the following corollary.

**Corollary 3.** \( HP' = \bigcup_{i=0}^{n} \bigcup_{j=1}^{m_{i+1}} Hw_jc(Q_i)P_{(0)} \) (disjoint union) where \( \{w_1, \ldots, w_{m_{i+1}}\} \) is a complete set of representatives of \( W_{K \cap H}(a_i) \cap W(a_i)_{m} \setminus W(a_i)_{m} \) in \( N_{K \cap M}(a_i) (W(a_i)_{m} = N_{K \cap M}(a_i)/Z_{K \cap M}(a_i)) \). Moreover we have

\[
H'P' = \bigcup_{i=0}^{n} \bigcup_{j=1}^{m_{i+1}} H'w_jc(Q_i)P_{(0)} \text{ (disjoint union)}.
\]

\section{1. Notations and preliminaries}

Let \( \mathbf{R} \) denote the set of real numbers and \( \mathbf{R}_+ \) the subset of \( \mathbf{R} \) defined by \( \mathbf{R}_+ = \{ t \in \mathbf{R} | t \geq 0 \} \). Let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \). For subsets \( s \) and \( t \) in \( \mathfrak{g} \) and a subset \( S \) in \( G \), \( Z_s(t) \) and \( N_s(t) \) are the subsets of \( \mathfrak{g}, G \) and \( G \) defined by

\[
Z_s(t) = \{ X \in s | [X, Y] = 0 \text{ for all } Y \in t \},
\]

\[
N_s(t) = \{ x \in S | \text{Ad}(x)Y = Y \text{ for all } Y \in t \}
\]

and

\[
N_s(t) = \{ x \in S | \text{Ad}(x)t = t \},
\]

respectively.

Let \( G \) be a connected real semisimple Lie group, \( \sigma \) an involutive automorphism of \( G \) (i.e. \( \sigma^2 = \text{identity} \)) and \( H \) a subgroup of \( G \) satisfying \( (G_{\sigma})_0 \subset H \subset G_{\sigma} \) where \( G_{\sigma} = \{ x \in G | \sigma(x) = x \} \) and \( (G_{\sigma})_0 \) is the identity component of \( G_{\sigma} \). Then the triple \((G, H, \sigma)\) is an affine symmetric space such that \( G \) is real semisimple.

Let \( \mathfrak{g} \) and \( \mathfrak{h} \) be the Lie algebras of \( G \) and \( H \) respectively, and the automorphism \( \sigma \) of \( \mathfrak{g} \) be the one induced from the automorphism \( \sigma \) of \( G \). There exists a Cartan involution \( \theta \) of \( \mathfrak{g} \) such that \( \sigma \theta = \theta \sigma \) ([2], cf. Lemmas 3 and 4 in [3]). Fix such a Cartan involution \( \theta \) of \( \mathfrak{g} \). Let \( \mathfrak{g} = \mathfrak{h} + \mathfrak{q} \) (resp. \( \mathfrak{g} = \mathfrak{t} + \mathfrak{p} \)) be the decomposition of \( \mathfrak{g} \) into the \(+1\) and \(-1\) eigenspaces for \( \sigma \) (resp. \( \theta \)). Then we have the following direct sum decomposition

\[
\mathfrak{g} = \mathfrak{t} \cap \mathfrak{h} + \mathfrak{t} \cap \mathfrak{q} + \mathfrak{p} \cap \mathfrak{h} + \mathfrak{p} \cap \mathfrak{q}
\]

of \( \mathfrak{g} \). Let \( K \) denote the analytic subgroup of \( G \) for \( \mathfrak{t} \).
Let $\alpha$ be a maximal abelian subspace of $p$. Then the space of real linear forms on $\alpha$ is denoted by $\alpha^*$. For an $\alpha \in \alpha^*$, let $g(\alpha; \alpha)$ denote the subspace of $g$ defined by

$$g(\alpha; \alpha) = \{X \in g \mid [Y, X] = \alpha(Y)X \quad \text{for all} \quad Y \in \alpha\}.$$ 

Then the root system $\Sigma(\alpha)$ of the pair $(g, \alpha)$ is the finite subset of $\alpha^*$ defined by

$$\Sigma(\alpha) = \{\alpha \in \alpha^* - \{0\} \mid g(\alpha; \alpha) \neq \{0\}\}.$$ 

Let $\Sigma(\alpha)^+$ be a positive system of $\Sigma(\alpha)$. Then we can define a minimal parabolic subalgebra $\Psi(\alpha, \Sigma(\alpha)^+)$ of $g$ and a minimal parabolic subgroup $P(\alpha, \Sigma(\alpha)^+)$ of $G$ by

$$\Psi(\alpha, \Sigma(\alpha)^+) = m + \alpha + n$$

and

$$P(\alpha, \Sigma(\alpha)^+) = MAN,$$

respectively, where $m = Z(\alpha)$, $M = Z_K(\alpha)$, $A = \exp a$, $n = \sum_{\alpha \in \Sigma(\alpha)^+} g(\alpha, \alpha)$ and $N = \exp n$.

Let $\Psi'$ be an arbitrary parabolic subalgebra of $g$ containing $\Psi(\alpha, \Sigma(\alpha)^+)$ and $P'$ the corresponding parabolic subgroup of $G$. Then there is a unique Langlands decomposition

$$\Psi' = m' + a' + n'$$

of $\Psi'$ such that $a' \subseteq a$. Let $a'_+$ denote the subset of $a$ defined by

$$a'_+ = \{Y \in a' \mid \alpha(Y) > 0 \quad \text{for all} \quad \alpha \in \Sigma(\alpha) \quad \text{such that} \quad g(\alpha; \alpha) \subseteq n'\}.$$ 

The corresponding Langlands decomposition of $P'$ is denoted by $P' = M'A'N'$. Let $P^0$ be a minimal parabolic subgroup of $G$ and $\Psi^0$ the corresponding minimal parabolic subalgebra of $g$. Then the factor space $G/P^0$ is identified with the set of minimal parabolic subalgebras of $g$ by the correspondence $xP^0 \to \text{Ad}(x)\Psi^0$, $x \in G$. Thus the $H$-orbits on $G/P^0$ are identified with the $H$-conjugacy classes of minimal parabolic subalgebras of $g$.

Here we review a main result of [3]. Let $\{a_i \mid i \in I\}$ be a complete set of representatives of the $K \cap H$-conjugacy classes of $\sigma$-stable maximal abelian subspace of $p$. Let $W(a_i) = N_K(a_i)/Z_K(a_i)$ be the Weyl group of $\Sigma(a_i)$ and $W_{K \cap H}(a_i)$ the subgroup of $W(a_i)$ defined by

$$W_{K \cap H}(a_i) = N_{K \cap H}(a_i)/Z_{K \cap H}(a_i).$$

**Proposition** (Corollary 1 of Theorem 1 in [3]). There is a one-to-one correspondence between the set of $H$-conjugacy classes of minimal parabolic subalgebras of $g$ and the set $\cup_{i \in I} W_{K \cap H}(a_i)W(a_i)$ (disjoint union). Fix a positive
system $\Sigma(a_i)^+$ of $\Sigma(a_i)$ for each $i \in I$. Then $W_{K \cap H}(a_i)w \in W_{K \cap H}(a_i)W(a_i)$ corresponds to the $H$-conjugacy class of minimal parabolic subalgebras of $g$ containing $\Psi(a_i, w\Sigma(a_i)^*)$.

§2. Theorem and its corollaries

Let $\Psi^0$ be an arbitrary parabolic subalgebra of $g$ containing $\Psi^0$ and $P^0$ the corresponding parabolic subgroup of $G$. Then we have a canonical surjection

$$f: H\backslash G/P^0 \longrightarrow H\backslash G/P^0'.$$

For every double coset $\varnothing = HxP^0 \in H\backslash G/P^0$ ($x \in G$), we want to study $f^{-1}(\varnothing) = H\backslash HxP^0/P^0$. It follows from Proposition in §1 that there exist an $h \in H$, a $\sigma$-stable maximal abelian subspace $a$ of $p$ and a positive system $\Sigma(a)^+$ of $\Sigma(a)$ such that $Ad(\alpha x)\Psi^0 = \Psi(a, \Sigma(a)^+)$. Thus we have only to study the double coset decomposition $H\backslash HP'/P$ for such a minimal parabolic subalgebra $\Psi = \Psi(a, \Sigma(a)^+)$ where $P$ is the minimal parabolic subgroup corresponding to $\Psi$ and $P' = h\Psi P^0 x^{-1} h^{-1}$.

Therefore we fix a $\sigma$-stable maximal abelian subspace $a$ of $p$ and a positive system $\Sigma(a)^+$ of $\Sigma(a)$. Put $\Psi = \Psi(a, \Sigma(a)^+)$ and let $\Psi'$ be the parabolic subalgebra of $g$ which is conjugate to $\Psi^0$ and contains $\Psi$. Notations $\Psi = m + a + n$, $P = MAN$, $\Psi' = m' + a' + n'$, $P' = M'A'N'$ and $a'$ are the same as in §1.

Since $H\backslash HP'$ is isomorphic to $H \cap P \backslash P'$, there is a canonical bijection

$$H \cap P \backslash P'/P \cong H\backslash HP'/P.$$  

Then the following theorem gives standard representatives for $H \cap P \backslash P'$ since $P'/P$ is identified with the set of minimal parabolic subalgebras of $g$ contained in $\Psi'$.

**Theorem.** Every minimal parabolic subalgebra of $g$ contained in $\Psi$ is $H \cap P'$-conjugate to a minimal parabolic subalgebra $\Psi_1$ of $g$ of the form

$$\Psi_1 = \Psi(a_1, \Sigma(a_1)^+),$$

where $a_1$ is a $\sigma$-stable maximal abelian subspace of $p$ such that $a_1 \supset a'$ and $\Sigma(a_1)^+$ is a positive system of $\Sigma(a_1)$ such that $\langle \Sigma(a_1)^+, a_1' \rangle \subset R_+$.

**Remark.** Conversely if $a_1$ and $\Sigma(a_1)^+$ satisfy the conditions in Theorem, then $\Psi_1 = \Psi(a_1, \Sigma(a_1)^+)$ is contained in $\Psi'$. In fact, write $\Psi_1 = m_1 + a_1 + n_1$ where $m_1 = \mathfrak{Z}(a_1)$ and $n_1 = \sum_{a \in \Sigma(a_1)^+} g(a_1; a)$. Note that

$$\Psi' = \sum_{a} g(a'; a)$$

(the sum is taken over all $a \in (a')^*$ such that $\langle a, a' \rangle \supset R_+$).
where \((\alpha')^\ast\) is the space of real linear forms on \(\alpha'\) and \(\mathfrak{g}(\alpha'; \alpha) = \{X \in \mathfrak{g} \mid [Y, X] = \alpha(Y)X\}\). Then it follows from the condition for \(\alpha_i\) that \(m_1 + a_1 \subset \mathfrak{g}(\alpha'; 0)\). On the other hand, it follows from the condition for \(\Sigma(\alpha_i)^+\) that \(\mathfrak{g}(\alpha_i; \alpha) \subset \mathfrak{g}(\alpha'; \alpha|_{\alpha}) \subset \Psi'\) for \(\alpha \in \Sigma(\alpha_i)^+\). Thus we have \(\Psi_1 \subset \Psi'\).

We use the following method of Lusztig and Vogan ([5], p. 29, Lemma 5.2). Let \(\pi: P' \to M'\) be the projection with respect to the Langlands decomposition \(P' = M'A'N'\). Then \(\pi\) is a group homomorphism and induces an isomorphism of \(P'/P\) onto \(M'/M' \cap P\). Put \(J = \pi(H \cap P')\). Then there is a canonical bijection

\[
(2.2) \quad H \cap P'|P \cong J|\frac{M'}{M'} \cap P.
\]

(In [5], \(G\) and \(H\) are complex groups and \(P'\) is a parabolic subgroup of \(G\) corresponding to a simple root of \(\Sigma(\alpha)^+\).)

Let \(J_0\) and \(M'_0\) be the identity components of \(J\) and \(M'\) respectively. Since \(M' \cap P \supseteq M\), every connected component of \(M'\) has a non-trivial intersection with \(M' \cap P\). Thus \(M'/M' \cap P\) is isomorphic to \(M'_0/M'_0 \cap P\) and we have a canonical surjection

\[
(2.3) \quad J_0|\frac{M'_0}{M'_0} \cap P \longrightarrow J|\frac{M'}{M'} \cap P.
\]

It is clear that the subalgebras \(m' \cap \Psi\) and \(m' \cap \sigma\Psi'\) are a minimal parabolic subalgebra and a parabolic subalgebra of \(m'\) respectively. Let \(\Sigma(\alpha)_m^\ast\) and \(\Sigma(\alpha)_n^\ast\) be the subsets of \(\Sigma(\alpha)\) defined by \(\Sigma(\alpha)_m^\ast = \{\alpha \in \Sigma(\alpha) \mid \langle \alpha, \alpha' \rangle = \{0\}\}\) and \(\Sigma(\alpha)_n^\ast = \{\alpha \in \Sigma(\alpha) \mid \langle \alpha, \alpha' \rangle \subset R_+ - \{0\}\}\) respectively. Then

\[
m' + \alpha' = m + a + \sum_{\alpha \in \Sigma(\alpha)_m} \mathfrak{g}(\alpha; \alpha)
\]

and

\[
n' = \sum_{\alpha \in \Sigma(\alpha)_n} \mathfrak{g}(\alpha; \alpha).
\]

Let

\[
m' \cap \sigma\Psi' = m'' + a'' + n''
\]

be the Langlands decomposition of \(m' \cap \sigma\Psi'\) such that \(a'' \subset a\). Let \(\Sigma(\alpha)_m^\ast\) and \(\Sigma(\alpha)_n^\ast\) be the subsets of \(\Sigma(\alpha)_m\) defined by \(\Sigma(\alpha)_m^\ast = \{\alpha \in \Sigma(\alpha) \mid \langle \alpha, \alpha' + \sigma\alpha' \rangle = \{0\}\}\) and \(\Sigma(\alpha)_n^\ast = \{\alpha \in \Sigma(\alpha)_m \mid \langle \alpha, \sigma\alpha' \rangle \subset R_+ - \{0\}\}\) respectively. Then we have

\[
m'' + a'' + a' = m + a + \sum_{\alpha \in \Sigma(\alpha)_m} \mathfrak{g}(\alpha; \alpha)
\]

and

\[
n'' = \sum_{\alpha \in \Sigma(\alpha)_n} \mathfrak{g}(\alpha; \alpha).
\]

**Lemma.** Let \(j\) be the Lie algebra of \(J\) and \(a''_j\) be the subspace of \(a''\) given by \(a''_j = \pi((a'' + a') \cap \mathfrak{h})\). Then

\[
\]
\[ j = m' \cap h + a'' + n'' \]

**Proof.** Put \( A_1 = \Sigma(a)_m \cap \sigma \Sigma(a)_m = \Sigma(a)_m \), \( A_2 = \Sigma(a)_m \cap \sigma \Sigma(a)_m = \Sigma(a)_m \), and \( A_3 = \Sigma(a)_m \cap \sigma \Sigma(a)_n \), and set
\[ \Psi_i = \sum_{\alpha \in A_i} (g(a; \alpha) + g(a; \sigma \alpha)) \cap h \quad (i = 1, 2, 3). \]

Then
\[ \Psi' \cap h = \Psi' \cap \sigma \Psi' \cap h = m \cap h + a \cap h + A_1 + A_2 + A_3. \]

Since \( \pi: \Psi' \rightarrow m' \) is the projection with respect to the decomposition \( \Psi' = m' + a' + n' \), we have
\[ j = \pi(\Psi' \cap h) = m \cap h + \pi(a \cap h) + A_1 + \sum_{\alpha \in A_2} g(a; \alpha) \]
\[ = m \cap h + m'' + a \cap h + a'' + A_1 + n'' = m'' \cap h + a'' + n''. \]

q.e.d.

Let \( W(a)_m \) and \( W(a)_m \) denote the subgroups of \( W(a) \) generated by the reflections with respect to the roots of \( \Sigma(a)_m \) and \( \Sigma(a)_m \) respectively.

**Proof of Theorem.** We have only to find a set of standard representatives \( S \subset M_0 \) of \( J_0 \backslash M_0 / M_0 \cap P \) since the set \( S \) becomes a set of representatives of \( H \backslash H P/P \) in view of the above arguments.

\( M_0 \cap P \) is a minimal parabolic subgroup of \( M_0 \) since \( m' \cap \Psi \) is a minimal parabolic subalgebra of \( m' \) and since \( Z_K \cap M(a) = M_0 \cap M \) is contained in \( M_0 \cap P \). In the same way \( M_0 \cap \sigma P' \) is proved to be a parabolic subgroup of \( M_0 \). Thus we have the Bruhat decomposition
\[ M_0 = \bigcup_{w \in W_1} (M_0 \cap \sigma P')w(M_0 \cap P) \]
where \( W_1 \) is a complete set of representatives of \( W(a)_m \backslash W(a)_m \) in \( N_K \cap M_0(a) \).

Let \( M_0 \cap \sigma P' = M'' A'' N'' \) be the Langlands decomposition of \( M_0 \cap \sigma P' \) corresponding to \( m' \cap \sigma \Psi' = m'' + a'' + n'' \). Then it follows from Lemma that
\[ (M_0 \cap \sigma P')w(M_0 \cap P) = J_0 M'' A'' w(M_0 \cap P) \]
for every \( w \in W_1 \). Therefore we have only to study the decomposition
\[ J_0 \cap M'' A'' \backslash M'' A'' / w P w^{-1} \cap M'' A''. \]
Since \( M'' A'' / w P w^{-1} \cap M'' A'' \) is isomorphic to \( M_0'/w P w^{-1} \cap M_0' (M_0' \) is the identity component of \( M' \) and since \( J_0 \cap M'' A'' = (M'' \cap H)_0 \exp a'' \) (Lemma), there is a canonical bijection
\[ (2.4) \quad (M'' \cap H)_0 \backslash M_0'/w P w^{-1} \cap M_0' \xrightarrow{\sim} J_0 \cap M'' A'' \backslash M'' A'' / w P w^{-1} \cap M'' A''. \]
Here we note that \( M'_0 \) is \( \sigma \)-stable. Thus the triple \((M'_0, (M'' \cap H)_0, \sigma)\) is an affine symmetric space such that \( M'_0 \) is a connected real reductive Lie group. Moreover \( wPw^{-1} \cap M'_0 \) is a minimal parabolic subgroup of \( M'_0 \). Therefore the result of [3] can be applied to the left hand side of (2.4). For every \( x \in M''_0 \) there is a \( y \in (M'' \cap H)_0 x(wPw^{-1} \cap M'_0) \) such that \( a''_x = \text{Ad}(y)(a \cap m'') \) is a \( \sigma \)-stable maximal abelian subspace of \( m'' \cap p \) (Proposition in § 1).

Thus we have proved the following. For an arbitrary \( x \in HP' \) there exists a \( w \in W_1 \) and a \( y \in M'_0 \) such that \( a'_1 = \text{Ad}(y)a \) is \( \sigma \)-stable and that \( yw \in HxP \). Then it is clear that \( a_1 \) and \( \Psi_1 = \text{Ad}(yw)\Psi = \Psi(a_1, (a(a_1))^+) \) satisfy the conditions of the theorem. Hence the theorem is proved. q.e.d.

For a \( \sigma \)-stable maximal abelian subspace \( a_1 \) of \( p \) satisfying \( a_1 \supset a' \), we can define subsets \( \Sigma(a_1)_+ \) and \( \Sigma(a_1)_{+\perp} \) of \( \Sigma(a_1) \) in the same manner as \( \Sigma(a)_+ \) and \( \Sigma(a)_{+\perp} \). If \( (a_1)^+ \) is a positive system of \( \Sigma(a_1) \), then \( \Sigma(a_1)^+ \) and \( \Sigma(a_1)^{+\perp} \) are defined by \( \Sigma(a_1)^{+\perp} = \Sigma(a_1)^+ \cap \Sigma(a_1)^+ \) and \( \Sigma(a_1)^{+\perp} = \Sigma(a_1)_{+\perp} \cap \Sigma(a_1)^+ \) respectively.

Now we consider closed \( H \)-orbits and open \( H \)-orbits on \( HP'/P \) with respect to the topology of \( HP'/P \).

**COROLLARY 1.** Retain the notations in Theorem.

(a) A minimal parabolic subalgebra \( \Psi_1 = \Psi(a_1, (a(a_1))^+) \) satisfying the conditions of Theorem is contained in a closed \( H \)-orbit on \( HP'/P \) (\( \Psi_1 \) is identified with a point in \( P'/P \)) if and only if the following three conditions are satisfied:

(i) \( \langle \Sigma(a_1)^+_+, \sigma a'_+ \rangle \subset \mathbb{R}_+ \),
(ii) \( \Sigma(a_1)^{+\perp} \) is \( \sigma \)-compatible (i.e. \( \alpha \in \Sigma(a_1)^{+\perp}, \alpha \mid_{m'' \cap a_1 \cap p} \neq 0 \Rightarrow \sigma \alpha \in \Sigma(a_1)^{+\perp} \)),
(iii) \( m'' \cap a_1 \cap p \) is maximal abelian in \( m'' \cap p \).

(b) A minimal parabolic subalgebra \( \Psi_1 = \Psi(a_1, (a(a_1))^+) \) satisfying the conditions of Theorem is contained in an open \( H \)-orbit on \( HP'/P \) if and only if the following three conditions are satisfied:

(i) \( \langle \Sigma(a_1)^{+\perp} \sigma \alpha'_+ \rangle \subset \mathbb{R}_+ \),
(ii) \( \Sigma(a_1)^{+\perp} \) is \( \sigma \)-\( \alpha \)-compatible (i.e. \( \alpha \in \Sigma(a_1)^{+\perp}, \alpha \mid_{m'' \cap a_1 \cap q} \neq 0 \Rightarrow \sigma \alpha \in \Sigma(a_1)^{+\perp} \)),
(iii) \( m'' \cap a_1 \cap q \) is maximal abelian in \( m'' \cap p \).

**PROOF.** Since the bijections (2.1) and (2.2) come from the topological isomorphisms \( H \cap P' \backslash P' \cong H \backslash HP' \) and \( P'/P \cong M'/M' \cap P \) respectively, we have only to consider closed double cosets and open double cosets in the decomposition

\[ J \setminus M'/M' \cap P. \]

For \( x \in M' \) and \( y \in J \), we have \( J_0 y x(M' \cap P) = y J_0 x(M' \cap P) \). Hence \( J x(M' \cap P) \) is closed (resp. open) in \( M' \) if and only if \( J_0 x(M' \cap P) \) is closed (resp. open) in \( M' \) and therefore we have only to consider closed double cosets and open double cosets in the decomposition
Consider the decomposition
\[ M'_0 = \bigcup_{w \in W} J_0 M'' A'' w (M'_0 \cap P). \]
Then open double cosets in \( J_0 \backslash M'_0 / M'_0 \cap P \) are contained in
\[ J_0 M'' A'' w_2 (M'_0 \cap P) = (M'_0 \cap \sigma P') w_2 (M'_0 \cap P) \]
where \( w_2 \) is the unique element in \( W_1 \) satisfying
\[ (m' \cap \sigma \Psi') + \text{Ad}(w_2)(m' \cap \Psi) = m'. \] (2.5)
On the other hand closed double cosets in \( J_0 \backslash M'_0 / M'_0 \cap P \) are contained in
\[ J_0 M'' A'' w_1 (M'_0 \cap P) \]
where \( w_1 \) is the unique element in \( W_1 \) satisfying
\[ \text{Ad}(w_1)(m' \cap \Psi) \approx n''. \] (2.6)
This is proved as follows. Let \( g: J_0 \to M'' A'' \cap J_0 \) be the projection with respect to the decomposition \( J_0 = (M'' A'' \cap J_0) N'' \). For \( x \in M'' A'' \) and \( w \in W_1 \), we have
\[ J_0 x w (M'_0 \cap P) / M'_0 \cap P \cong J_0 / J_0 \cap x w (M'_0 \cap P) w^{-1} x^{-1}. \]
Then the map \( g \) induces a projection
\[ J_0 / J_0 \cap x w (M'_0 \cap P) w^{-1} x^{-1} \longrightarrow (M'' A'' \cap J_0) / \text{Ad}(w)(x w (M'_0 \cap P) w^{-1} x^{-1}) \]
with fibres isomorphic to \( F = N'' / N'' \cap w (M'_0 \cap P) w^{-1} x^{-1} \). Since \( x^{-1} N'' x = N'' \), we have \( F \cong N'' / N'' \cap w (M'_0 \cap P) w^{-1} \). If we apply Lemma 1.1.4.1 in [6] to \( n'' \) and \( n'' \cap \text{Ad}(w)(m' \cap \Psi) \), it follows easily that \( F \) is topologically isomorphic to \( \mathbb{R}^k \) where \( k = \dim n'' - \dim (n'' \cap \text{Ad}(w)(m' \cap \Psi)) \). If the double coset \( J_0 x w (M'_0 \cap P) \) is closed in \( M'_0 \), then \( J_0 x w (M'_0 \cap P) / (M'_0 \cap P) \) is compact and therefore \( k = 0 \). Hence \( \text{Ad}(w)(m' \cap \Psi) \approx n'' \) and \( w = w_1 \).

The assertion (a) is proved as follows. Since the canonical map
\[ M'_0 / w_1 P w_1^{-1} \cap M'_0 \longrightarrow M'' A'' / w_1 P w_1^{-1} \cap M'' A'' \]
is a topological isomorphism and since (2.5) is a bijection, we have only to consider closed double cosets in
\[ (M'' \cap H) \backslash M'' / w_1 P w_1^{-1} \cap M'' \] (2.7)
For each double coset in (2.7), take a representative \( x \in M'_0 \) so that \( \text{Ad}(x)(m'' \cap \alpha) = \alpha'' \) is \( \sigma \)-stable. Then \( x \) is contained in a closed double coset in (2.7) if and only
if \( \alpha''_1 \cap h \) is maximal abelian in \( m'' \cap p \cap h \) and the positive system \( \Sigma(\alpha'')^+ \) of \( \Sigma(\alpha'') \) corresponding to \( xw_1, p_{w^{-1}}^{-1} x^{-1} \cap M_0'' \) is \( \sigma \)-compatible ([3], § 3, Proposition 2).

Put \( a_i = \text{Ad}(x)a \) and \( \Psi_i = \text{Ad}(x)w_1, \Psi = \Psi(a_i, \Sigma(\alpha'_1)^+) \).
Then it is clear that (2.6) is equivalent to the condition (i) in (a) and that the above conditions for \( \alpha''_1 \) (= \( a_1 \cap m'' \)) and \( \Sigma(\alpha''_1)^+ \) are equivalent to the conditions (ii) and (iii) in (a). Hence the assertion (a) is proved.

The assertion (b) is proved by a similar argument using (2.5) and Proposition 1 in [3]. q.e.d.

For an affine symmetric space \((G, H, \sigma)\) such that \( G \) is semisimple, the associated affine symmetric space \((G, H', \sigma')\) is defined by \( H' = (K \cap H) \exp(p \cap q) \).
Then there exists a one-to-one correspondence between the double coset decompositions \( H \bs G / P \) and \( H' \bs G / P' \). If \( \alpha \) is a \( \sigma \)-stable maximal abelian subspace of \( p \), an \( H \)-orbit containing \( \Psi(a, \Sigma(\alpha'^+)) \) corresponds to the \( H' \)-orbit containing the same \( \Psi(a, \Sigma(\alpha'^+)) \) ([3], Corollary 2 of Theorem 1).

COROLLARY 2. (a) In this correspondence, \( H \bs H' P / P \) corresponds to \( H' \bs H' P / P \). Moreover closed \( H \)-orbits on \( H' P / P \) correspond to open \( H' \)-orbits on \( H' P / P \) and open ones to closed ones.

(b) Let \( P'' \) be a parabolic subgroup of \( G \) containing \( P' \).
Then there is a one-to-one correspondence between \( H \bs H' P'' / P' \) and \( H' \bs H' P'' / P' \) which is compatible with the canonical surjections \( f : H \bs H' P'' / P' \rightarrow H \bs H P'' / P' \) and \( f' : H' \bs H' P'' / P' \rightarrow H' \bs H' P'' / P' \) and with the correspondence \( H \bs H' P'' / P \simeq H' \bs H' P'' / P' \).
In this correspondence closed \( H \)-orbits on \( H' P'' / P' \) correspond to open \( H' \)-orbits on \( H' P'' / P' \) and open ones to closed ones.

PROOF. The first assertion in (a) is clear from Theorem. The second assertion in (a) is clear from Corollary 1. Since a double coset \( H x P' \) in \( H P'' \) is closed (resp. open) in \( H P'' \) if and only if \( H x P' \) contains a closed (resp. open) double coset \( H y P \) in \( H P'' \), and since the same holds for \( H' \), the assertions in (b) are clear from (a). q.e.d.

REMARK. Let \( a^e \) be a \( \sigma \)-stable maximal abelian subspace of \( p \) such that \( a^e \cap q \) is maximal abelian in \( p \cap q \) and let \( \Sigma(a^e)^+ \) be a \( \sigma \)-compatible positive system of \( \Sigma(a^e) \).
Then \( \Psi^e = \Psi(a^e, \Sigma(a^e)^+) \) is contained in an open \( H \)-orbit on \( G / P \).
Let \( \Psi^o \) be a parabolic subalgebra of \( g \) containing \( \Psi^e \) and \( W_\Psi^o \) the subgroup of \( W(a^e) \) corresponding to \( \Psi^o \).
Then it follows easily from Theorem and [3], Proposition 1 that there is a one-to-one correspondence between the set of open double cosets in \( H \bs G / P^o \) and

\[
W_{K \cap H}(a^e) / W(a^e) / W_\Psi(a^e) \cap W_\Psi^o.
\]
where \( W_\sigma(a^\sigma) = \{ w \in W(a^\sigma) \mid w\sigma = \sigma w \} \). This fact is also proved in [4], Corollary 16.

Let \( a^\sigma \) be a \( \sigma \)-stable maximal abelian subspace of \( \mathfrak{p} \) such that \( a^\sigma \cap \mathfrak{h} \) is maximal abelian in \( \mathfrak{p} \cap \mathfrak{h} \) and let \( \Sigma(a^\sigma)^+ \) be a \( \sigma \)-compatible positive system of \( \Sigma(a^\sigma) \). Let \( \mathfrak{p}'^\sigma \) be a parabolic subalgebra of \( \mathfrak{g} \) containing \( \mathfrak{p}^\sigma = \mathfrak{p}(a^\sigma, \Sigma(a^\sigma)^+) \) and \( W_\mathfrak{p}' \), the subgroup of \( W(a^\sigma) \) corresponding to \( \mathfrak{p}'^\sigma \). Then there is a one-to-one correspondence between the set of closed double cosets in \( \mathcal{H}/G/P' \) and

\[
W_{\mathcal{K}\mathcal{H}}(a^\sigma)/W_\mathfrak{p}^\sigma(a^\sigma) \cap W_\mathfrak{p}'^\sigma,
\]

where \( W_\mathfrak{p}(a^\sigma) = \{ w \in W(a^\sigma) \mid w\sigma = \sigma w \} \) (Theorem and [3], Proposition 2).

In the following we shall give an explicit formula for the decomposition \( \mathcal{H}/\mathcal{H}P'/P \) applying the method used in §2 of [3]. Let \( a_0 \) be a \( \sigma \)-stable maximal abelian subspace of \( \mathfrak{p} \) such that \( a_0 \supseteq a^\sigma \) and that \( m'^\sigma \cap a_0 \cap \mathfrak{h} \) is maximal abelian in \( m'^\sigma \cap \mathfrak{p} \cap \mathfrak{h} \). Such a subspace \( a_0 \) of \( \mathfrak{p} \) is constructed as follows. Let \( a_0^\sigma \) be a maximal abelian subspace of \( m'^\sigma \cap \mathfrak{p} \cap \mathfrak{h} \) and \( a_0'^\sigma \) a maximal abelian subspace of \( m'' \cap \mathfrak{p} \) containing \( a_0^\sigma \). Then \( a_0 = a_0'^\sigma + \mathfrak{a}'' + a'^\sigma \) is a desired one. By [3], p. 341, Lemma 7, all the maximal abelian subspaces \( a'' \) of \( m'' \cap \mathfrak{p} \cap \mathfrak{h} \) such that \( \mathfrak{a}'' \cap \mathfrak{h} \) is maximal abelian in \( m'' \cap \mathfrak{p} \cap \mathfrak{h} \) are mutually \( (M'' \cap H)_0 \)-conjugate. Thus the choice of \( a_0 \) is unique up to \( (M'' \cap H)_0 \)-conjugacy. Fix a positive system \( \Sigma(a_0)^+ \) of \( \Sigma(a_0) \) such that \( \langle \Sigma(a_0)^+, a_i^\sigma \rangle \subseteq \mathbb{R}_+ \). Then \( \mathfrak{p}(0) = \mathfrak{p}(a_0, \Sigma(a_0)^+) \) is contained in \( \mathfrak{p}' \). Let \( P(0) \) be the corresponding minimal parabolic subgroup of \( G \).

Let \( \bar{a} \) be a \( \sigma \)-stable maximal abelian subspace of \( \mathfrak{p} \) such that \( \bar{a} \cap \mathfrak{h} \) is maximal abelian in \( \mathfrak{p} \cap \mathfrak{h} \), \( \bar{a} \supseteq a_0 \cap \mathfrak{h} \) and \( \bar{a} \cap \mathfrak{q} \subseteq a_0 \cap \mathfrak{q} \). The existence of such a subspace \( \bar{a} \) of \( \mathfrak{p} \) is an easy consequence of [3], p. 342, Lemma 8. Put \( \tau = \{ Y \in \bar{a} \cap \mathfrak{h} \mid B(Y, a_0 \cap \mathfrak{h}) = \{0\} \} \). Then \( \bar{a} \cap \mathfrak{h} = a_0 \cap \mathfrak{h} + \tau \) (direct sum).

Put \( \Sigma_\mathfrak{h}(a_0)^+ = \{ z \in \Sigma(a_0)^+ \mid H_z \in m'' \cap a_0 \cap \mathfrak{h} \} \) where \( H_z \in a_0 \) is defined by \( B(H_z, Y) = \alpha(Y) \) for all \( Y \in a_0 \). Then a set of root vectors \( Q = \{ X_1, \ldots, X_n \} \) is said to be a \( \varphi \)-orthogonal system of \( \Sigma_\mathfrak{h}(a_0)^+ \) if the following two conditions are satisfied:

(i) \( \alpha_i \in \Sigma_\mathfrak{h}(a_0)^+ \) and \( [X_{a_i}, X_{a_j}, \theta X_{a_i}] = 0 \) for \( i = 1, \ldots, k \).

(ii) \( [X_{a_i}, X_{a_j}] = [X_{a_i}, \theta X_{a_j}] = 0 \) for \( i \neq j \).

We normalize \( X_{a_i} \) \( i = 1, \ldots, k \) so that \( 2\alpha_i(H_{a_i})B(X_{a_i}, \theta X_{a_i}) = -1 \). Define an element \( c(Q) \) of \( M_0 \) by

\[
c(Q) = \exp (\pi/2)(X_{a_1} + \theta X_{a_1}) \cdots \exp (\pi/2)(X_{a_k} + \theta X_{a_k}).
\]

Then \( a_1^\sigma = \text{Ad}(c(Q))a_0 \) is a \( \sigma \)-stable maximal abelian subspace of \( \mathfrak{p} \) such that \( a_1^\sigma \supseteq a'^\sigma \).

Let \( \{ Q_0, \ldots, Q_n \} \) \( (Q_0 = \phi) \) be a complete set of representatives of \( \varphi \)-orthogonal
systems of $\Sigma_0(a_0)$ with respect to the following equivalence relation $\sim$. For two $q$-orthogonal systems $Q = \{X_1, \ldots, X_m\}$ and $Q' = \{X_1', \ldots, X_m'\}$ of $\Sigma_0(a_0)$, $Q \sim Q'$ if and only if there exists a $w \in W_{K \cap H}(\mathfrak{a}) = N_{K \cap H}(\mathfrak{a})/Z_{K \cap H}(\mathfrak{a})$ such that 

$$w(r + \sum_{j=1}^k H_{a_j}) = r + \sum_{j=1}^k H_{a_j}.$$ 

Put $a_i = \text{Ad}(c(Q_i))a_0$, $i = 1, \ldots, n$. Then the following is a trivial consequence of Theorem in this paper, Corollary 1 of Theorem 1 in [3] (Proposition in § 1) and Theorem 2 in [3].

**Corollary 3.** $HP' = \bigcup_{i=0}^n \bigcup_{j=1}^{m(i)} Hw_j^i c(Q_i)P(0)$ (disjoint union) where $\{w_1^i, \ldots, w_m^i\}$ is a complete set of representatives of $W_{K \cap H}(a_i) \cap W(a_m) \backslash W(a_m)$ in $N_{K \cap M}(a_i)$. Moreover we have 

$$H'P' = \bigcup_{i=0}^n \bigcup_{j=1}^{m(i)} H'w_j^i c(Q_i)P(0)$$ (disjoint union).

**Example 1.** Suppose that $G = G_1 \times G_1$ where $G_1$ is a connected real semi-simple Lie group with Lie algebra $\mathfrak{g}_1$ and that $H = AG = \{(x, x) \in G \times G \mid x \in G_1\}$. Let $\mathfrak{g}_1 = \mathfrak{t}_1 + \mathfrak{p}_1$ be a Cartan decomposition of $\mathfrak{g}_1$ and put $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_1$. Then a $\sigma$-stable maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is of the form $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_1$ where $\mathfrak{a}_1$ is a maximal abelian subspace of $\mathfrak{p}_1$. Let $\mathfrak{t}^0$ be a minimal parabolic subalgebra of $\mathfrak{g}$ of the form $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{p}_1$ where $\Phi = \Phi(\mathfrak{a}_1, \Sigma(\mathfrak{a}_1))$ for some positive system $\Sigma(\mathfrak{a}_1)$. Then there is a one-to-one correspondence 

$$\Delta W(\mathfrak{a}_1)\backslash W(\mathfrak{a}_1) \times W(\mathfrak{a}_1) \xrightarrow{\sim} H\backslash G/P^0$$ 

which is induced by the map $(w_1, w_2) \mapsto \text{Ad}(w_1)\Phi_1 + \text{Ad}(w_2)\Phi_1$ where $\Delta W(\mathfrak{a}_1) = \{(w_1, w_2) \in W(\mathfrak{a}_1) \times W(\mathfrak{a}_1) \mid w \in W(\mathfrak{a}_1)\}$. If we identify $H\backslash G$ with $G$ by the map $(x, y) \mapsto x^{-1}y (x, y \in G_1)$, the decomposition $H\backslash G/P^0$ is equivalent to the Bruhat decomposition 

$$P_1 \backslash G_1 \times P_1 \cong W(\mathfrak{a}_1).$$ 

Fix $(w_1, w_2) \in W(\mathfrak{a}) = W(\mathfrak{a}_1) \times W(\mathfrak{a}_1)$ and put $\mathfrak{p} = \text{Ad}(w_1)\Phi_1 + \text{Ad}(w_2)\Phi_1$. Let $\mathfrak{p}^0 = \Phi_1 + \Phi_1'$ be an arbitrary parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}^0$ and let $W_{\Phi_1}$ and $W_{\Phi_1'}$ be the subgroups of $W(\mathfrak{a}_1)$ corresponding to $\Phi_1$ and $\Phi_1'$ respectively. The parabolic subalgebra $\mathfrak{p}' = \text{Ad}(w_1)\Phi_1 + \text{Ad}(w_2)\Phi_1'$ contains $\mathfrak{p}$ and then $W(\mathfrak{a}) = W_1, W_2, w_1, w_2$ contains $\mathfrak{p}$ and then $W(\mathfrak{a}_m) = w_1 W_{\Phi_1} w_1^{-1} \times w_2 W_{\Phi_1} w_2^{-1}$. Thus the minimal parabolic subalgebras of $\mathfrak{g}$ given in Theorem are of the form $\text{Ad}(w_1 w_1')\Phi_1 + \text{Ad}(w_2 w_2')\Phi_1$ where $w_1' \in W_{\Phi_1}, w_2' \in W_{\Phi_1'}$. Hence there is a bijection 

$$\Delta W(\mathfrak{a}_1)\backslash W(\mathfrak{a}_1) \times W(\mathfrak{a}_1)/W_{\Phi_1} \times W_{\Phi_1} \xrightarrow{\sim} H\backslash G/P^0.'$$ 

If we identify $H\backslash G$ with $G_1$, the above decomposition $H\backslash G/P^0$ is equivalent to the well-known decomposition.
$P'_1 \backslash G_1 / P'_1 \cong W_{g_1} \backslash W(a_1) / W_{g_1}$.

Example 2 ([5], p. 29, Lemma 5.2). Let $G$ be a connected complex semi-simple Lie group and $\sigma$ a complex linear involution of $G$. Then $H$ is a complex subgroup of $G$. A Cartan involution $\theta$ is a conjugation of $g$ with respect to a compact real form $\mathfrak{f}$ of $g$ and $p = (-1)^{1/2}\mathfrak{f}$. Let $a$ be a $\sigma$-stable maximal abelian subspace of $p$ and $\Sigma(a)^+$ a positive system of $\Sigma(a)$. Then $\Psi = \Psi(a, \Sigma(a)^+)$ is a Borel subalgebra of $g$. Let $\Psi'$ be a parabolic subalgebra of $g$ corresponding to a simple root $\alpha$ of $\Sigma(a)^+$. Then the simple root $\alpha$ is called (i) compact imaginary if $g(\alpha; \alpha) \subset \mathfrak{h}$, (ii) non-compact imaginary if $g(\alpha; \alpha) \subset \mathfrak{q}$, (iii) real if $\sigma \alpha = -\alpha$ and (iv) complex if $\sigma \alpha \neq \pm \alpha$. In [5], $H \backslash H P' / P = H \backslash G / P$ is determined in each case (i)~(iv). Therefore $f^{-1}(f(\theta))$ is determined for an arbitrary $\theta \in H \backslash G / P$ if $P'$ is a parabolic subgroup of $G$ corresponding to a simple root.

References


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