HIROSHIMA MATH. J.
12 (1982), 529–541

An analogue of Peter-Weyl theorem for the infinite dimensional unitary group

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(Received April 15, 1982)

Introduction

In the previous paper [6] we proved an analogue of the Peter-Weyl theorem for the infinite dimensional rotation group $O(\boldsymbol{E})$ (for the definition see [6]). In this paper we shall prove the analogue of Peter-Weyl theorem for $U(\boldsymbol{E})$ (for the definition see § 1). In the paper [6], for a proof of the irreducibility of the representation $\pi_{n,\rho}$ we used the results of A. M. Vershik, I. M. Gel'fand and M. I. Graev [9] in which they used the results of A. A. Kirillov [4]. In this paper we shall prove the irreducibility of the representation $\pi_{p,q,\rho,\delta}$, where the method of proof also works for $\pi_{n,\rho}$. Thus the arguments in our previous papers [5] and [6] are complete in our framework. It is plausible that the representations $U_{p,q,\rho,\delta}$ of the group of diffeomorphisms on a compact riemannian manifold would be able to be constructed in such a way that they should correspond to the representations $U_{n,\rho}$ (for the definition see [6]) given by A. M. Vershik, I. M. Gel'fand and M. I. Graev.

§ 1. Irreducibility of representations $\pi_{p,q,\rho,\delta}$

Let M be a compact riemannian manifold. We denote by $C^\infty(M, \boldsymbol{R})$ the space of all real valued C^∞-functions on M and $L^2(M, \boldsymbol{R})$ the Hilbert space of all square integrable real valued functions on M. We denote by $C^\infty(M, \boldsymbol{R})^*$ the dual space of $C^\infty(M, \boldsymbol{R})$. Let $C^\infty(M)$ be the space of all complex valued C^∞-functions on M. We denote by $L^2(M)$ the Hilbert space of all square integrable complex valued functions on M. We write $\boldsymbol{E}$, $\boldsymbol{H}$ and $\boldsymbol{E}^*$ instead of $C^\infty(M)$, $L^2(M)$ and $C^\infty(M)^*$ respectively, where $C^\infty(M)^*$ denotes the dual space of $C^\infty(M)$. We denote by $U(\boldsymbol{E})$ the group of all linear homeomorphisms of $\boldsymbol{E}$ which are isometries of $\boldsymbol{H}$. Let $L^2(M \times \cdots \times M)$ be the Hilbert space of all square integrable complex valued functions on $M \times \cdots \times M$ (r-times). We write simply $L^2(M: r)$ instead of $L^2(M \times \cdots \times M)$. Let V be a finite dimensional vector space with an inner product and $L^2(M \times \cdots \times M, V)$ the Hilbert space of all V-valued functions f on $M \times \cdots \times M$ (r-times) such that

$$\|f\|^2 = \int_{M \times \cdots \times M} \|f(u_1, \ldots, u_r)\|_V^2 du_1 \cdots du_r < +\infty.$$

We also write simply $L^2(M: r, V)$ instead of $L^2(M\times\cdots\times M, V)$.

As in the previous paper [6], we shall consider a Gel'fand triple

$$C^\infty(M\times M)\subset L^2(M\times M)\subset C^\infty(M\times M)^*.$$

We can identify $C^\infty(M\times M)$, $L^2(M\times M)$ and $C^\infty(M\times M)^*$ with $\boldsymbol{E}\hat{\otimes}\boldsymbol{E}$, $\boldsymbol{H}\bar{\otimes}\boldsymbol{H}$ and $(\boldsymbol{E}\hat{\otimes}\boldsymbol{E})^*$ respectively, where $\boldsymbol{E}\hat{\otimes}\boldsymbol{E}$ and $\boldsymbol{H}\bar{\otimes}\boldsymbol{H}$ denote the completion of $\boldsymbol{E}\otimes\boldsymbol{E}$ and $\boldsymbol{H}\otimes\boldsymbol{H}$ respectively. Then we get a complex Gaussian measure ν on $(\boldsymbol{E}\hat{\otimes}\boldsymbol{E})^*$ such that for any ζ in $\boldsymbol{E}\hat{\otimes}\boldsymbol{E}$

$$e^{-\|\zeta\|^2}=\int_\Omega K(z;\zeta)d\nu(z),\quad K(z;\zeta)=e^{i\{\langle z,\zeta\rangle+(\langle z,\zeta\rangle)^-\}},$$

where $\Omega=(\boldsymbol{E}\hat{\otimes}\boldsymbol{E})^*$, (for a function f on Ω we often use the notation $(f(z))^-$ instead of the complex conjugation of $f(z)$). Let $\boldsymbol{N}$ be the set of all positive integers. For any p and q in $\boldsymbol{N}\cup\{0\}$, we consider the complex Hermite polynomial;

$$H_{p,q}(t,\bar{t})=(-1)^{p+q}e^{t\bar{t}}\frac{\partial^{p+q}}{\partial\bar{t}^p\partial t^q}e^{-t\bar{t}}\qquad (t\in\boldsymbol{C}).$$

In the following we fix, once for all, an orthonormal basis $\{\xi_j; j\in\boldsymbol{N}\}$ of $L^2(M)$ such that $\xi_j\in C^\infty(M,\boldsymbol{R})$ for any $j\in\boldsymbol{N}$. Then $\{\xi_i\otimes\xi_j;\ i,j\in\boldsymbol{N}\}$ is an orthonormal basis contained in $C^\infty(M,\boldsymbol{R})\hat{\otimes}C^\infty(M,\boldsymbol{R})$. We put

$$\mathfrak{B}_{p,q}=\{\textstyle\prod_{i,j=1}^\infty(p_{ij}!q_{ij}!)^{-1/2}H_{p_{i,j},q_{i,j}}(\langle z,\xi_i\otimes\xi_j\rangle,(\langle z,\xi_i\otimes\xi_j\rangle)^-);$$
$$\textstyle\sum_{i,j=1}^\infty p_{ij}=p,\ \sum_{i,j=1}^\infty q_{ij}=q\}.$$

Then it is known that $\bigcup_{n=0}^\infty(\bigcup_{p+q=n}\mathfrak{B}_{p,q})$ is an orthonormal basis of $L^2(\Omega,\nu)$. We denote by $\mathfrak{H}_{p,q}$ the closed subspace spanned by $\mathfrak{B}_{p,q}$. Then we have

$$L^2(\Omega,\nu)=\textstyle\sum_{n=0}^\infty\oplus\sum_{p+q=n}\oplus\,\mathfrak{H}_{p,q}\qquad \text{(Wiener-Itô decomposition)}.$$

We denote by $P_{p,q}$ the orthogonal projection of $L^2(\Omega,\nu)$ onto $\mathfrak{H}_{p,q}$. We consider the transformation $\mathscr{T}$ defined by

$$(\mathscr{T}f)(\zeta)=\int_\Omega K(z;\zeta)(f(z))^-d\nu(z)\qquad (f\in L^2(\Omega,\nu),\ \zeta\in\boldsymbol{E}\otimes\boldsymbol{E}),$$

(see [2]). And we define a transformation $\mathscr{T}_*$ by

$$(\mathscr{T}_*f)(\zeta)=e^{\|\zeta\|^2}\textstyle\sum_{n=0}^\infty i^{-n}\sum_{p+q=n}\mathscr{T}(P_{p,q}f)(\zeta).$$

Then $\mathscr{T}_*$ is injective. In case $\mathscr{T}_*f=\phi$, we write $f=\phi^\#$.

For any g in $U(\boldsymbol{E})$ we can define linear mappings L_g and R_g of $\boldsymbol{E}\hat{\otimes}\boldsymbol{E}$ into itself by

$$L_g(\xi\otimes\eta)=(g\xi)\otimes\eta,\quad R_g(\xi\otimes\eta)=\xi\otimes(g\eta).$$

We denote by gz and zg the dual actions of $U(\boldsymbol{E})$ on Ω defined by

$$\langle gz, \zeta\rangle = \langle z, L_{g^{-1}}\zeta\rangle, \quad \langle zg, \zeta\rangle = \langle z, R_g\zeta\rangle \qquad (\zeta \in \boldsymbol{E}\hat{\otimes}\boldsymbol{E}).$$

For each g in $U(\boldsymbol{E})$ we shall identify g with the linear form on $\boldsymbol{E}\hat{\otimes}\boldsymbol{E}$ defined by

$$\xi_i \otimes \xi_j \longmapsto (\xi_i, g\xi_j) \qquad (i, j \in \boldsymbol{N}).$$

Thus we regard the group $U(\boldsymbol{E})$ as a subset of Ω. Let $C[X_{ij}, \overline{X}_{kl};\, i, j, k, l \in \boldsymbol{N}]$ be the polynomial ring of infinite variables X_{ij}, $\overline{X}_{kl}$ over $\boldsymbol{C}$. Let $C(\Omega)$ be the set of all continuous functions on Ω. We denote by $C(U(\boldsymbol{E}))$ the set of functions given by the restriction of functions in $C(\Omega)$ to $U(\boldsymbol{E})$. We consider the mapping from $C[X_{ij}, \overline{X}_{kl};\, i, j, k, l \in \boldsymbol{N}]$ to $C(\Omega)$ defined by the map: $F \mapsto f$, where $F((X_{ij}, \overline{X}_{kl})) \in C[X_{ij}, \overline{X}_{kl};\, i, j, k, l \in \boldsymbol{N}]$ and $f(z) = F((\langle z, \xi_i \otimes \xi_j\rangle, (\langle z, \xi_k \otimes \xi_l\rangle)^-))$. We shall denote by $F(\Omega)$ the image of this mapping. We call functions in $F(\Omega)$ polynomials on Ω. We put $F(U(\boldsymbol{E})) = F(\Omega)|_{U(\boldsymbol{E})}$. We also call functions in $F(U(\boldsymbol{E}))$ polynomials on $U(\boldsymbol{E})$. It is easy to see that the restriction mapping is injective. Hence for each polynomial f on $U(\boldsymbol{E})$ there exists a unique polynomial $\tilde{f}$ on Ω such that $f = \tilde{f}|_{U(\boldsymbol{E})}$. In the following we use the same notation f instead of $\tilde{f}$. Since

$$g_{ij} = (g\xi_j, \xi_i) = (\langle g, \xi_i \otimes \xi_j\rangle)^-,$$

g_{ij} can be regarded as a function on $U(\boldsymbol{E})$ which is clearly a restriction to $U(\boldsymbol{E})$) of the function $g_{ij}(z) = (\langle z, \xi_i \otimes \xi_j\rangle)^-$ on Ω. As

$$\begin{aligned}\mathscr{T}_*(\textstyle\prod_{i,j} H_{p_{ij}, q_{ij}}(\langle\cdot, \xi_i\otimes\xi_j\rangle, (\langle\cdot, \xi_i\otimes\xi_j\rangle)^-))(\zeta) &= \textstyle\prod_{i,j} (\xi_i\otimes\xi_j, \zeta)^{p_{ij}}(\zeta, \xi_i\otimes\xi_j)^{q_{ij}} \\ = \textstyle\prod_{i,j} ((g_{ij}(\zeta))^-)^{p_{ij}}(g_{ij}(\zeta))^{q_{ij}} &= (\textstyle\prod_{i,j} (\bar{g}_{ij})^{p_{ij}}(g_{ij})^{q_{ij}})(\zeta),\end{aligned}$$

we have

$$(\textstyle\prod_{i,j} (\bar{g}_{ij})^{p_{ij}}(g_{ij})^{q_{ij}})^{\#}(z) = \textstyle\prod_{i,j} H_{p_{ij}, q_{ij}}(\langle z, \xi_i\otimes\xi_j\rangle, (\langle z, \xi_i\otimes\xi_j\rangle)^-).$$

Let $\mathfrak{S}_r$ be the group of all permutations of $\{1,\dots, r\}$. Let (ρ, V_ρ) and (δ, V_δ) be irreducible unitary finite dimensional representations of $\mathfrak{S}_p$ and $\mathfrak{S}_q$ respectively. We denote by $\hat{\mathfrak{S}}_r$ the set of all equivalence classes of irreducible unitary representations of $\mathfrak{S}_r$. The group $\mathfrak{S}_p \times \mathfrak{S}_q$ acts on $M \times \cdots \times M$ ($(p+q)$-times) on the right by

$$(u_1,\dots, u_p, v_1,\dots, v_q)\cdot(\sigma, \tau) = (u_{\sigma(1)},\dots, u_{\sigma(p)}, v_{\tau(1)},\dots, v_{\tau(q)}).$$

We write simply $u\cdot\sigma$ and $v\cdot\tau$ instead of $(u_{\sigma(1)},\dots, u_{\sigma(p)})$ and $(v_{\tau(1)},\dots, v_{\tau(q)})$ respectively. The right action of $\mathfrak{S}_p \times \mathfrak{S}_q$ induces an action $\tilde{\lambda}(\sigma, \tau)$ on $L^2(M\colon p+q)$ defined by

$$(\tilde{\lambda}(\sigma, \tau)f)(u, v) = f(u\cdot\sigma, v\cdot\tau) \qquad (f \in L^2(M\colon p+q)).$$

For any irreducible unitary representations (ρ, V_ρ) of $\mathfrak{S}_p$ and (δ, V_δ) of $\mathfrak{S}_q$, we denote by $\operatorname{Hom}(V_\delta, V_\rho)$ the space of all linear mappings of V_δ to V_ρ. We give the space $\operatorname{Hom}(V_\delta, V_\rho)$ the natural inner product induced by those of V_ρ and V_δ. We put

$$\mathscr{H}_{p,q,\rho,\delta} = \{f \in L^2(M\colon p + q, \operatorname{Hom}(V_\delta, V_\rho));$$
$$f(u \cdot \sigma, v \cdot \tau) = \rho(\sigma)^{-1} f(u, v)\delta(\tau),\ \sigma \in \mathfrak{S}_p,\ \tau \in \mathfrak{S}_q\}.$$

By the canonical isomorphism ι we have

$$L^2(M\colon p+q) \cong L^2(M)\bar{\otimes}\cdots\bar{\otimes}L^2(M) \qquad ((p+q)\text{-times}).$$

For any g in $U(\boldsymbol{E})$ we define a unitary operator $\hat{\pi}_{p,q}(g)$ on $L^2(M)\bar{\otimes}\cdots\bar{\otimes}L^2(M)$ by

$$\hat{\pi}_{p,q}(g)(\xi_{i_1}\otimes\cdots\otimes\xi_{i_p}\otimes\xi_{k_1}\otimes\cdots\otimes\xi_{k_q}) = (g\xi_{i_1})\otimes\cdots\otimes(g\xi_{i_p})\otimes(g^*\xi_{k_1})\otimes\cdots\otimes(g^*\xi_{k_q}),$$

where g^* denotes the adjoint operator of g. We denote by $\tilde{\pi}_{p,q}(g)$ the unitary operator on $L^2(M\colon p+q)$ which corresponds to $\hat{\pi}_{p,q}(g)$. For any σ in $\mathfrak{S}_p$ and τ in $\mathfrak{S}_q$ we define the action $\lambda(\sigma)\otimes\lambda(\tau)$ on $L^2(M)\bar{\otimes}\cdots\bar{\otimes}L^2(M)$ by

$$(\lambda(\sigma)\otimes\lambda(\tau))(\xi_{i_1}\otimes\cdots\otimes\xi_{i_p}\otimes\xi_{k_1}\otimes\cdots\otimes\xi_{k_q}) = \xi_{i_{\sigma(1)}}\otimes\cdots\otimes\xi_{i_{\sigma(p)}}\otimes\xi_{k_{\tau(1)}}\otimes\cdots\otimes\xi_{k_{\tau(q)}}.$$

We denote by $\tilde{\pi}_{p,q}\otimes I$ the unitary representation of $U(\boldsymbol{E})$ on $L^2(M\colon p+q)\otimes \operatorname{Hom}(V_\delta, V_\rho)$, where I denotes the identity operator on $\operatorname{Hom}(V_\delta, V_\rho)$. Using the canonical isomorphism we have

$$L^2(M\colon p+q, \operatorname{Hom}(V_\delta, V_\rho)) \cong L^2(M\colon p+q) \otimes \operatorname{Hom}(V_\delta, V_\rho).$$

Hence we obtain the unitary representation $\tilde{\pi}_{p,q,\rho,\delta}$ of $U(\boldsymbol{E})$ on $L^2(M\colon p+q, \operatorname{Hom}(V_\delta, V_\rho))$ which corresponds to the representation $\tilde{\pi}_{p,q}\otimes I$. Since $\tilde{\pi}_{p,q}(g)\otimes I$ commutes with $\tilde{\lambda}(\sigma, \tau)\otimes I$, $\mathscr{H}_{p,q,\rho,\delta}$ is $\tilde{\pi}_{p,q,\rho,\delta}(U(\boldsymbol{E}))$-invariant. So that we get the subrepresentation $\pi_{p,q,\rho,\delta}$ of $U(\boldsymbol{E})$ on $\mathscr{H}_{p,q,\rho,\delta}$. We put $(M\times\cdots\times M)' = \{(u_1,\dots, u_r) \in M\times\cdots\times M;\ u_m \neq u_n\ (m\neq n)\}$, and write simply $(M\colon r)'$ instead of $(M\times\cdots\times M)'$ (r-times). It is easy to see that there exist open subsets F_p in $M\times\cdots\times M$ (p-times) and F_q in $M\times\cdots\times M$ (q-times) which satisfy the following conditions. The mapping ϕ:

$$F_p\times F_q\times\mathfrak{S}_p\times\mathfrak{S}_q \ni (u, v, \sigma, \tau) \longmapsto (u\cdot\sigma, v\cdot\tau) \in (M\colon p)'\times(M\colon q)'$$

is injective and

$$\phi(F_p\times F_q\times\mathfrak{S}_p\times\mathfrak{S}_q) = (M\colon p)'\times(M\colon q)'.$$

Let $L^2(\mathfrak{S}_r)$ be the space of all functions on $\mathfrak{S}_r$. We introduce an inner product defined by the normalized Haar measure on $\mathfrak{S}_r$. Then by the Peter-Weyl theorem for $\mathfrak{S}_r$, we have

$$L^2(\mathfrak{S}_r) = \sum_\rho V_\rho \otimes V_\rho^*,$$

where $\sum_\rho$ is taken over all ρ in $\hat{\mathfrak{S}}_r$. We note that

$$r! = \dim(L^2(\mathfrak{S}_r)) = \sum_\rho (\dim V_\rho)^2$$

which we need in the proof of Theorem 1. Now we obtain the following

$$\begin{aligned} L^2(M\colon p+q) &\cong L^2(F_p \times F_q \times \mathfrak{S}_p \times \mathfrak{S}_q) \cong (L^2(F_p)\otimes L^2(\mathfrak{S}_p))\bar{\otimes}(L^2(F_q)\otimes L^2(\mathfrak{S}_q)) \\ &\cong (\sum_\rho L^2(F_p)\otimes V_\rho \otimes V_\rho^*)\bar{\otimes}(\sum_\delta L^2(F_q)\otimes V_\delta \otimes V_\delta^*) \\ &\cong \sum_\rho \sum_\delta (L^2(F_p)\bar{\otimes} L^2(F_q)\otimes V_\rho \otimes V_\delta^*)\otimes V_\rho^* \otimes V_\delta \\ &\cong \sum_\rho \sum_\delta L^2(F_p \times F_q, \mathrm{Hom}(V_\delta, V_\rho))\otimes V_\rho^* \otimes V_\delta. \end{aligned}$$

Hence we get

$$L^2(M\colon p+q) \cong \sum_\rho \sum_\delta \mathscr{H}_{p,q,\rho,\delta} \otimes V_\rho^* \otimes V_\delta.$$

For any d in $\boldsymbol{N}$ we put $\boldsymbol{N}_d = \{1,\dots, d\}$, $(\boldsymbol{N}_d \times \boldsymbol{N}_d)^r = (\boldsymbol{N}_d \times \boldsymbol{N}_d) \times \cdots \times (\boldsymbol{N}_d \times \boldsymbol{N}_d)$ (r-times). As for elements in $(\boldsymbol{N}_d \times \boldsymbol{N}_d)^p \times (\boldsymbol{N}_d \times \boldsymbol{N}_d)^q$ we often use the symbolical expression $((i, j), (k, l))$ instead of $((i_1, j_1),\dots, (i_p, j_p), (k_1, l_1),\dots, (k_q, l_q))$. $\mathfrak{S}_p$ and $\mathfrak{S}_q$ act on $(\boldsymbol{N}_d \times \boldsymbol{N}_d)^p \times (\boldsymbol{N}_d \times \boldsymbol{N}_d)^q$ on the right by

$$((i\cdot\sigma, j\cdot\sigma), (k\cdot\tau, l\cdot\tau)) = ((i_{\sigma(1)}, j_{\sigma(1)}),\dots, (i_{\sigma(p)}, j_{\sigma(p)}), (k_{\tau(1)}, l_{\tau(1)}),\dots, (k_{\tau(q)}, l_{\tau(q)}),$$

where $\sigma \in \mathfrak{S}_p$ and $\tau \in \mathfrak{S}_q$. We denote by $Z_d(r)$ the set of all $d \times d$ matrices $\alpha = (\alpha_{mn})$ which satisfy the following conditions;

$$\alpha_{mn} \in \boldsymbol{N} \cup \{0\} \quad (m, n \in \boldsymbol{N}_d), \qquad \sum_{m,n=1}^d \alpha_{mn} = r.$$

For each $(i, j) = ((i_1, j_1),\dots, (i_r, j_r)) \in (\boldsymbol{N}_d \times \boldsymbol{N}_d)^r$ we assign an element $\alpha = (\alpha_{mn}) \in Z_d(r)$ by the following rule and we put $T_r(i, j) = \alpha$. α_{mn} is the number of the components (i_s, j_s) $(s = 1,\dots, r)$ such that $i_s = m$ and $j_s = n$. And we define the mapping $T_p \times T_q$ from $(\boldsymbol{N}_d \times \boldsymbol{N}_d)^p \times (\boldsymbol{N}_d \times \boldsymbol{N}_d)^q$ to $Z_d(p) \times Z_d(q)$ by $(T_p \times T_q)((i, j), (k, l)) = (T_p(i, j), T_q(k, l))$. In case $T_r(i, j) = T_r(i', j')$ we write $(i, j) \sim (i', j')$. It is easy to show the following lemma.

LEMMA 1. *$(i, j) \sim (i', j')$ holds if and only if there exists a σ in $\mathfrak{S}_r$ such that $(i', j') = (i\cdot\sigma, j\cdot\sigma)$, where $(i, j), (i', j') \in (\boldsymbol{N}_d \times \boldsymbol{N}_d)^r$.*

From this lemma we see that the number of elements of $T_r^{-1}(\alpha)$ is equal to $r!(\prod_{m,n=1}^d \alpha_{mn}!)^{-1}$. We put $\boldsymbol{N}^r = \boldsymbol{N} \times \cdots \times \boldsymbol{N}$ (r-times). For any i in $\boldsymbol{N}^p$ and k in $\boldsymbol{N}^q$ we write simply $\xi_i \otimes \xi_k$ instead of $\xi_{i_1} \otimes \cdots \otimes \xi_{i_p} \otimes \xi_{k_1} \otimes \cdots \otimes \xi_{k_q}$ in $L^2(M)\bar{\otimes}\cdots\bar{\otimes}L^2(M)$. Then for any g in $U(\boldsymbol{E})$. and for any j in $\boldsymbol{N}^p$ and l in $\boldsymbol{N}^q$, we have

$$\begin{aligned} \hat{\pi}_{p,q}(g)(\xi_j \otimes \xi_l) &= (g\xi_j)\otimes(g^*\xi_l) = g\xi_{j_1}\otimes\cdots\otimes g\xi_{j_p}\otimes g^*\xi_{l_1}\otimes\cdots\otimes g^*\xi_{l_q} \\ &= \sum_{i,k}(g_{i_1 j_1}\cdots g_{i_p j_p}\bar{g}_{k_1 l_1}\cdots \bar{g}_{k_q l_q})\xi_i \otimes \xi_k, \end{aligned}$$

where $\sum_{i,k}$ means $\sum_{i_1=1}^{\infty}\cdots\sum_{i_p=1}^{\infty}\sum_{k_1=1}^{\infty}\cdots\sum_{k_q=1}^{\infty}$. In the following we write symbolically $g_{ij}\bar{g}_{kl}$ instead of $g_{i_1j_1}\cdots g_{i_pj_p}\bar{g}_{k_1l_1}\cdots\bar{g}_{k_ql_q}$. For any i, j in $\boldsymbol{N}^p$ and k, l in $\boldsymbol{N}^q$, we denote by $\max\{i, j, k, l\}$ a maximum number included in the set $\{i_1,\dots, i_p, j_1,\dots, j_p, k_1,\dots, k_q, l_1,\dots, l_q\}$. We put $d=\max\{i, j, k, l\}$, $T_p(i, j)=\alpha$ and $T_q(k, l)=\beta$, where $\alpha\in Z_d(p)$ and $\beta\in Z_d(q)$. Since $(g_{ij}\bar{g}_{kl})^\#(z)=\prod_{m,n=1}^{d} H_{\alpha_{mn},\beta_{mn}}(\langle z, \xi_m\otimes\xi_n\rangle, (\langle z, \xi_m\otimes\xi_n\rangle)^-)$, the following lemma is obvious.

LEMMA 2. *For any i, j, i', j' in $\boldsymbol{N}^p$ and for any k, l, k', l' in $\boldsymbol{N}^q$ let $d=\max\{i, j, k, l, i', j', k', l'\}$, $(\alpha, \beta)=(T_p(i, j), T_q(k, l))$ and $(\alpha', \beta')=(T_p(i', j'), T_q(k', l'))$. Then it follows that*

$$\int_\Omega (g_{ij}\bar{g}_{kl})^\#(z)((g_{i'j'}\bar{g}_{k'l'})^\#(z))^- dv(z)$$
$$=\begin{cases} \prod_{m,n=1}^{d}(\alpha_{mn}!\beta_{mn}!\alpha'_{mn}!\beta'_{mn}!) & ((i, j)\sim(i', j') \text{ and } (k, l)\sim(k', l')), \\ 0 & ((i, j)\nsim(i', j') \text{ or } (k, l)\nsim(k', l')). \end{cases}$$

THEOREM 1. 1) *If ρ and δ are both irreducible, then $(\pi_{p,q,\rho,\delta}, \mathscr{H}_{p,q,\rho,\delta})$ is irreducible.*

2) *Two representations $(\pi_{p,q,\rho,\delta}, \mathscr{H}_{p,q,\rho,\delta})$ and $(\pi_{p',q',\rho',\delta'}, \mathscr{H}_{p',q',\rho',\delta'})$ are equivalent if and only if $p=p'$, $q=q'$, $\rho\simeq\rho'$, $\delta\simeq\delta'$.*

PROOF. We denote by $\mathrm{Hom}_{U(\boldsymbol{E})}(L^2(M\colon p+q), L^2(M\colon p+q))$ the space of all intertwining operators on $L^2(M\colon p+q)$. Using the isometry $\iota\colon L^2(M\colon p+q)\to L^2(M)\bar{\otimes}\cdots\bar{\otimes}L^2(M)$ $((p+q)$-times), for any $\tilde{A}$ in $\mathrm{Hom}_{U(\boldsymbol{E})}(L^2(M\colon p+q), L^2(M\colon p+q))$ there exists an operator A on $L^2(M)\bar{\otimes}\cdots\bar{\otimes}L^2(M)$ such that $A\cdot\iota=\iota\cdot\tilde{A}$. We put

$$A(\xi_i\otimes\xi_k)=\sum_{j,l}a_{jl}^{ik}\xi_j\otimes\xi_l,$$

where $i, j\in\boldsymbol{N}^p$ and $k, l\in\boldsymbol{N}^q$. For any g in $U(\boldsymbol{E})$ by definition of $\hat{\pi}_{p,q}(g)$ we have $A\cdot\hat{\pi}_{p,q}(g)=\hat{\pi}_{p,q}(g)\cdot A$. Then we have

$$\begin{aligned}\hat{\pi}_{p,q}(g)A(\xi_i\otimes\xi_k) &= \hat{\pi}_{p,q}(g)(\textstyle\sum_{j,l}a_{jl}^{ik}\xi_j\otimes\xi_l)\\ &= \textstyle\sum_{j,l}a_{jl}^{ik}(g\xi_j\otimes g^*\xi_l)=\sum_{s,t}\sum_{j,l}a_{jl}^{ik}g_{sj}\bar{g}_{tl}\xi_s\otimes\xi_t,\end{aligned}$$

where $s\in\boldsymbol{N}^p$ and $t\in\boldsymbol{N}^q$. On the other hand we have

$$\begin{aligned}A\hat{\pi}_{p,q}(g)(\xi_i\otimes\xi_k) &= A(g\xi_i\otimes g^*\xi_k)=A(\textstyle\sum_{m,n}g_{mi}\bar{g}_{nk}\xi_m\otimes\xi_n)\\ &= \textstyle\sum_{s,t}\sum_{m,n}a_{st}^{mn}g_{mi}\bar{g}_{nk}\xi_s\otimes\xi_t,\end{aligned}$$

where $m\in\boldsymbol{N}^p$ and $n\in\boldsymbol{N}^q$. Since $A\cdot\hat{\pi}_{p,q}(g)=\hat{\pi}_{p,q}(g)\cdot A$, we conclude that for any s and t

$$\sum_{j,l}a_{jl}^{ik}g_{sj}\bar{g}_{tl}=\sum_{m,n}a_{st}^{mn}g_{mi}\bar{g}_{nk}.$$

Fix any j' in $\boldsymbol{N}^p$ and l' in $\boldsymbol{N}^q$ then we have

$$\int_{\Omega}(\sum_{j,l} a_{jl}^{ik} g_{sj}\bar{g}_{tl})^{\#}(z)((g_{sj'}\bar{g}_{tl'})^{\#}(z))^{-}dv(z)$$
$$=\int_{\Omega}(\sum_{m,n} a_{st}^{mn} g_{mi}\bar{g}_{nk})^{\#}(z)((g_{sj'}\bar{g}_{tl'})^{\#}(z))^{-}dv(z).$$

For any $s=(s_1,\dots, s_p)$ and $t=(t_1,\dots, t_q)$ such that $s_h \neq s_{h'}$ $(h \neq h')$ and $t_h \neq t_{h'}$ $(h \neq h')$, from Lemma 2, we get

$$a_{j'l'}^{ik}=\int_{\Omega}(\sum_{j,l} a_{jl}^{ik} g_{sj}\bar{g}_{tl})^{\#}(z)((g_{sj'}\bar{g}_{tl'})^{\#}(z))^{-}dv(z)$$
$$=\int_{\Omega}(\sum_{m,n} a_{st}^{mn} g_{mi}\bar{g}_{nk})^{\#}(z)((g_{sj'}\bar{g}_{tl'})^{\#}(z))^{-}dv(z) = \sum_{m,n}^{1} a_{st}^{mn},$$

where $\sum_{m,n}^{1}$ means the summation which is taken over m and n such that $(m, i)\sim(s, j')$ and $(n, k)\sim(t, l')$. From Lemma 1 this implies that $a_{j'l'}^{ik}=0$ unless there exist σ and τ such that $j'=i\cdot\sigma$ and $l'=k\cdot\tau$. Thus we obtain

$$a_{j'l'}^{ik} = \sum_{\sigma,\tau}^{2} a_{st}^{(s\cdot\sigma^{-1})(t\cdot\tau^{-1})} = \sum_{\sigma,\tau}^{2} a_{(s\cdot\sigma)(t\cdot\tau)}^{st},$$

where $\sum_{\sigma,\tau}^{2}$ means the summation which is taken over σ and τ such that $j'=i\cdot\sigma$ and $l'=k\cdot\tau$. Now we assume that $i=(i_1,\dots, i_p)$ and $k=(k_1,\dots, k_q)$ satisfy the following conditions;

$$i_h \neq i_{h'} \quad (h \neq h') \quad \text{and} \quad k_h \neq k_{h'} \quad (h \neq h').$$

Then we have

$$a_{(i\cdot\sigma)(k\cdot\tau)}^{ik} = a_{(s\cdot\sigma)(t\cdot\tau)}^{st},$$

so that we can write

$$a_{(i\cdot\sigma)(k\cdot\tau)}^{ik} = a_{\sigma,\tau}.$$

Thus for any i and k we get

$$a_{jl}^{ik} = \sum_{\sigma,\tau}^{3} a_{\sigma,\tau},$$

where $\sum_{\sigma,\tau}^{3}$ means the summation which is taken over σ and τ such that $j=i\cdot\sigma$ and $l=k\cdot\tau$. It follows that

$$A(\xi_i\otimes\xi_k) = \sum_{j,l} a_{jl}^{ik}\xi_j\otimes\xi_l = \sum_{\sigma,\tau} a_{\sigma,\tau}\xi_{i\cdot\sigma}\otimes\xi_{k\cdot\tau}.$$

Hence we conclude that

$$A = \sum_{\sigma,\tau} a_{\sigma,\tau}\lambda(\sigma)\otimes\lambda(\tau),$$

where $\sum_{\sigma,\tau}$ is taken over all σ in $\mathfrak{S}_p$ and τ in $\mathfrak{S}_q$. We denote by $\mathscr{I}_{p,q}$ the space of all operators on $L^2(M)\bar{\otimes}\cdots\bar{\otimes}L^2(M)$ ($(p+q)$-times) spanned by the set $\{\lambda(\sigma)\otimes$

$\lambda(\tau)$; $\sigma \in \mathfrak{S}_p$, $\tau \in \mathfrak{S}_q\}$ over $\boldsymbol{C}$. Then for any g in $U(\boldsymbol{E})$ it is clear that $A \cdot \hat{\pi}_{p,q}(g) = \hat{\pi}_{p,q}(g) \cdot A$. Thus we obtain

$$\mathrm{Hom}_{U(\boldsymbol{E})}(L^2(M\colon p+q), L^2(M\colon p+q)) \cong \mathscr{I}_{p,q}.$$

This implies that

$$\dim \mathrm{Hom}_{U(\boldsymbol{E})}(L^2(M\colon p+q), L^2(M\colon p+q)) = p!q!.$$

We remark that $L^2(M\colon p+q) = \sum_\rho \sum_\delta \mathscr{H}_{p,q,\rho,\delta} \otimes V_\rho^* \otimes V_\delta$, and that $\sum_\rho \sum_\delta (\dim V_\rho^* \otimes V_\delta)^2 = (\sum_\rho (\dim V_\rho)^2)(\sum_\delta (\dim V_\delta)^2) = p!q!$. Now the assertion of the theorem follows immediately from the following lemma.

LEMMA 3. *Let* $(\pi, \mathscr{H})$ *be a unitary representation (of a group* G*) such that* $\mathscr{H}$ *is a direct sum of closed invariant irreducible subspaces. Suppose that* $\mathscr{H} = \sum_{k=1}^s m_k W_k$ *(orthogonal decomposition) where* $m_k W_k = W_k + \cdots + W_k$ *(*m_k*-times) and* W_k $(k=1,\dots,s)$ *are closed invariant subspaces. Further assume that* $\dim \mathrm{Hom}_G(\mathscr{H}, \mathscr{H}) = \sum_{k=1}^s (m_k)^2$. *Then* W_k $(k=1,\dots,s)$ *are irreducible and* W_k *is equivalent to* $W_{k'}$ *if and only if* k *is equal to* k'.

PROOF. Let $V_1, \dots, V_l$ be the representatives of irreducible subspaces which occur in $\mathscr{H}$. We put

$$W_k = \sum_{i=1}^l n_i^k V_i,$$

where $n_i^k \in \boldsymbol{N} \cup \{0\}$. Then we have

$$\mathscr{H} = \sum_{i=1}^l \left(\sum_{k=1}^s m_k n_i^k\right) V_i.$$

Thus we get

$$\begin{aligned} \sum_{i=1}^l \left(\sum_{k=1}^s m_k n_i^k\right)^2 &= \sum_{i=1}^l \sum_{k_1=1}^s \sum_{k_2=1}^s m_{k_1} m_{k_2} (n_i^{k_1})(n_i^{k_2}) \\ &= \dim \mathrm{Hom}_G(\mathscr{H}, \mathscr{H}) = \sum_{k=1}^s (m_k)^2. \end{aligned}$$

Since m_k $(k=1,\dots,s)$ are positive integers, we have the following

$$\sum_{i=1}^l (n_i^{k_1})^2 = 1 \qquad (k_1 = k_2), \quad \sum_{i=1}^l (n_i^{k_1})(n_i^{k_2}) = 0 \qquad (k_1 \neq k_2).$$

Hence l is equal to s and W_k $(k=1,\dots,s)$ are irreducible and W_k is not equivalent to $W_{k'}$ $(k \neq k')$.

§ 2. Peter-Weyl theorem for $U(\boldsymbol{E})$

We denote by $L^2(M \times M\colon p+q)^\wedge$ the Hilbert space of all square integrable functions F on $(M \times M) \times \cdots \times (M \times M)$ ($(p+q)$-times) such that for any σ in $\mathfrak{S}_p$ and τ in $\mathfrak{S}_q$

$$F((u^1_{\sigma(1)}, u^2_{\sigma(1)}),\dots, (u^1_{\sigma(p)}, u^2_{\sigma(p)}), (v^1_{\tau(1)}, v^2_{\tau(1)}),\dots, (v^1_{\tau(q)}, v^2_{\tau(q)}))$$
$$= F((u^1_1, u^2_1),\dots, (u^1_p, u^2_p), (v^1_1, v^2_1),\dots, (v^1_q, v^2_q)).$$

For any f in $\mathfrak{H}_{p,q}$ there exists a unique F in $L^2(M\times M\colon p+q)^\wedge$ such that

$$(\mathscr{T}_* f)(\zeta) = \int_{(M\times M)\times\cdots\times(M\times M)} F((u^1_1, u^2_1),\dots, (u^1_p, u^2_p), (v^1_1, v^2_1),\dots, (v^1_q, v^2_q))$$
$$\times (\zeta(u^1_1, u^2_1))^-\cdots(\zeta(u^1_p, u^2_p))^-\zeta(v^1_1, v^2_1)\cdots\zeta(v^1_q, v^2_q)du^1_1 du^2_1\cdots dv^1_q dv^2_q, \text{ (see [2])}.$$

As is easily seen the measure ν is $U(\boldsymbol{E})$-invariant. For any g in $U(\boldsymbol{E})$ we define

$$(\pi_L(g)f)(z) = f(g^{-1}z), \quad (\pi_R(g)f)(z) = f(zg),$$

where $f\in L^2(\Omega, \nu)$. Then π_L and π_R are unitary representations of $U(\boldsymbol{E})$. For any (g_1, g_2) in $U(\boldsymbol{E})\times U(\boldsymbol{E})$ we put

$$(\omega_*(g_1, g_2)f)(z) = f(g_1^{-1}zg_2).$$

Then ω_* is a unitary representation of $U(\boldsymbol{E})\times U(\boldsymbol{E})$. Clearly $\mathfrak{H}_{p,q}$ is $\omega_*(U(\boldsymbol{E})\times U(\boldsymbol{E}))$-invariant. We obtain a unitary subrepresentation $(\omega_{p,q}, \mathfrak{H}_{p,q})$ of $U(\boldsymbol{E})\times U(\boldsymbol{E})$.

THEOREM 2 (*Peter-Weyl theorem for* $U(\boldsymbol{E})$). *The unitary representation* ω_* *of* $U(\boldsymbol{E})\times U(\boldsymbol{E})$ *is decomposed as follows:*

$$L^2(\Omega, \nu) = \textstyle\sum_{n=0}^\infty \oplus \sum_{p+q=n} \oplus \sum_\rho \sum_\delta \mathscr{H}_{p,q,\rho,\delta} \bar{\oplus} \mathscr{H}^*_{p,q,\rho,\delta},$$

where $\omega_{p,q}(g_1, g_2)$ *corresponds to* $\pi_{p,q,\rho,\delta}(g_1)\otimes\pi^*_{p,q,\rho,\delta}(g_2)$ *for each* (g_1, g_2) *in* $U(\boldsymbol{E})\times U(\boldsymbol{E})$.

PROOF. We put $\mathfrak{H}^\wedge_{p,q} = \{f\in L^2(M\colon p+q)\bar{\otimes} L^2(M\colon p+q); (\tilde\lambda(\sigma,\tau)\otimes\tilde\lambda(\sigma,\tau))f = f, (\sigma,\tau)\in\mathfrak{S}_p\times\mathfrak{S}_q\}$. Then we have the canonical isomorphism $\iota_{p,q}\colon L^2(M\times M\colon p+q)^\wedge \to \mathfrak{H}^\wedge_{p,q}$. As we saw in the previous section, we have

$$L^2(M\colon p+q) \cong \textstyle\sum_\rho \sum_\delta \mathscr{H}_{p,q,\rho,\delta}\otimes V^*_\rho\otimes V_\delta.$$

We remark that the unitary operator $\tilde\lambda(\sigma,\tau)$ corresponds to $I\otimes\rho^*(\sigma)\otimes\delta(\tau)$ where I denotes the identiry operator on $\mathscr{H}_{p,q,\rho,\delta}$. Thus we have

$$\mathfrak{H}_{p,q} \cong \mathfrak{H}^\wedge_{p,q} \cong \{\gamma\in \textstyle\sum_{\rho_1}\sum_{\delta_1}\sum_{\rho_2}\sum_{\delta_2} (\mathscr{H}_{p,q,\rho_1,\delta_1}\otimes V^*_{\rho_1}\otimes V_{\delta_1})\bar{\otimes}(\mathscr{H}_{p,q,\rho_2,\delta_2}\otimes V^*_{\rho_2}\otimes V_{\delta_2});$$
$$(I\otimes\rho_1^*(\sigma)\otimes\delta_1(\tau)\otimes I\otimes\rho_2^*(\sigma)\otimes\delta_2(\tau))\gamma = \gamma,\ \sigma\in\mathfrak{S}_p,\ \tau_q\in\mathfrak{S}_q\}.$$

Using the Schur's lemma we obtain the following

$$\dim\{w\in V^*_{\rho_1}\otimes V_{\delta_1}\otimes V^*_{\rho_2}\otimes V_{\delta_2};\ (\rho_1^*(\sigma)\otimes\delta_1(\tau)\otimes\rho_2^*(\sigma)\otimes\delta_2(\tau))w=w,\ \sigma\in\mathfrak{S}_p,\ \tau\in\mathfrak{S}_q\}$$

$$=\begin{cases}0 & (\rho_1\not\simeq\rho_2^* \quad\text{or}\quad \delta_1^*\not\simeq\delta_2),\\ 1 & (\rho_1\simeq\rho_2^* \quad\text{and}\quad \delta_1^*\simeq\delta_2).\end{cases}$$

Hence we get

$$\mathfrak{H}_{p,q}=\sum_\rho\sum_\delta \mathscr{H}_{p,q,\rho,\delta}\bar{\otimes}\mathscr{H}_{p,q,\rho^*,\delta^*}=\sum_\rho\sum_\delta \mathscr{H}_{p,q,\rho,\delta}\bar{\otimes}\mathscr{H}^*_{p,q,\rho,\delta}.$$

§3 Polynomial representations of discrete class

Let $(\pi, \mathfrak{H})$ be a unitary representation of $U(\boldsymbol{E})$. For v and w in $\mathfrak{H}$ we define a function $\phi^\pi_{v,w}(g)$ on $U(\boldsymbol{E})$ by

$$\phi^\pi_{v,w}(g)=(v,\ \pi(g)w).$$

We call $(\pi, \mathfrak{H})$ a polynomial representation of $U(\boldsymbol{E})$ if there exists an orthonormal basis $\{v_i;\ i\in\boldsymbol{N}\}$ of $\mathfrak{H}$ such that $\phi^\pi_{i,j}(g)=(v_i,\ \pi(g)v_j)\ (i, j\in\boldsymbol{N})$ are polynomials. We denote by $\mathfrak{H}_f$ the space of all finite linear combinations of $v_i\ (i\in\boldsymbol{N})$. We call $(\pi, \mathfrak{H})$ of discrete class if the multilinear functional B:

$$\mathfrak{H}_f\times\mathfrak{H}_f\times\mathfrak{H}_f\times\mathfrak{H}_f\ni(v,\ w,\ v',\ w')\longmapsto\int_\Omega \phi^{\pi\#}_{v,w}(z)(\phi^{\pi\#}_{v',w'}(z))^- d\nu(z)\in\boldsymbol{C}$$

is continuous. The following proposition can be proved similarly to the case of $O(\boldsymbol{E})$, (see [6], Proposition 3).

PROPOSITION 1. 1) *Let $(\pi, \mathfrak{H})$ be an irreducible unitary polynomial representation of discrete class. Then there exists a positive constant c such that*

$$\int_\Omega \phi^{\pi\#}_{v,w}(z)(\phi^{\pi\#}_{v',w'}(z))^- d\nu(z)=c(v,\ v')(w,\ w')\qquad (v,\ w,\ v',\ w'\in\mathfrak{H}_f).$$

2) *Let $(\pi, \mathfrak{H})$ and $(\pi', \mathfrak{H}')$ be irreducible unitary polynomial representations of discrete class. If π and π' are non-equivalent, then*

$$\int_\Omega \phi^{\pi\#}_{v,w}(z)(\phi^{\pi'\#}_{v',w'}(z))^- d\nu(z)=0\qquad (v,\ w\in\mathfrak{H}_f,\ v',\ w'\in\mathfrak{H}'_f).$$

THEOREM 3. *For any p and q in $\boldsymbol{N}\cup\{0\}$ and for any irreducible unitary representations (ρ, V_ρ) of $\mathfrak{S}_p$ and (δ, V_δ) of $\mathfrak{S}_q$, $(\pi_{p,q,\rho,\delta}, \mathscr{H}_{p,q,\rho,\delta})$ is an irreducible unitary polynomial representation of discrete class. Conversely for any irreducible unitary polynomial representation of discrete class $(\pi, \mathfrak{H})$, there exist p and q in $\boldsymbol{N}\cup\{0\}$ and irreducible unitary representations (ρ, V_ρ) of $\mathfrak{S}_p$ and (δ, V_δ) of $\mathfrak{S}_q$ such that $(\pi, \mathfrak{H})$ is equivalent to $(\pi_{p,q,\rho,\delta}, \mathscr{H}_{p,q,\rho,\delta})$.*

PROOF. Let $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_s\}$ be an orthonormal basis of V_ρ and $\{\boldsymbol{f}_1,\dots,\boldsymbol{f}_t\}$ an orthonormal basis of V_δ^*. Then we have an orthonomal basis $\mathfrak{B}_{p\,q,\rho,\delta}=\{\xi_{i_1}\otimes\cdots\otimes\xi_{i_p}\otimes\xi_{k_1}\otimes\cdots\otimes\xi_{k_q}\otimes\boldsymbol{e}_{i_0}\otimes\boldsymbol{f}_{k_0};\ i_0,\dots,i_p,\ k_0,\dots,k_q\in\boldsymbol{N}\}$ of $L^2(M)\bar{\otimes}\cdots\bar{\otimes}L^2(M)\otimes V_\rho\otimes V_\delta^*$. It is easy to see that $\sum_{\sigma,\tau}\lambda(\sigma)\otimes\lambda(\tau)\otimes\rho(\sigma)^{-1}\otimes\delta(\tau)$ defines the orthogonal projection of $L^2(M)\bar{\otimes}\cdots\bar{\otimes}L^2(M)\otimes V_\rho\otimes V_\delta^*$ onto the subspace which is equivalent to $\mathscr{H}_{p,q,\rho,\delta}$. Hence for the proof of "only if" part of the theorem it is sufficient to prove that $\hat{\pi}_{p,q}\otimes I\otimes I$ is a polynomial representation of discrete class. We put

$$v_{ik}=\xi_{i_1}\otimes\cdots\otimes\xi_{i_p}\otimes\xi_{k_1}\otimes\cdots\otimes\xi_{k_q}\otimes\boldsymbol{e}_{i_0}\otimes\boldsymbol{f}_{k_0},$$
$$v_{jl}=\xi_{j_1}\otimes\cdots\otimes\xi_{j_p}\otimes\xi_{l_1}\otimes\cdots\otimes\xi_{l_q}\otimes\boldsymbol{e}_{j_0}\otimes\boldsymbol{f}_{l_0},$$
$$g\xi_{j_h}=\textstyle\sum_{m_h}g_{m_hj_h}\xi_{m_h},\quad g^*\xi_{l_h}=\sum_{n_h}\bar{g}_{n_hl_h}\xi_{n_h}.$$

And we put

$$\phi_{ik;jl}(g)=(v_{ik},(\pi_{p,q}(g)\otimes I\otimes I)v_{jl}).$$

Then we have

$$\phi_{ik;jl}(g)=\delta_{i_0j_0}\delta_{k_0l_0}g_{i_1j_1}\cdots g_{i_pj_q}\bar{g}_{k_1l_1}\cdots\bar{g}_{k_ql_q},$$

where $\delta_{i_0j_0}$ and $\delta_{k_0l_0}$ are Kronecker's δ. Thus $\phi_{ik;jl}$ is a polynomial on $U(\boldsymbol{E})$. Next we shall prove that the functional B is continuous. For any v, w, v', w' in $L^2(M)\bar{\otimes}\cdots\bar{\otimes}L^2(M)\otimes V_\rho\otimes V_\delta^*$ we put

$$v=\textstyle\sum_{i,k}a_{ik}v_{ik},\quad w=\sum_{j,l}b_{jl}v_{jl},\quad v'=\sum_{i',j'}c_{i'j'}v_{i'j'},\quad w'=\sum_{j',l'}d_{j'l'}v_{j'l'}.$$

Then we have

$$\begin{aligned}\phi_{v,w}(g)&=(v,(\hat{\pi}_{p,q}(g)\otimes I\otimes I)w)=\textstyle\sum_{i,k}\sum_{j,l}\delta_{i_0j_0}\delta_{k_0l_0}a_{ik}b_{jl}g_{i_1j_1}\cdots g_{i_pj_p}\bar{g}_{k_1l_1}\cdots\bar{g}_{k_ql_q}\\&=\textstyle\sum_{i,k}\sum_{j,l}\delta_{i_0j_0}\delta_{k_0l_0}a_{ik}b_{jl}g_{ij}\bar{g}_{kl}.\end{aligned}$$

It follows that

$$\int_\Omega(g_{ij}\bar{g}_{kl})^\#(z)((g_{i'j'}\bar{g}_{k'l'})^\#(z))^-d\nu(z)=0$$

unless $(i,j)\sim(i',j')$ and $(k,l)\sim(k',l')$.
We put

$$d=\max\{i,j,k,l,i',j',k',l';\ a_{ik},b_{jl},c_{i'k'},d_{j'l'}\neq 0\}.$$

Using the Schwarz inequality we have

$$\begin{aligned}|B(v,w,v',w')|&=\left|\int_\Omega\phi_{v,w}^\#(z)(\phi_{v',w'}^\#(z))^-d\nu(z)\right|\\&\leqq\textstyle\sum_{i,k}\sum_{j,l}\sum_{i',k'}\sum_{j',l'}\delta_{i_0j_0}\delta_{k_0l_0}\delta_{i_0'j_0'}\delta_{k_0'l_0'}|a_{ik}||b_{jl}||c_{i'k'}||d_{j'l'}|\end{aligned}$$

$$\times \left|\int_\Omega (g_{ij}\bar{g}_{kl})^\#(z)((g_{i'j'}\bar{g}_{k'l'})^\#(z))^- d\nu(z)\right|$$

$$\leqq \sum_\alpha \sum_\beta \sum_{i,k}^\alpha \sum_{j,l}^\beta |a_{ik}||c_{ik}||b_{jl}||d_{jl}|(\prod_{i,k}\alpha_{ik}!)(\prod_{j,l}\beta_{jl}!)$$

(where $\sum_\alpha$ and $\sum_\beta$ mean the summations which are taken over all α in $Z_d(p)$ and β in $Z_d(q)$ respectively, $\sum_{i,k}^\alpha$ and $\sum_{j,l}^\beta$ mean the summations which are taken over (i, k) in $T_p^{-1}(\alpha)$ and (j, l) in $T_q^{-1}(\beta)$ respectively,)

$$= \sum_\alpha \sum_\beta \{p!(\prod_{i,k}\alpha_{ik}!)^{-1}(\sum_{i,k}^\alpha a_{ik}^2)^{1/2}(\sum_{i,k}^\alpha c_{ik}^2)^{1/2}\}$$

$$\times \{q!(\prod_{j,l}\beta_{jl}!)^{-1}(\sum_{j,l}^\beta b_{jl}^2)^{1/2}(\sum_{j,l}^\beta d_{jl}^2)^{1/2}\}(\prod_{i,k}\alpha_{ik}!)(\prod_{j,l}\beta_{jl}!)$$

$$\leqq p!q!\{(\sum_{i,k} a_{ik}^2)(\sum_{i,k} c_{ik}^2)\}^{1/2}\{(\sum_{j,l} b_{jl}^2)(\sum_{j,l} d_{jl}^2)\}^{1/2}$$

$$= p!q!\|v\|\ \|w\|\ \|v'\|\ \|w'\|.$$

Thus we have

$$|B(v, w, v', w')| \leqq p!q!\|v\|\ \|w\|\ \|v'\|\ \|w'\|.$$

Conversely let $(\pi, \mathfrak{H})$ be an irreducible unitary polynomial representation of discrete class. Then by definition, there exists an orthonormal basis $\{v_i;\ i \in \boldsymbol{N}\}$ of $\mathfrak{H}$ which satisfies the following conditions; $\phi_{i,j}^\pi(g) = (v_i, \pi(g)v_j)\,(i, j \in \boldsymbol{N})$ are polynomials and B:

$$\mathfrak{H}_f \times \mathfrak{H}_f \times \mathfrak{H}_f \times \mathfrak{H}_f \ni (v, w, v', w') \longmapsto \int_\Omega \phi_{v,w}^{\pi\#}(z)(\phi_{v',w'}^{\pi\#}(z))^- d\nu(z) \in \boldsymbol{C}$$

is continuous. From Proposition 1 there exists a positive constant c such that

$$B(v, w, v', w') = c(v, v')(w, w'),$$

where $v, w, v', w' \in \mathfrak{H}_f$. Now we fix v_0, and for any v in $\mathfrak{H}_f$ we define a linear operator A by

$$(Av)(z) = \phi_{v,v_0}^{\pi\#}(z).$$

Since B is continuous, A defines a bounded linear operator of $\mathfrak{H}$ into $L^2(\Omega, \nu)$. As is easily seen we get the following

$$(A\pi(g)v)(z) = \phi_{\pi(g)v,v_0}^{\pi\#}(z) = \phi_{v,v_0}^{\pi\#}(g^{-1}z) = (\pi_L(g)Av)(z).$$

This implies that A is an intertwining operator of $\mathfrak{H}$ into $L^2(\Omega, \nu)$. Thus $(\pi, \mathfrak{H})$ is equivalent to a subrepresentation of $(\pi_L, L^2(\Omega, \nu))$. On the other hand, from Theorem 2, we can prove that any subrepresentation of $(\pi_L, L^2(\Omega, \nu))$ is equivalent to $(\pi_{p,q,\rho,\delta}, \mathscr{H}_{p,q,\rho,\delta})$ for some p and q in $\boldsymbol{N} \cup \{0\}$ and ρ in $\hat{\mathfrak{S}}_p$, δ in $\hat{\mathfrak{S}}_q$. This completes the proof of the theorem.

REMARK. Using the similar argument we improve on the inequality:

$$|B(v, w, v', w')| \leqq (n!)^2\|v\| \|w\| \|v'\| \|w'\|,$$

in the proof of Theorem 2 in [6] as follows:

$$|B(v, w, v', w')| \leqq n!\|v\| \|w\| \|v'\| \|w'\|.$$

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