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A stochastic method for solving quasilinear parabolic equations and its application to an ecological model

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Introduction

We are concerned with the following Cauchy problem for a quasilinear parabolic equation:

$$(1.1)\quad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\triangle u + \sum_{i=1}^{n} b_i(t, x; u)\dfrac{\partial u}{\partial x_i} + c(t, x; u)u, \quad t > 0,\ x \in \boldsymbol{R}^n, \\ u(0, x) = f(x) \geqq 0, \end{cases}$$

where $b_i(t, x; \cdot)$, $1 \leqq i \leqq n$, and $c(t, x; \cdot)$ are mappings defined for some functions $u: [0, \infty) \times \boldsymbol{R}^n \to \boldsymbol{R}$. We assume that the coefficients $b_i(t, x; u)$, $1 \leqq i \leqq n$, and $c(t, x; u)$ are independent of the future $\{u(s, y): s > t, y \in \boldsymbol{R}^n\}$ for each t. (See §1 for precise definition.)

Our main results are stated in §1 and §2. They are summarized as follows. The equation (1.1) has a unique solution which has a nice probabilistic expression (1.2) based upon an n-dimensional Brownian motion $\{B_t = (B_t^1, \dots, B_t^n), t \geqq 0\}$:

$$(1.2)\quad u(t, x) = E_x[f(B_t) \exp\left\{\int_0^t c(t-s, B_s; u)ds\right\} M_t(u)],$$

$$M_t(u) = \exp\left\{\sum_{i=1}^{n} \int_0^t b_i(t-s, B_s; u)dB_s^i - \frac{1}{2}\sum_{i=1}^{n} \int_0^t b_i(t-s, B_s; u)^2 ds\right\},$$

under some suitable conditions. In a special case where $b_i(t, x; u) = b_i(t, x, u(t, x))$, $1 \leqq i \leqq n$, and $c(t, x; u) = c(t, x, u(t, x))$, Freidlin [2] solved the Cauchy problem (1.1) by finding the unique solution of (1.2). Our results can be regarded as a generalization of Freidlin's. In §3, our theorem is applied to the equation

$$(3.1)\quad \frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial x}\left[\left(\int_x^{x+r} v(t, y)dy - \int_{x-r}^{x} v(t, y)dy\right)v\right], \quad t > 0,\ x \in \boldsymbol{R},$$

which appears in an ecological model. It can be proved that there exists a unique solution of (3.1) for each r, which is bounded for $0 \leqq t < \infty$ and continuous in the parameter $r \in [0, \infty]$. Here the expression (1.2) of the solution plays an essential role. We make two remarks on some related problems in §4; the one is on time-lag systems and the other is on Neumann problems.

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§ 1. The generalized solution

A function $u=u(t, x)$ is called a *generalized solution* for the Cauchy problem (1.1) if u satisfies the equation (1.2) for all (t, x). In this section we show the existence and the uniqueness of the generalized solution for (1.1).

For each $t>0$, let $\mathscr{U}_t$ be the Banach space of bounded measurable functions u: $[0, t]\times \boldsymbol{R}^n \to \boldsymbol{R}$ with the norm $\|u\|_t=\sup\{|u(s, x)|: (s, x)\in[0, t]\times\boldsymbol{R}^n\}$.

Fix $T>0$. Throughout the paper we assume that the coefficients $b_i(t, x; u)$ and $c(t, x; u)$ are defined for $u\in\mathscr{U}_T$ and $(t, x)\in[0, T]\times\boldsymbol{R}^n$, and they are measurable in (t, x) for each $u\in\mathscr{U}_T$. We also assume that the coefficients are independent of the future: If $u_1(s, x)=u_2(s, x)$, $(s, x)\in[0, t]\times\boldsymbol{R}^n$, then $b_i(s, x; u_1)=b_i(s, x; u_2)$, $1\leqq i\leqq n$, and $c(s, x; u_1)=c(s, x; u_2)$, $(s, x)\in[0, t]\times\boldsymbol{R}^n$ for each $t\leqq T$. Hence for each $(t, x)\in[0, T]\times\boldsymbol{R}^n$ these coefficients can be considered as functions on $\mathscr{U}_t$.

We put the following conditions:

(I) For any $K>0$, there exist C_1, $C_2>0$ satisfying

$$\text{(i)}\quad |b_i(t, x; u)| \leqq C_1,\ 1\leqq i\leqq n,$$
$$|c(t, x; u)| \leqq C_1,\quad (t, x)\in[0, T]\times\boldsymbol{R}^n,$$

for $u\in\mathscr{U}_T$ with $\|u\|_T\leqq K$,

$$\text{(ii)}\quad |b_i(t, x; u)-b_i(t, x; v)| \leqq C_2\|u-v\|_t,\ 1\leqq i\leqq n,$$
$$|c(t, x; u)-c(t, x; v)| \leqq C_2\|u-v\|_t,\quad (t, x)\in[0, T]\times\boldsymbol{R}^n,$$

for $u, v\in\mathscr{U}_T$ with $\|u\|_T$, $\|v\|_T\leqq K$.

(II) There exists a constant $C>0$ such that $c(t, x; u)\leqq C$ for each $(t, x)\in[0, T]\times\boldsymbol{R}^n$ and for $u\in\mathscr{U}_T$ with $u\geqq 0$.

THEOREM 1. *Assume that the coefficients $b_i(t, x; u)$, $1\leqq i\leqq n$, and $c(t, x; u)$ satisfy the conditions* (I) *and* (II). *Then the equation* (1.2) *has a unique solution in $\mathscr{U}_T$ for any bounded measurable function f.*

Before proving Theorem 1, we prepare the following

LEMMA 1. *Let $\{B_t=(B_t^1,\dots, B_t^n), t\geqq 0\}$ be an n-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, P)$ and $\mathscr{F}_t$ be the σ-field generated by $\{B_s: 0\leqq s\leqq t\}$ and all P-null sets. Suppose that $\mathscr{F}_t$-adapted processes $\{b_{ki}(t), 0\leqq t\leqq T\}$, $1\leqq i\leqq n$, $k=1, 2$ satisfy $|b_{ki}(t, \omega)|\leqq C_*$ for $(t, \omega)\in$*

$[0, T]\times\Omega$. *Then for functionals*

$$(1.3)\qquad a_k(t) = \sum_{i=1}^n \int_0^t b_{ki}(s)dB_s^i - (1/2)\sum_{i=1}^n \int_0^t b_{ki}(s)^2 ds, \quad k = 1, 2,$$

it holds that

$$(1.4)\quad E[|\exp(a_1(t)) - \exp(a_2(t))|] \leqq M\left\{E\left[\int_0^t \sum_{i=1}^n |b_{1i}(s) - b_{2i}(s)|^4 ds\right]\right\}^{1/4},$$

for some $M = M(C_*, T, n) > 0$.

PROOF. Let $\chi_1(\omega)$ be the characteristic function of the set $\{\omega: |a_1(t) - a_2(t)| \leqq 1\}$. Using $|\exp x - 1| \leqq 2|x|$ for $|x| \leqq 1$, we obtain

$$\begin{aligned}(1.5)\qquad & E[|\exp(a_1(t)) - \exp(a_2(t))|] \\ & \leqq \{E[\exp(2a_2(t))]\}^{1/2}\{E[|\exp(a_1(t) - a_2(t)) - 1|^2]\}^{1/2} \\ & \leqq \{E[\exp(2a_2(t))]\}^{1/2}\Big[4E[|a_1(t) - a_2(t)|^2] \\ & \quad + \{E[|\exp(a_1(t) - a_2(t)) - 1|^4]\}^{1/2}\{E[(1 - \chi_1)^2]\}^{1/2}\Big]^{1/2}.\end{aligned}$$

The martingale property of $\exp(a_2(t))$ yields that

$$(1.6)\qquad E[\exp(2a_2(t))] \leqq \exp(nC_*^2 t).$$

By the boundedness of $b_{ki}(t)$, we have

$$\begin{aligned}(1.7)\qquad E[|a_1(t) - a_2(t)|^2] & \leqq 2E\left[\left|\sum_{i=1}^n \int_0^t (b_{1i}(s) - b_{2i}(s))dB_s^i\right|^2\right] \\ & \quad + 2E\left[\left|(1/2)\sum_{i=1}^n \int_0^t \{b_{1i}(s)^2 - b_{2i}(s)^2\}ds\right|^2\right] \\ & \leqq 2(1 + C_*^2 nt)E\left[\int_0^t \sum_{i=1}^n |b_{1i}(s) - b_{2i}(s)|^2 ds\right] \\ & \leqq 2(1 + C_*^2 nt)(nt)^{1/2}\left\{E\left[\int_0^t \sum_{i=1}^n |b_{1i}(s) - b_{2i}(s)|^4 ds\right]\right\}^{1/2}.\end{aligned}$$

Next we estimate the last term of (1.5);

$$\begin{aligned}(1.8)\qquad & E[(1 - \chi_1)^2] = P(\omega: |a_1(t) - a_2(t)| > 1) \leqq E[|a_1(t) - a_2(t)|^4] \\ & \leqq 8E\left[\left|\sum_{i=1}^n \int_0^t (b_{1i}(s) - b_{2i}(s))dB_s^i\right|^4\right] \\ & \quad + 8E\left[\left|(1/2)\sum_{i=1}^n \int_0^t (b_{1i}(s)^2 - b_{2i}(s)^2)ds\right|^4\right].\end{aligned}$$

Set $b_i(s) = b_{1i}(s) - b_{2i}(s)$ and $X_t = \sum_{i=1}^n \int_0^t b_i(s)dB_s^i$. Then

$$E[X_t^4] = 6E\Big[\sum_{i=1}^n \int_0^t X_s^2 b_i(s)^2 ds\Big] = 6\int_0^t E[X_s^2 \sum_{i=1}^n b_i(s)^2]\,ds$$
$$\leqq 6n^{1/2}\int_0^t (E[X_s^4])^{1/2}(E[\sum_{i=1}^n b_i(s)^4])^{1/2} ds$$
$$\leqq 6n^{1/2}\Big(\int_0^t E[X_s^4]ds\Big)^{1/2}\Big(\int_0^t E[\sum_{i=1}^n b_i(s)^4]ds\Big)^{1/2}.$$

Putting $A(s)=E[X_s^4]$ and $\lambda=6n^{1/2}\Big(\int_0^t E[\sum_{i=1}^n b_i(s)^4]ds\Big)^{1/2}$, we have

$$A(s)^2 \leqq \lambda^2 \int_0^s A(z)dz, \quad 0 \leqq s \leqq t.$$

Since $A(s)$ is bounded on $[0, t]$, it follows that

$$A(s) \leqq \lambda^2 s/2, \quad 0 \leqq s \leqq t.$$

Therefore we obtain

$$(1.9) \qquad E\Big[\Big|\sum_{i=1}^n \int_0^t (b_{1i}(s) - b_{2i}(s))dB_s^i\Big|^4\Big] \leqq 18ntE\Big[\int_0^t \sum_{i=1}^n |b_{1i}(s) - b_{2i}(s)|^4 ds\Big].$$

On the other hand,

(1.10)

$$E\Big[\Big|(1/2)\sum_{i=1}^n \int_0^t \{b_{1i}(s)^2 - b_{2i}(s)^2\}ds\Big|^4\Big] \leqq n^3 C_*^4 t^3 E\Big[\int_0^t \sum_{i=1}^n |b_{1i}(s) - b_{2i}(s)|^4 ds\Big].$$

Finally we show the boundedness of $E[|\exp(a_1(t)-a_2(t))-1|^4]$. By Itô's formula

$$E[|\exp(a_1(t)-a_2(t))-1|^4]$$
$$= E\Big[\int_0^t 4\{\exp(a_1(s)-a_2(s)) - 1\}^3 \exp(a_1(s)-a_2(s)) \sum_{i=1}^n b_{2i}(s)(b_{2i}(s) - b_{1i}(s))ds\Big]$$
$$+ E\Big[\int_0^t 6\{\exp(a_1(s)-a_2(s)) - 1\}^2 \exp(2a_1(s)-2a_2(s)) \sum_{i=1}^n (b_{1i}(s) - b_{2i}(s))^2 ds\Big]$$
$$\leqq 32nC_*^2 \int_0^t E[\exp(4a_1(s)-4a_2(s)) + 1]ds.$$

Using Itô's formula again, we obtain

$$E[\exp(4a_1(t)-4a_2(t))] \leqq 1 + 36nC_*^2 \int_0^t E[\exp(4a_1(s)-4a_2(s))]ds,$$

which implies

$$E[\exp(4a_1(t)-4a_2(t))] \leqq \exp(36nC_*^2 t).$$

Hence

$$E[|\exp(a_1(t)-a_2(t))-1|^4] \leqq 32nC_*^2(1+\exp(36nC_*^2t))t. \tag{1.11}$$

Summarizing the estimates from (1.5) to (1.11), we reach

$$\begin{aligned} &E[|\exp(a_1(t))-\exp(a_2(t))|] \\ &\leqq \exp(nC_*^2T/2)\Big[8(1+nC_*^2T)(nT)^{1/2} \\ &\quad + \{32nC_*^2(1+\exp(36nC_*^2T))T\}^{1/2}\{8(18+n^2C_*^4T)nT^2\}^{1/2}\Big]^{1/2} \\ &\quad \times \left\{\int_0^t E[\textstyle\sum_{i=1}^n |b_{1i}(s)-b_{2i}(s)|^4 ds]\right\}^{1/4}, \quad 0\leqq t\leqq T, \end{aligned}$$

as was to be proved.

Proof of Theorem 1. Owing to the assumptions (I) and (II), take C for fixed $T>0$ and C_1, $C_2>0$ for $K=\|f\|_\infty \exp(CT)$. Define the operator Φ: $\mathscr{U}_T \to \mathscr{U}_T$ by

$$(\Phi u)(t, x) = E_x[f(B_t)\exp\left\{\int_0^t c(t-s, B_s; u)ds\right\}M_t(u)]. \tag{1.12}$$

Let $\mathscr{U}_t^K$ be the complete subset of $\mathscr{U}_t$ consisting of u with $\|u\|_t \leqq K$ and $u\geqq 0$ for $t\leqq T$. Then Φ maps $\mathscr{U}_t^K$ into itself. For $u, v\in\mathscr{U}_t^K$ and $(t, x)\in[0, T]\times \boldsymbol{R}^n$, we get

$$\begin{aligned} &|(\Phi u)(t, x)-(\Phi v)(t, x)| \\ &\leqq E_x\Big[f(B_t)\Big|\exp\left\{\int_0^t c(t-s, B_s; u)ds\right\}-\exp\left\{\int_0^t c(t-s, B_s; v)ds\right\}\Big|M_t(u)\Big] \\ &\quad + E_x\Big[f(B_t)\exp\left\{\int_0^t c(t-s, B_s; v)ds\right\}|M_t(u)-M_t(v)|\Big] \\ &= I_1+I_2. \end{aligned}$$

Using Lemma 1, we have

$$\begin{aligned} I_2 &\leqq \|f\|_\infty \exp(CT)M(C_1, T, n)\left\{E_x\Big[\int_0^t \textstyle\sum_{i=1}^n |b_i(t-s, B_s; u)-b_i(t-s, B_s; v)|^4 ds\Big]\right\}^{1/4} \\ &\leqq \|f\|_\infty \exp(CT)M(C_1, T, n)(nt)^{1/4}C_2\|u-v\|_t. \end{aligned}$$

On the other hand,

$$\begin{aligned} I_1 &\leqq \|f\|_\infty \exp(Ct+nC_1^2t/2) \\ &\quad \times \left\{E_x\Big[\Big|\exp\Big(\int_0^t (c(t-s, B_s; u)-c(t-s, B_s; v))ds\Big)-1\Big|^2\Big]\right\}^{1/2} \\ &= \|f\|_\infty \exp(Ct+nC_1^2t/2)\Big\{E_x\Big[\Big|\int_0^t (c(t-s, B_s; u)-c(t-s, B_s; v)) \end{aligned}$$

$$\times \exp\left(\int_0^s (c(t-z, B_z; u) - c(t-z, B_z; v))dz\right)ds\Big|^2\Big]\Big\}^{1/2}$$

$$\leqq \|f\|_\infty \exp(CT + 2C_1T + nC_1^2T/2)tC_2\|u - v\|_t.$$

Combining the above estimates, we obtain

$$|(\Phi u)(t, x) - (\Phi v)(t, x)|$$

$$\leqq \|f\|_\infty \exp(CT)\{T^{3/4}\exp(2C_1T + nC_1^2T/2) + n^{1/4}M(C_1, T, n)\}C_2t^{1/4}\|u - v\|_t,$$

for $(t, x) \in [0, T] \times \boldsymbol{R}^n$. Therefore there exists $t_0 > 0$ depending only on K, T, n, C_1 and C_2 such that the operator Φ is contractive on $\mathscr{U}_{t_0}^K$. The equation (1.2) has a unique solution u_0 on $\mathscr{U}_{t_0}^K$.

For the continuation of u_0, let the operator Φ' on $\mathscr{U}'_{T-t_0}$ be

$$(\Phi' v)(t, x) = E_x\left[u_0(t_0, B_t) \exp\left\{\int_0^t c'(t-s, B_s; v)ds\right\}M'_t(v)\right]$$

with

$$\mathscr{U}'_t = \{v \in \mathscr{U}_t : v(0, x) = u_0(t_0, x)\},\ 0 \leqq t \leqq T - t_0$$

$$M'_t(v) = \exp\left\{\sum_{i=1}^n \int_0^t b'_i(t-s, B_s; v)dB_s^i - (1/2)\sum_{i=1}^n \int_0^t b'_i(t-s, B_s; v)^2 ds\right\},$$

$$b'_i(t, x; v) = b_i(t + t_0, x; v^*),$$

$$c'(t, x; v) = c(t + t_0, x; v^*),$$

$$v^*(t, x) = u_0(t, x)\chi_{[0,t_0]}(t) + v(t - t_0, x)\chi_{[t_0, T-t_0]}(t).$$

Then by the assumptions (I), (II) and the estimate

$$\|u_0(t_0, \cdot)\|_\infty \exp(C(T - t_0)) \leqq \|f\|_\infty \exp(CT) = K,$$

Φ' is again contractive on $(\mathscr{U}'_{t_0})^K$ for the same t_0. Let $v_0 \in (\mathscr{U}'_{t_0})^K$ be the unique fixed point of Φ'. In consideration of Markov property of $\{B_t\}$, we can regard $(v_0)^*$ as a unique continued solution of (1.2) for $0 \leqq t \leqq 2t_0$. The rest of the proof is a routine work.

§2. The classical solution

In this section we prove that, under some additional conditions, the generalized solution constructed in §1 is a solution of (1.1) in the classical sense.

Let $\mathscr{C}_t(\alpha)$ be the space of bounded continuous functions $u: [0, t] \times \boldsymbol{R}^n \to \boldsymbol{R}$, which are Hölder continuous in $x \in \boldsymbol{R}^n$ with exponent $\alpha > 0$. Besides the conditions (I) and (II) in §1, we put the following conditions:

(III) If $u \in \mathscr{C}_T(\alpha)$, then the coefficients $b_i(\cdot, \cdot; u)$, $1 \leqq i \leqq n$, and $c(\cdot, \cdot; u)$

are in $\mathscr{C}_T(\alpha)$.

(IV) For any $K>0$ there exists a constant $C_3>0$ satisfying

$$|b_i(t, x; u) - b_i(t, y; u)| \leqq C_3 L|x-y|^{\alpha},\ 1 \leqq i \leqq n,$$
$$|c(t, x; u) - c(t, y; u)| \leqq C_3 L|x-y|^{\alpha},\quad 0 \leqq t \leqq T,\quad x, y \in \boldsymbol{R}^n,$$

for $u \in \mathscr{C}_T(\alpha)$ with Hölder coefficient L and $\|u\|_T \leqq K$.

THEOREM 2. *Assume* (I)~(IV). *If an initial function f is bounded and satisfies $|f(x)-f(y)| \leqq L_0|x-y|^{\alpha_0}$, $x, y \in \boldsymbol{R}^n$ for some $\alpha_0, L_0>0$ then for each $T>0$ there exists a unique solution $u \in \mathscr{C}_T(\alpha_0)$ of* (1.1) *in the classical sense. The solution u is characterized by*

$$u(t, x) = \int_{\boldsymbol{R}^n} f(y) U(t, x; 0, y; u) dy, \tag{2.1}$$

where $U(t, x; s, y; w)$ is the fundamental solution of the equation

$$v_t = (1/2)\triangle v + \textstyle\sum_{i=1}^n b_i(t, x; w) v_{x_i} + c(t, x; w) v, \tag{2.2}$$

for each $w \in \mathscr{C}_T(\alpha_0)$.

The proof of Theorem 2 is essentially based on the following lemma. Let $\mathscr{C}_t(\alpha, L)$ be the collection of $u \in \mathscr{C}_t(\alpha)$ whose Hölder coefficients are not bigger than L and $\mathscr{C}_t^K(\alpha, L)$ be the collection of $u \in \mathscr{C}_t(\alpha, L)$ satisfying $\|u\|_t \leqq K$ and $u \geqq 0$.

LEMMA 2. *Assume* (I)~(IV) *and that f satisfies the conditions stated in Theorem 2. Put $K = \|f\|_\infty \exp(CT)$. Then Φ defined by* (1.12) *maps $\mathscr{C}_t^K(\alpha_0, L)$ into $\mathscr{C}_t^K(\alpha_0, NL_0 + JLt^{1/4})$ for each $t \leqq T$, where N and J are positive constants depending only on K and T.*

PROOF. We prove the lemma only for $n=1$ writing $b(t, x; u) = b_1(t, x; u)$ and $B_t = B_t^1$. Take C_1, C_2 in the condition (I) and C_3 in (IV) for K. For $u \in \mathscr{C}_T^K(\alpha_0, L)$ and $0 \leqq s \leqq t \leqq T$, $x, y \in \boldsymbol{R}$, we have

$$\begin{aligned}
&|(\Phi u)(t, x) - (\Phi u)(s, y)| \\
&= \Big| E_0\Big[f(B_t+x) \exp\Big\{\int_0^t c(t-z, B_z+x; u) dz\Big\} M_t(x; u)\Big] \\
&\quad - E_0\Big[f(B_s+y) \exp\Big\{\int_0^s c(s-z, B_z+y; u) dz\Big\} M_s(y; u)\Big]\Big| \\
&\leqq E_0\Big[|f(B_t+x) - f(B_s+y)| \exp\Big\{\int_0^t c(t-z, B_z+x; u) dz\Big\} M_t(x; u)\Big]
\end{aligned}$$

$$+ E_0\Big[f(B_s+y)\Big| \exp\Big\{\int_0^t c(t-z,\ B_z+x;\ u)dz\Big\} - \exp\Big\{\int_0^s c(s-z,\ B_z+y;\ u)dz\Big\}\Big| M_t(x;\ u)\Big]$$

$$+ E_0\Big[f(B_s+y)\exp\Big\{\int_0^s c(s-z,\ B_z+y;\ u)dz\Big\}|M_t(x;\ u) - M_s(y;\ u)|\Big]$$

$$= I_1 + I_2 + I_3,$$

where

$$M_t(x;\ u) = \exp\Big\{\int_0^t b(t-z,\ B_z+x;\ u)dB_z - (1/2)\int_0^t b(t-z,\ B_z+x;\ u)^2 dz\Big\}.$$

It follows that

$$\begin{aligned} I_1 &\leqq L_0\{E_0[|B_t+x-B_s-y|^{2\alpha_0}]\}^{1/2}\exp(Ct+C_1^2t/2) \\ &\leqq L_0[2|x-y|^{\alpha_0} + 2|t-s|^{\alpha_0/2}\kappa]\exp(Ct+C_1^2t/2), \end{aligned}$$

where $\kappa^2 = \int_{\boldsymbol{R}} z^{2\alpha_0}(2\pi)^{-1/2}\exp(-z^2/2)dz$. Next we have

$$\begin{aligned} I_2 &\leqq \|f\|_\infty \exp(Cs+C_1^2t/2) \\ &\quad\times\Big\{E_0\Big[\Big|\exp\Big(\int_0^t c(t-z,\ B_z+x;\ u)dz - \int_0^s c(s-z,\ B_z+y;\ u)dz\Big)-1\Big|^2\Big]\Big\}^{1/2} \\ &\leqq \|f\|_\infty \exp(Cs+2C_1t+C_1^2t/2) \\ &\quad\times\Big\{2C_1^2|t-s|^2 + 2sE_0\Big[\int_0^s |c(t-z,\ B_z+x;\ u) - c(s-z,\ B_z+y;\ u)|^2dz\Big]\Big\}^{1/2}. \end{aligned}$$

For the last term, we obtain

$$\begin{aligned} I_3 \leqq \|f\|_\infty &\exp(Ct)M\Big\{C_1^4|t-s| \\ &+ E_0\Big[\int_0^s |b(t-z,\ B_z+x;\ u) - b(s-z,\ B_z+y;\ u)|^4dz\Big]\Big\}^{1/4}, \end{aligned}$$

where $M=M(C_1,\ T,\ 1)$ is the constant in Lemma 1. By the estimates of I_1, I_2 and I_3, we can easily see that Φu is continuous in $(t,\ x)\in[0,\ T]\times\boldsymbol{R}$. Set $t=s$. Then it follows that

$$\begin{aligned} &|(\Phi u)(t,\ x) - (\Phi u)(t,\ y)| \\ &\quad\leqq 2L_0|x-y|^{\alpha_0}\exp(CT+C_1^2T/2) \\ &\qquad+ \|f\|_\infty\exp(CT+2C_1T+C_1^2T/2)2t^{1/2} \\ &\qquad\times\Big\{E_0\Big[\int_0^t |c(t-z,\ B_z+x;\ u) - c(t-z,\ B_z+y;\ u)|^2dz\Big]\Big\}^{1/2} \end{aligned}$$

$$+ \|f\|_\infty \exp(CT)M\left\{E_0\left[\int_0^t |b(t-z, B_z+x; u) - b(t-z, B_z+y; u)|^4 dz\right]\right\}^{1/4}$$

$$\leqq 2\exp(CT+C_1^2T/2)L_0|x-y|^{\alpha_0}$$

$$+ \|f\|_\infty \exp(CT)\{2T^{3/4}\exp(2C_1T+C_1^2T/2)+M\}C_3t^{1/4}L|x-y|^{\alpha_0}.$$

Therefore Lemma 2 has been proved.

PROOF OF THEOREM 2. We first show, by iteration, that the equation (1.2) has a unique solution in $\mathscr{C}_T(\alpha_0)$. Put $\Phi^{(0)}=I$, $\Phi^{(n)}=\Phi\circ\Phi^{(n-1)}$, $n\geqq 1$, and $K=\|f\|_\infty \exp(CT)$. Using Lemma 2 n-times, we see that $\Phi^{(n)}$ maps $\mathscr{C}_t^K(\alpha_0, L_0)$ into $\mathscr{C}_t^K(\alpha_0, L_n)$ for each $t\leqq T$ with $L_n=\{N+\cdots+N(Jt^{1/4})^{n-1}+(Jt^{1/4})^n\}L_0$.

Let $t_0>0$ be the same as in the proof of Theorem 1, and put $t_1=\min\{t_0, (2J)^{-4}\}$. Then Φ is contractive and has a unique fixed point $u_0=\lim_{n\to\infty}\Phi^{(n)}f$ in $\mathscr{C}_{t_1}^K(\alpha_0, 2NL_0)$. We can uniquely prolong the solution u_0 to $u\in\mathscr{C}_T(\alpha_0)$ in the same manner as in the proof of Theorem 1.

Next we prove that the solution u is unique one of (1.1) in the classical sense, by showing (2.1). Since $b_i(\cdot, \cdot; u)$, $1\leqq i\leqq n$, and $c(\cdot, \cdot; u)$ are in $\mathscr{C}_T(\alpha_0)$, the linear equation (2.2) for $w=u$ has a fundamental solution $U(t, x; s, y; u)$ (cf. [3]). Putting $v(t, x)=\int_{R^n} f(y)U(t, x; 0, y; u)dy$, we can conclude that $v=u$ as follows. Let

$$Y_s = \sum_{i=1}^n \int_0^s b_i(t-z, B_z; u)dB_z^i - (1/2)\sum_{i=1}^n \int_0^s b_i(t-z, B_z; u)^2 dz$$

$$+ \int_0^s c(t-z, B_z; u)dz$$

for $s\in[0, t]$. Then, using Itô's formula, we have

$$E_x[v(t-s, B_s)\exp(Y_s)] = E_x[v(t, B_0)\exp(Y_0)], \qquad s\leqq t. \tag{2.3}$$

For $s=t$, (2.3) becomes

$$E_x\left[f(B_t)\exp\left\{\int_0^t c(t-z, B_z; u)dz\right\}M_t(u)\right] = v(t, x),$$

which implies $v=u$. This theorem has been proved.

§3. Application to an ecological model

We consider the following equation intoroduced by M. Mimura:

$$
(3.1)\quad \begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 v}{\partial x^2} - \dfrac{\partial}{\partial x}\left[\left(\displaystyle\int_x^{x+r} v(t, y)dy - \int_{x-r}^{x} v(t, y)dy\right)v\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t > 0,\ x \in \boldsymbol{R}, \\ v(0, x) = v_0(x) \geqq 0, \quad \displaystyle\int_{\boldsymbol{R}} v_0(x)dx = 1. \end{cases}
$$

This equation (3.1) is regarded as a spatially aggregating population model consisting of a forward equation for the probability density $v(t, x)$, which drifts to the right (left) direction if

$$
\int_{x-r}^{x} v(t, y)dy < \int_{x}^{x+r} v(t, y)dy \qquad (>, \text{ respectively}).
$$

In case of $r=0$, (3.1) is reduced to the heat equation. As an application of our theorem, we have the following

PROPOSITION 1. *Assume that v_0 is bounded Lipschitz continuous. Then the equation* (3.1) *has a unique solution $v^{(r)}$ in $\mathscr{C}_T(1)$ for each $T>0$, and $v^{(r)}$ satisfies*

$$
(3.2)\qquad 0 \leqq v^{(r)}(t, x) \leqq \|v_0\|_\infty + 1/2, \qquad (t, x) \in [0, \infty) \times \boldsymbol{R},
$$

$0 \leqq r \leqq \infty$, *and*

$$
(3.3)\qquad \lim_{r\to r_0} v^{(r)}(t, x) = v^{(r_0)}(t, x),
$$

uniformly in $(t, x) \in [0, T] \times \boldsymbol{R}$ for each $r_0 \in [0, \infty]$ and each $T>0$.

We give the outline of the proof. If (3.1) has a solution v, then $u(t, x) = \int_{-\infty}^{x} v(t, y)dy$ satisfies

$$
(3.4)\qquad u_t = (1/2)u_{xx} + b^{(r)}(t, x; u)u_x,
$$

where $b^{(r)}(t, x; u) = 2u(t, x) - u(t, x+r) - u(t, x-r)$, $0 \leqq r < \infty$, and $b^{(\infty)}(t, x; u) = 2u(t, x) - 1$. By Theorem 2 in §2, the equation (3.4) has a unique solution $u^{(r)}(t, x)$ for each r. We can see that $v^{(r)} = (u^{(r)})_x$ is the unique solution of (3.1) and

$$
0 \leqq v^{(r)}(t, x) \leqq \|v_0\|_\infty + 2u^{(r)}(t, x)\{1 - u^{(r)}(t, x)\},
$$

which implies (3.2). Notice that $\lim_{x\to\pm\infty} v^{(r)}(t, x) = 0$, uniformly in $t \in [0, T]$ and $r \in [0, \infty]$. Hence we obtain (3.3).

REMARK. Let us consider more generally

$$
(3.5)\quad \begin{cases} v_t = (1/2)v_{xx} - \left[\left(\left(\int_R k(x-y)v(t,y)dy\right)v\right]_x, & t>0,\ x\in \boldsymbol{R}, \\ v(0,x) = v_0(x) \geqq 0, \quad \int_R v_0(x)dx = 1. & \end{cases}
$$

Here the function k satisfies the following conditions: (i) k is bounded, (ii) k is differentiable except for a finite number of discontinuous points of the first kind and $k' \in L^1$, and (iii) the limits $\lim_{x\to\pm\infty} k(x)$ exist. In this case, the same kind of method as in (3.1) is also applicable: the equation (3.5) has a unique solution in $\mathscr{C}_T(1) \cap \{v \geqq 0: \int_R v(t,x)dx = 1,\ t\in[0,T]\}$ for each $T>0$ and for any bounded Lipschitz continuous function v_0.

§4. Related problems

1. The time-lag systems. In the theory of the population dynamics, the time-lag systems appear very often. As a simple case, we consider

$$
(4.1)\quad \begin{cases} u_t = (1/2)u_{xx} + b(t,x,u(t-r,x))u_x + c(t,x,u(t-r,x))u, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t>0,\ x\in\boldsymbol{R}, \\ u(s,x) = f(s,x), \quad -r \leqq s \leqq 0,\ x\in\boldsymbol{R}, \end{cases}
$$

for some $r>0$. We put the following conditions:

(A) $b(t,x,y)$ and $c(t,x,y)$ are continuous functions on $[0,\infty)\times\boldsymbol{R}\times\boldsymbol{R}$.

(B) For any $T>0$, there exist constants $C_1, C_2>0$ such that

(i) $|b(t,x,0)|,\ |c(t,x,0)| \leqq C_1, \quad 0\leqq t\leqq T,\ x\in\boldsymbol{R}$,

(ii) $|b(t,x_1,y_1) - b(t,x_2,y_2)| \leqq C_2(|x_1-x_2|) + |y_1-y_2|)$,

$|c(t,x_1,y_1) - c(t,x_2,y_2)| \leqq C_2(|x_1-x_2| + |y_1-y_2|)$,

for $x_1, x_2, y_1, y_2 \in \boldsymbol{R}$, $0\leqq t\leqq T$.

Using Theorem 2 in §2, we obtain

PROPOSITION 2. *Assume* (A) *and* (B). *If the initial function* $f(s,x)$ *is bounded continuous and satisfies* $|f(s,x_1)-f(s,x_2)|\leqq L|x_1-x_2|$, $x_1, x_2\in\boldsymbol{R}$, $-r\leqq s\leqq 0$, *for some* $L>0$, *then the equation* (4.1) *has a unique solution in* $\mathscr{C}_T(1)$ *for each* $T>0$.

In a special case where $b(t,x,y)=b(x,y)$ and $c(t,x,y)=c(x,y)$, the solution of (4.1) is expressed by

$$
u(t,x) = \textstyle\sum_{n=0}^{\infty} (\Psi^{n+1}f^*)(t-nr)\chi_{[nr,(n+1)r)}(t),
$$

with

$$\Psi^1 = \Psi, \; \Psi^{n+1} = \Psi \circ \Psi^n$$

$$(\Psi v)(t, x) = E_x\left[v(r, B_t) \exp\left\{\int_0^t c(B_s, v(t-s, B_s))ds\right\} M_t(v)\right], \; v \in \mathscr{C}_r(1),$$

$$M_t(v) = \exp\left\{\int_0^t b(B_s, v(t-s, B_s))dB_s - (1/2)\int_0^t b(B_s, v(t-s, B_s))^2 ds\right\},$$

$$f^*(t, x) = f(t-r, x), \; 0 \leqq t \leqq r, \; x \in \boldsymbol{R}.$$

2. The Neumann problem. The Neumann problem analogous to (1.1) is formulated as follows:

$$\text{(4.2)} \quad \begin{cases} u_t = (1/2)\triangle u + \sum_{i=1}^n b_i(t, x; u)u_{x_i} + c(t, x; u)u, & t > 0, \; x \in D, \\ u(0, x) = f(x), \quad u_{x_1}(t, x)|_{x_1=0} = 0, \end{cases}$$

where $D = \{x = (x_1, \ldots, x_n) \in \boldsymbol{R}^n : x_1 > 0\}$. If the equation (4.2) has a solution u, then u satisfies

$$\text{(4.3)} \quad u(t, x) = E_x\left[f(X_t) \exp\left\{\int_0^t c(t-s, X_s; u)ds\right\} M_t(u)\right],$$

$$M_t(u) = \exp\left\{\sum_{i=1}^n \int_0^t b_i(t-s, X_s; u)dB_s^i - (1/2)\sum_{i=1}^n \int_0^t b_i(t-s, X_s; u)^2 ds\right\},$$

$$X_t = (X_t^1, B_t^2, \ldots, B_t^n), \quad X_t^1 = B_t^1 - \min\{B_s^1 \wedge 0 : 0 \leqq s \leqq t\},$$

where $\{(B_t^1, \ldots, B_t^n), t > 0\}$ is an n-dimensional Brownian motion. For the equations (4.2) and (4.3), we can obtain the analogous theorems to Theorems 1 and 2.

Let $\mathscr{U}_t^+$ be the Banach space of bounded measurable functions u: $[0, t] \times D \to \boldsymbol{R}$ with the norm $\|u\|_t = \sup\{|u(s, x)| : (s, x) \in [0, t] \times D\}$, and $\mathscr{C}_t^+(\alpha)$ be the space of continuous functions $u \in \mathscr{U}_t^+$, which are Hölder continuous in $x \in D$ with exponent $\alpha > 0$.

Here $b_i(t, x; \cdot)$ and $c(t, x; \cdot)$ are considered as functions on $\mathscr{U}_t^+$. The conditions (I)$\sim$(IV) are called (I)$^+$ $\sim$(IV)$^+$ respectively if $\mathscr{U}_t$, $\boldsymbol{R}^n$, and $\mathscr{C}_t(\alpha)$ in (I)$\sim$(IV) are replaced by $\mathscr{U}_t^+$, D and $\mathscr{C}_t^+(\alpha)$ respectively.

THEOREM 1'. *Under the conditions* (I)$^+$ *and* (II)$^+$ *the equation* (4.3) *has a unique solution in* $\mathscr{U}_T^+$ *for each bounded measurable function* $f \geqq 0$ *and each* $T > 0$.

THEOREM 2'. *Assume* (I)$^+$ $\sim$(IV)$^+$. *If the initial function f is bounded and satisfies* $|f(x) - f(y)| \leqq L_0|x-y|^{\alpha_0}$, $x, y \in D$, *for some* α_0, L_0, *then for each* $T > 0$ *there exists a unique solution* $u \in \mathscr{C}_T^+(\alpha_0)$ *of* (4.2) *in the classical sense.*

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