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On the relatively smooth subhyperalgebras of hyperalgebras

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Introduction

Let H be a hyperalgebra (i.e., a cocommutative irreducible Hopf algebra) over a field k. Then for any cocommutative coalgebra C over k the set $\mathrm{Hom}_{coal}(C, H)$ of all coalgebra homomorphisms of C into H has a group structure. If ρ: $H \to J$ is a homomorphism of hyperalgebras, then ρ induces a group homomorphism of $\mathrm{Hom}_{coal}(C, H)$ into $\mathrm{Hom}_{coal}(C, J)$. It is known that if

$$(*) \qquad k \longrightarrow G \longrightarrow H \longrightarrow J \longrightarrow k$$

is an exact sequence of hyperalgebras, then the induced sequence

$$e \longrightarrow \mathrm{Hom}_{coal}(C, G) \longrightarrow \mathrm{Hom}_{coal}(C, H) \longrightarrow \mathrm{Hom}_{coal}(C, J)$$

of groups is exact for any cocommutative coalgebra C ([13, Proposition 14.12]).

In [15] Yanagihara showed that if the exact sequence $(*)$ is split, then the induced sequence

$$(**) \quad e \longrightarrow \mathrm{Hom}_{coal}(C, G) \longrightarrow \mathrm{Hom}_{coal}(C, H) \longrightarrow \mathrm{Hom}_{coal}(C, J) \longrightarrow e$$

is exact and split for any C. On the other hand, it is proved by Dieudonné in [1, Proposition 8, Chapter 2] that when H and J in $(*)$ are of finite type and reduced over a perfect field k, the sequence $(**)$ is exact for any C if and only if G is reduced. Moreover Takeuchi showed in [10, Theorem 1.8.1] that if the homomorphism $H \to J$ is smooth in the sense of [10], then $(**)$ is exact for any C. (Actually, the smoothness of the homomorphism is stronger than the exactness of $(**)$.)

In this paper we will generalize the above results and give several characterizations for $(**)$ to be exact.

Let G be a subhyperalgebra of a hyperalgebra H and $J = H/HG^+$ the quotient coalgebra. Consider the sequence

$$k \longrightarrow G \longrightarrow H \longrightarrow J \longrightarrow k,$$

where $G \to H$ is the inclusion and $H \to J$ is the natural projection. We will prove in Section 1 that the following conditions are equivalent: (1) The induced map $\mathrm{Hom}_{coal}(C, H) \to \mathrm{Hom}_{coal}(C, J)$ is surjective for any cocommutative coalgebra

C; (2) J has a coalgebra splitting; (3) G has a coalgebra retraction; (4) H is isomorphic as a coalgebra to $J\otimes G$ canonically (Proposition 1.2 and Theorem 1.3).

We will call a subhyperalgebra G of H relatively smooth if G has a coalgebra retraction. This terminology could be justified if we show that every smooth subhyperalgebra of any hyperalgebra is always relatively smooth, and that if H is smooth then every relatively smooth subhyperalgebra of H is itself smooth. These will be proved in the final section (Theorems 3.6, 3.8).

In Section 2 we consider sequences of divided powers in connection with relative smoothness. We will prove a structure theorem (Theorem 2.14): When H is a stable hyperalgebra over a perfect field, a subhyperalgebra G of H is relatively smooth if and only if every primitive element of G has the same coheight in G as in H.

Throughout this paper we fix a ground field k. All coalgebras and Hopf algebras we consider and their tensor products are defined over k. We further assume that all coalgebras in this paper are pointed, irreducible and cocommutative. For such a coalgebra C we denote the unique grouplike element of C by 1_C (or simply by 1 if we have no confusion on C). Many other notations (e.g., the structure map of coalgebras, Σ-notation) will follow those in the book of Sweedler [7].

Finally, the author wishes to take this opportunity of expressing his gratitude to Prof. S. Tôgô for his valuable advices and continual encouragement.

1. Coalgebra splittings and retractions

We begin with the following proposition which will play an important role in this paper with Proposition 1.2. However, the proof is so easy that we omit it.

PROPOSITION 1.1. *Let C, D be coalgebras and $\rho: C\to D$ a coalgebra map. Then the following two conditions are equivalent:*

(1) *For each coalgebra F and each coalgebra map $f: F\to D$ there is a coalgebra map $g: F\to C$ such that $f=\rho\circ g$.*

(2) *ρ has a coalgebra splitting, that is, there is a coalgebra map $\lambda: D\to C$ such that $\rho\circ\lambda=\mathrm{id}_D$.*

Let $\rho: C\to D$ be a coalgebra map. The *h-kernel* of ρ is defined as

$$h\text{-}\ker\rho = \{c\in C \mid (\mathrm{id}_C\otimes\rho)\Delta(c) = c\otimes 1_D\}.$$

h-$\ker\rho$ is a subcoalgebra of C and it is the largest subcoalgebra contained in $k1_C+\ker\rho$. It is easy to see that a subcoalgebra E of C is contained in h-$\ker\rho$ if and only if $\rho(E)=k1_D$ or equivalently the restriction of ρ to E coincides with the counit ε of E.

For a coalgebra C we denote by $P(C)$ the space of primitive elements of C:

$$P(C) = \{c \in C \mid \Delta(c) = 1 \otimes c + c \otimes 1\}.$$

It is convenient to take notice of the following facts:

$$\ker \rho \cap P(C) = h\text{-}\ker \rho \cap P(C) = P(h\text{-}\ker \rho)$$

for any coalgebra map ρ of C into any coalgebra.

We say that a sequence

$$\cdots \longrightarrow C_{i-1} \xrightarrow{\rho_i} C_i \xrightarrow{\rho_{i+1}} C_{i+1} \longrightarrow \cdots$$

of coalgebras and coalgebra maps is exact if $\rho_i(C_{i-1}) = h\text{-}\ker \rho_{i+1}$ for all i. It is easy to see that a sequence

$$k \longrightarrow C \xrightarrow{\rho} D \qquad (\text{resp. } C \xrightarrow{\rho} D \longrightarrow k)$$

is exact if and only if ρ is injective (resp. surjective).

Let $\rho: C \to D$ be a coalgebra map. Then C has a natural D-comodule structure with the structure map $\tilde{\rho}: C \to D \otimes C$ given by $\tilde{\rho}(c) = \sum \rho(c_{(1)}) \otimes c_{(2)}$. The following proposition gives a (partial) generalization of Theorem 1.8.1 in [10].

PROPOSITION 1.2. *Let $\rho: C \to D$ be a surjective coalgebra map with h-kernel E. Then the following conditions are equivalent:*

(1) *E has a coalgebra retraction in C (that is, there is a coalgebra map η: $C \to E$ which is identical on E), and C is injective as a D-comodule.*

(2) *There is a coalgebra isomorphism $\theta: C \to D \otimes E$ such that* $(\mathrm{id}_D \otimes \varepsilon)\theta = \rho$.

PROOF. (1)$\Rightarrow$(2). Let η be a retraction of E in C. It follows from the UMP of the tensor product of coalgebras that there is a coalgebra map θ: $C \to D \otimes E$ such that the diagram

$$\begin{tikzcd} & E \\ C \arrow[ur, "\eta"] \arrow[r, "\theta"] \arrow[dr, "\rho"'] & D \otimes E \arrow[u, "\varepsilon \otimes \mathrm{id}_E"'] \arrow[d, "\mathrm{id}_D \otimes \varepsilon"] \\ & D \end{tikzcd}$$

is commutative. To prove that θ is injective we may verify that the restriction of θ to $P(C)$ is injective. Let x be any element of $P(C)$ such that $\theta(x) = 0$. Then we have $\rho(x) = 0$. Since x is a primitive element of C, it follows that $x \in h\text{-}\ker \rho = E$. Thus $x = \eta(x) = (\varepsilon \otimes \mathrm{id}_E)\theta(x) = 0$. This proves the injectivity of θ.

The surjectivity of θ is proved as in the proof of Theorem 1.8.1 in [10]. Since θ is a coalgebra map it is also a D-comodule map, where $D \otimes E$ has a D-

comodule structure via $\mathrm{id}_D \otimes \varepsilon$. The assumption that C is injective implies that there is a D-subcomodule V of $D \otimes E$ such that

$$D \otimes E = \theta(C) \otimes V.$$

On the other hand, it is easily seen that the sequence

$$k \longrightarrow E \xrightarrow{\iota} D \otimes E \xrightarrow{\mathrm{id}_D \otimes \varepsilon} D \longrightarrow k$$

is exact, where $\iota(x) = 1_D \otimes x$ for all $x \in E$. In particular, h-ker $(\mathrm{id}_D \otimes \varepsilon) = k1_D \otimes E$. We see that $\theta(E) \subset h$-ker $(\mathrm{id}_D \otimes \varepsilon)$ by $(\mathrm{id}_D \otimes \varepsilon)\theta = \rho$. Therefore for $x \in E$ there is $y \in E$ such that $\theta(x) = 1_D \otimes y$. Applying $\varepsilon \otimes \mathrm{id}_E$ to this, we have $y = \eta(x) = x$. Thus we have $\theta(x) = 1_D \otimes x$ for all $x \in E$, and in particular $\theta(E) = k1_D \otimes E$. It follows that $k1_D \otimes E \subset \theta(C)$, and then we have

$$(k1_D \otimes E) \cap V = 0.$$

From Lemma 1.8.3 in [10] it follows that $V = 0$, so that $\theta(C) = D \otimes E$. This proves the surjectivity of θ.

(2)$\Rightarrow$(1). Since θ is a coalgebra isomorphism and $(\mathrm{id}_D \otimes \varepsilon)\theta = \rho$, we have

$$\theta(E) = \theta(h\text{-ker}\, \rho) = h\text{-ker}\, (\mathrm{id}_D \otimes \varepsilon) = k1_D \otimes E.$$

Therefore θ induces an isomorphism $\theta|_E \colon E \to k1_D \otimes E$. If we put

$$\eta = (\theta|_E)^{-1}(\varepsilon \otimes \mathrm{id}_E)\theta \colon C \longrightarrow E,$$

then η is a coalgebra map which is identical on E.

It is well known that if $D \otimes E$ is equipped with a D-comodule structure by $\Delta_D \otimes \mathrm{id}_E$, then $D \otimes E$ is injective. Hence, to prove that C is an injective D-comodule we have only to show that θ is a D-comodule map, that is, the diagram

$$\begin{array}{ccc} C & \xrightarrow{\tilde{\rho}} & D \otimes C \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_D \otimes \theta} \\ D \otimes E & \xrightarrow[\Delta_D \otimes \mathrm{id}_E]{} & D \otimes D \otimes E \end{array}$$

is commutative. We have

$$\begin{aligned} (\mathrm{id}_D \otimes \theta)\tilde{\rho} &= (\mathrm{id}_D \otimes \theta)(\rho \otimes \mathrm{id}_C)\Delta_C \\ &= (\mathrm{id}_D \otimes \theta)\,[(\mathrm{id}_D \otimes \varepsilon_E)\theta \otimes \mathrm{id}_C]\Delta_C \\ &= (\mathrm{id}_D \otimes \varepsilon_E \otimes \mathrm{id}_{D \otimes E})(\theta \otimes \theta)\Delta_C \\ &= (\mathrm{id}_D \otimes \varepsilon_E \otimes \mathrm{id}_D \otimes \mathrm{id}_E)\Delta_{D \otimes E}\theta \end{aligned}$$

since θ is a coalgebra map. By the definition of the comultiplication of the tensor product coalgebra we have

$$(\mathrm{id}_D \otimes \varepsilon_E \otimes \mathrm{id}_D \otimes \mathrm{id}_E)\Delta_{D\otimes E} = \Delta_D \otimes \mathrm{id}_E.$$

Therefore we have

$$(\mathrm{id}_D \otimes \theta)\tilde{\rho} = (\Delta_D \otimes \mathrm{id}_E)\theta,$$

which proves the commutativity of the diagram. It follows that C is injective. This completes the proof.

If C and D are coalgebras, we denote by $\mathrm{Hom}_{coal}(C, D)$ the set of all coalgebra maps of C into D.

A Hopf algebra is called a hyperalgebra if the underlying coalgebra is pointed, irreducible and cocommutative. Let H be a hyperalgebra. Then, for any coalgebra C, the set $\mathrm{Hom}_{coal}(C, H)$ has a group structure: the multiplication of f and g in $\mathrm{Hom}_{coal}(C, H)$ is given by $f*g=\mu(f\otimes g)\Delta$, the unit element is ε and the inverse of f is $S\circ f$, where S is the antipode of H. If J is a hyperalgebra and ρ: $H\to J$ is a hyperalgebra map, then ρ induces a group homomorphism ρ_C: $\mathrm{Hom}_{coal}(C, H)\to \mathrm{Hom}_{coal}(C, J)$, $\rho_C(f)=\rho\circ f$ for each $f\in \mathrm{Hom}_{coal}(C, H)$.

A sequence of hyperalgebras is said to be exact if it is exact as a sequence of coalgebras.

The following fact is proved in [15, Lemma 6]: A sequence

$$k \longrightarrow G \xrightarrow{j} H \xrightarrow{\rho} J$$

of hyperalgebras is exact if and only if the induced sequence

$$e \longrightarrow \mathrm{Hom}_{coal}(C, G) \xrightarrow{j_C} \mathrm{Hom}_{coal}(C, H) \xrightarrow{\rho_C} \mathrm{Hom}_{coal}(C, J)$$

of groups is exact for all coalgebra C.

Proposition 1.1 asserts that ρ_C is surjective (for all C) if and only if ρ has a coalgebra splitting.

In the following we give some characterizations for ρ to have a coalgebra splitting. Let H be a hyperalgebra, I a left ideal coideal of H and ρ: $H\to H/I$ the natural map. Note that H/I has a left H-module structure and a coalgebra structure, and ρ is an H-linear coalgebra map, that is, ρ is an H-module map as well as a coalgebra map. Let G be the h-kernel of ρ. Then G is a subhyperalgebra of H and $I=HG^+$, the left ideal generated by $G^+=G\cap \ker\varepsilon$. The correspondence $I\mapsto G$ gives a bijection between the set of all left ideal coideals and the set of all subhyperalgebras ([5]). We denote by $H//G$ the quotient (H-module) coalgebra H/HG^+.

The following result has been proved by Takeuchi (see [12, Theorem 4] and

[11, Proposition A.2.2]): Let $J=H//G$ be the quotient H-module coalgebra. Then H is an injective J-comodule.

We now show the following as our main theorem.

THEOREM 1.3. *Let H be a hyperalgebra, J a coalgebra with a left H-module structure and $\rho\colon H\to J$ a surjective H-linear coalgebra map with h-kernel G. Then the following conditions are equivalent:*

(1) *ρ has a coalgebra splitting.*

(2) *G has a coalgebra retraction in H.*

(3) *There is a coalgebra isomorphism $\theta\colon H\to J\otimes G$ such that $(\mathrm{id}_J\otimes\varepsilon)\theta=\rho$.*

PROOF. The implication (3)⇒(1) is clear, and (2)⇒(3) follows directly from Takeuchi's result and Proposition 1.2.

(1)⇒(2). Let λ be a coalgebra splitting of ρ. Put $\eta=S\lambda\rho*\mathrm{id}_H$. Since H is cocommutative, it is easy to see that η is a coalgebra map of H into itself. We first show that the image of η is actually contained in G. To see this, it suffices to prove that $S\eta(H)\subset G$ because G is a subhyperalgebra and in particular G is stable under S and $S^2=\mathrm{id}_H$ [7, Proposition 4.0.1]. Since G is the h-kernel of ρ and since $S\eta(H)$ is a subcoalgebra of H, it is enough to verify that $\rho(S\eta(H))=k$. Now let h be any element of H. Then we have

$$\begin{aligned}\rho S\eta(h) &= \rho(\Sigma\, S(h_{(1)})\lambda\rho(h_{(2)}))\\ &= \Sigma\, S(h_{(1)})\rho\lambda\rho(h_{(2)}) \quad \text{(since } \rho \text{ is } H\text{-linear)}\\ &= \Sigma\, S(h_{(1)})\rho(h_{(2)})\\ &= \rho(\Sigma\, S(h_{(1)})h_{(2)})\\ &= \varepsilon(h).\end{aligned}$$

η is identical on G. Indeed, since $\rho=\varepsilon$ on G we have for any x in G

$$\begin{aligned}\eta(x) &= \Sigma\, S\lambda\rho(x_{(1)})x_{(2)}\\ &= \Sigma\, \varepsilon(x_{(1)})x_{(2)}\\ &= x.\end{aligned}$$

This completes the proof.

In the proof of (1)⇒(2) of the above theorem η is furthermore a right G-linear map. For any $h\in H$, $x\in G$ we have

$$\begin{aligned}\eta(hx) &= \Sigma\, S\lambda\rho(h_{(1)}x_{(1)})h_{(2)}x_{(2)}\\ &= \Sigma\, S\lambda(h_{(1)}\rho(x_{(1)}))h_{(2)}x_{(2)}\\ &= \Sigma\, S\lambda(h_{(1)}\varepsilon(x_{(1)})\rho(1))h_{(2)}x_{(2)} \quad \text{(since } x_{(1)}\in G)\end{aligned}$$

$$= \Sigma\, S\lambda\rho(h_{(1)})h_{(2)}x$$
$$= \eta(h)x.$$

If we put $\eta' = S\eta S$, then it follows that η' is a coalgebra map of H into G which is left G-linear. As a consequence of Theorem 1.3 we have

COROLLARY 1.4. *Let G be a subhyperalgebra of a hyperalgebra H. If G has a coalgebra retraction, then there is also a right/left G-linear coalgebra retraction.*

REMARK 1.5. If G has a right G-linear coalgebra retraction, then it is easily shown that H gives rise to a right G-Hopf module. And then with the aid of Theorem 4.1.1 in [7] we can also prove that H is actually isomorphic to $J \otimes G$ as coalgebras, where $J = H//G$.

DEFINITION 1.6. Let H be a hyperalgebra and G a subhyperalgebra of H. We call G *relatively smooth in H* if G has a coalgebra retraction.

DEFINITION 1.7. We call an exact sequence

$$k \longrightarrow G \longrightarrow H \longrightarrow J \longrightarrow k$$

of hyperalgebras *strongly exact* if the induced sequence

$$e \longrightarrow \mathrm{Hom}_{coal}(C, G) \longrightarrow \mathrm{Hom}_{coal}(C, H) \longrightarrow \mathrm{Hom}_{coal}(C, J) \longrightarrow e$$

of groups is exact for every coalgebra C.

By Proposition 1.1 and Theorem 1.3 we have one of the main theorems of this paper which generalizes Corollary to Lemma 6 in [15].

THEOREM 1.8. *An exact sequence*

$$k \longrightarrow G \longrightarrow H \longrightarrow J \longrightarrow k$$

of hyperalgebras is strongly exact if and only if G is relatively smooth in H.

We should notice that Theorem 1.3 has a similar form to Theorem 2.2 in [6]. And, in fact, we have some results which are analogous to those in [6, §2].

PROPOSITION 1.9. *Consider a commutative diagram of hyperalgebras*

$$\begin{array}{ccccccccc}
k & \longrightarrow & G_1 & \xrightarrow{j_1} & H_1 & \xrightarrow{\rho_1} & J_1 & \longrightarrow & k \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
k & \longrightarrow & G_2 & \xrightarrow{j_2} & H_2 & \xrightarrow{\rho_2} & J_2 & \longrightarrow & k,
\end{array}$$

where the two rows are exact. Then:

(1) *If the upper row is strongly exact and there is a coalgebra map* γ': $J_2 \to J_1$ *such that* $\gamma\gamma' = \mathrm{id}_{J_2}$, *then the lower row is also strongly exact.*

(2) *If the lower row is strongly exact and there is a coalgebra map* α': $G_2 \to G_1$ *such that* $\alpha'\alpha = \mathrm{id}_{G_1}$, *then the upper row is also strongly exact.*

PROOF. These follow from Theorems 1.3 and 1.8.

A subhyperalgebra N of H is said to be *normal* if $HN^+ = N^+H$ or equivalently if $\Sigma x_{(1)}yS(x_{(2)}) \in N$ for all $x \in H$, $y \in N$. In this case the left ideal coideal HN^+ is a Hopf ideal (i.e., a two-sided ideal coideal) and so the quotient $H//N$ is a hyperalgebra.

COROLLARY 1.10. *Let N, G be normal subhyperalgebras of a hyperalgebra H such that $NG = H$. If $N \cap G$ is relatively smooth in N, then G is relatively smooth in H.*

PROOF. We have a commutative diagram with exact rows

$$\begin{tikzcd}
k \arrow[r] & N \cap G \arrow[r] \arrow[d, hook] & N \arrow[r] \arrow[d, hook] & N//(N \cap G) \arrow[r] \arrow[d, "\gamma"] & k \\
k \arrow[r] & G \arrow[r] & H \arrow[r] & H//G \arrow[r] & k .
\end{tikzcd}$$

Then γ is an isomorphism by Theorem 3 in [14]. The assertion follows from Theorem 1.8 and Proposition 1.9.

COROLLARY 1.11. *Let N, G be normal subhyperalgebras of a hyperalgebra H and $N \cap G$. If G is relatively smooth in H, then $G//N$ is relatively smooth in $H//N$.*

PROOF. The corollary follows from the isomorphism

$$\frac{H//N}{G//N} \cong H//G$$

of [14, Corollary 1 to Theorem 2].

2. Coheight condition

In this section we use sequences of divided powers to give a necessary condition for a subhyperalgebra to be relatively smooth. And we show that, under a certain condition, it is also sufficient. Throughout this section k is a perfect field of positive characteristic p.

Let C be a coalgebra. Then we have the V-map V of C defined by the equation ([2]):

$$\langle c^*, V(c)\rangle = \langle c^{*p}, c\rangle^{\frac{1}{p}}$$

for each $c \in C$ and $c^* \in C^*$, where C^* is the dual algebra of C. V is a $\frac{1}{p}$-linear coalgebra map of C into itself.

For a nonnegative integer r the iteration V^r of V by r times is a $\left(\frac{1}{p^r}\text{-linear}\right)$ coalgebra map and so its kernel $\ker V^r$ is a coideal. We denote by $C_{(r)}$ the h-kernel of the natural coalgebra map $C \to C/\ker V^r$, and call it the r-th Frobenius subcoalgebra of C.

LEMMA 2.1. *Let C be a coalgebra. Then*:

(1) $C_{(0)} = k \subset C_{(1)} \subset C_{(2)} \subset \cdots ;\ C = \bigcup_r C_{(r)}$.

(2) $V(C_{(r)}) \subset C_{(r-1)}$ *for* $r = 1, 2, \ldots$.

(3) *If C' is a subcoalgebra of C, then* $C'_{(r)} = C' \cap C_{(r)}$ *for* $r = 0, 1, 2, \ldots$.

(4) *If D is a coalgebra and $\rho\colon C \to D$ is a coalgebra map, then* $\rho(C_{(r)}) \subset D_{(r)}$ *for* $r = 0, 1, 2, \ldots$.

PROOF. Since, by definition, $V^0 = \mathrm{id}_C$, we have $C_{(0)} = k$. The inclusion $C_{(r)} \subset C_{(r+1)}$ holds since $\ker V^r \subset \ker V^{r+1}$ for every r. Any element of C generates a finite-dimensional subcoalgebra C' whose dual (commutative) algebra contains a nilpotent maximal ideal $\mathfrak{m}$ of codimension one. If $\mathfrak{m}^{p^r} = 0$, it can be seen that $C' \subset k1_C + \ker V^r$. Hence we have $C' \subset C_{(r)}$, which proves $C \subset \bigcup_r C_{(r)}$.

(2) follows directly from the definition of Frobenius subcoalgebras.

(3) and (4) follow from the fact that any coalgebra map commutes with the V-maps [2, Proposition 4.1.5].

For a primitive element x of C a sequence

$$1 = {}^0x,\ x = {}^1x,\ {}^2x, \ldots$$

of elements of C is called a sequence of divided powers over x if

$$\Delta({}^nx) = \textstyle\sum_{i=0}^{n} {}^ix \otimes {}^{n-i}x$$

for $n = 0, 1, 2, \ldots$. Then we know ([2, Proposition 4.1.9]) that

$$V({}^nx) = \begin{cases} 0 & \text{if } p \nmid n \\ {}^{\frac{n}{p}}x & \text{if } p \mid n . \end{cases}$$

LEMMA 2.2. *Let C be a coalgebra and $1 = {}^0x, {}^1x, {}^2x, \ldots$ be a sequence of divided powers over ${}^1x \neq 0$. Then*:

(1) ${}^nx \in C_{(\|n\|+1)}$, *where $\|n\|$ is the integer such that* $p^{\|n\|} \leq n < p^{\|n\|+1}$.

(2) ${}^{p^r}x \in C_{(r+1)}$ *but* ${}^{p^r}x \notin C_{(r)}$ *for* $r = 0, 1, 2, \ldots$.

PROOF. By the above remark, if $1 \leq j \leq n$, we have $V^{\|n\|+1}({}^j x)=0$. This implies that ${}^n x \in h$-ker $(C \to C/\ker V^{\|n\|+1})$, which proves (1).

(2) We see that ${}^{p^r}x \in C_{(r+1)}$ by (1) because $\|p^r\|=r$. Finally assume that ${}^{p^r}x \in C_{(r)}$. It follows that $V^r({}^{p^r}x)=0$. On the other hand, since $V^r({}^{p^r}x)={}^1x$ by the above remark, we have ${}^1x=0$, which contradicts the assumption.

Note that if H is a hyperalgebra, then Frobenius subcoalgebras of H are subhyperalgebras since the V-map of H is a $\left(\frac{1}{p}\text{-linear}\right)$ Hopf algebra map. Moreover we know by Theorem 1 in [8] that $H_{(1)}$ is equal to the restricted enveloping algebra of the restricted Lie algebra $P(H)$.

A primitive element x of a hyperalgebra H is said to have coheight t if x belongs to $V^t(H)$, and to have infinite coheight if x belongs to $V^\infty(H)=\bigcap_{n\geq 0} V^n(H)$. We denote the (maximal) coheight of x by (max-)$\mathrm{coh}_H\, x$.

As for the connection between coheights and sequences of divided powers we know some important results due to Sweedler ([8], [9]) and Newman ([3]). We list them as a lemma.

LEMMA 2.3. *Let H be a hyperalgebra.*

(1) *If $1={}^0x, {}^1x,\dots, {}^{p^{n+1}-1}x$ is a sequence of divided powers over ${}^1x$, then ${}^1x$ has coheight n ($n=0, 1, 2,\dots, \infty$).*

(2) *If a primitive element x has coheight $n(<\infty)$, then there is a sequence of divided powers over x of length $p^{n+1}-1$.*

(3) *If $V(V^\infty(H))=V^\infty(H)$ and a primitive element x has infinite coheight, then there is an infinite sequence of divided powers over x.*

(4) *If $V(V^\infty(H))=V^\infty(H)$ and H has a Sweedler basis $\{x_i\}_{i\in I}$ (see* [3] *for the definition) with I ordered and if*

$$1 = {}^0x_i,\ x_i = {}^1x_i,\ {}^2x_i,\dots,\ {}^{p^{t_i+1}-1}x_i\ (t_i=\text{max-coh}_H x_i)$$

is a sequence of divided powers over x_i for each i, then the ordered monomials

$$\prod{}^a x = {}^{a_1}x_{i_1}\, {}^{a_2}x_{i_2} \cdots {}^{a_r}x_{i_r}$$

form a linear basis for H, where $i_1<i_2<\cdots<i_r$ in I and $a_j<p^{t_{i_j}+1}$ for $j=1, 2,\dots, r$.

DEFINITION 2.4. Let H be a hyperalgebra and G a subhyperalgebra of H. We say that G satisfies *the coheight condition* (*in H*) provided that for all primitive elements x of G we have

$$\text{max-coh}_G x = \text{max-coh}_H x.$$

REMARK 2.5. To say that G satisfies the coheight condition is equivalent to saying that if x is a primitive elements of G then, for any integer t not exceeding max-coh$_H x$, there is a sequence of divided powers over x of length $p^{t+1}-1$ lying in G.

PROPOSITION 2.6. *Let G be a subhyperalgebra of a hyperalgebra H. If G is relatively smooth in H, then G satisfies the coheight condition.*

PROOF. By definition there is a coalgebra map $\eta: H \to G$ which is identical on G. Let x be any primitive element of G. If $1={}^0x$, ${}^1x=x,\dots$ is a sequence of divided powers over x in H, then it is clear that their images $1=\eta(1)$, $x=\eta(x)$, $\eta({}^2x),\dots$ under η form a sequence of divided powers over x lying in G with the same length as the original one. Therefore x has the same maximal coheight in G as in H. This proves the proposition.

A hyperalgebra is said to be *bounded* if the coheights of primitive elements are bounded.

LEMMA 2.7. *A hyperalgebra H is bounded if and only if $H=H_{(r)}$ for some integer r.*

PROOF. If H is bounded, then there is an integer r such that all sequences of divided powers lie in $H_{(r)}$ by Lemma 2.2. In this case we have $P(H) \cap V^r(H)=0$. In fact, if $x \neq 0$ belongs to $P(H) \cap V^r(H)$, then by Lemma 2.3 (2) there is a sequence of divided powers over x of length $p^{r+1}-1$. However, Lemma 2.2 (2) asserts ${}^{p^r}x \notin H_{(r)}$, which contradicts the choice of r. It follows that $V^r(H)=k1$ by Corollary 11.0.2. in [7] since $V^r(H)^+$ is a coideal of H. Therefore $H \subset k1+\ker V^r$, which implies $H=H_{(r)}$. The converse is clear.

The following theorems gives a partial converse to Proposition 2.6. The proof is based on a theorem of Newman [5].

THEOREM 2.8. *Let H be a bounded hyperalgebra. If a subhyperalgebra G of H satisfies the coheight condition, then G is relatively smooth in H.*

PROOF. By Lemma 1.1 in [5] H has a Sweedler basis B containing a Sweedler basis B' of G. For $x_\alpha \in B$, let $n_\alpha=$max-coh$_H x_\alpha$ and select sequences of divided powers over x_α with maximal length:

$$1 = {}^0g_\alpha,\ x_\alpha = {}^1g_\alpha,\ {}^2g_\alpha,\dots,\ {}^{p^{n_\alpha+1}-1}g_\alpha \quad \text{in} \quad G\ (x_\alpha \in B')$$

and

$$1 = {}^0h_\beta,\ x_\beta = {}^1h_\beta,\ {}^2h_\beta,\dots,\ {}^{p^{n_\beta+1}-1}h_\beta \quad \text{in} \quad H\ (x_\beta \in B-B').$$

Then by Theorem 1.3 in [5] the set of ordered products $\{\prod^b h \prod^a g\}$ forms a basis for H. And we see that the comultiplication Δ of H is given by the formula

$$\Delta(\prod^b h \prod^a g) = \sum_{b'+b''=b,\, a'+a''=a} \prod^{b'} h \prod^{a'} g \otimes \prod^{b''} h \prod^{a''} g .$$

Therefore the map η defined by $\prod^b h \prod^a g \mapsto \varepsilon(\prod^b h) \prod^a g$ is easily seen to be a coalgebra map of H into G which is identical on G. Hence G is relatively smooth in H.

LEMMA 2.9. *For $x \in P(H)$, we have*

$$\text{max-coh}_{H_{(r)}} x = \min \{r-1, \text{max-coh}_H x\}$$

for $r=1, 2, \dots$.

PROOF. Let $t=\min\{r-1, \text{max-coh}_H x\}$. Then there is a sequence of divided powers over x of length $p^{t+1}-1$ in H, say,

$$1,\ x = {}^1x,\ {}^2x, \dots,\ {}^{p^{t+1}-1}x.$$

Since $\|p^{t+1}-1\|=t$ this sequence lies in $H_{(r)}$ by Lemma 2.2, whence we have $\text{max-coh}_{H_{(r)}} x \geq t$. The inverse inequality is clear by Lemma 2.2. This proves the lemma.

PROPOSITION 2.10. *Let H be a hyperalgebra and G a subhyperalgebra of H. Then G satisfies the coheight condition in H if and only if, for every r, $G_{(r)}$ satisfies the coheight condition in $H_{(r)}$.*

PROOF. Assume that G satisfies the coheight condition in H and let x be any primitive element of G. Then by Lemma 2.9 and by the assumption we have for each r

$$\begin{aligned} \text{max-coh}_{G_{(r)}} x &= \min \{r - 1, \text{max-coh}_G x\} \\ &= \min \{r - 1, \text{max-coh}_H x\} \\ &= \text{max-coh}_{H_{(r)}} x. \end{aligned}$$

Conversely assume that $G_{(r)}$ satisfies the coheight condition in $H_{(r)}$ for every r. Let x be an element in $P(G)$. If $\text{max-coh}_H x < \infty$, then we have

$$\text{max-coh}_H x = \text{max-coh}_{H_{(r)}} x$$

for a sufficiently large r by Lemma 2.9. Thus by the assumption we have

$$\begin{aligned} \text{max-coh}_{H_{(r)}} x &= \text{max-coh}_{G_{(r)}} x \\ &\leq \text{max-coh}_G x. \end{aligned}$$

Consequently we have

$$\text{max-coh}_H x = \text{max-coh}_G x.$$

On the other hand if $\text{max-coh}_H\, x = \infty$, we have, for any $n > 0$,

$$\begin{aligned} \text{max-coh}_G\, x &\geq \text{max-coh}_{G_{(n)}}\, x \\ &= \text{max-coh}_{H_{(n)}}\, x \\ &= n-1. \end{aligned}$$

This implies that $\text{max-coh}_G\, x = \infty$. Therefore we have, in each case,

$$\text{max-coh}_G\, x = \text{max-coh}_H x,$$

which proves the proposition.

LEMMA 2.11. *Let A, B be subhyperalgebras of a hyperalgebra H satisfying the following conditions*:

(1) *$A \subset B$ and A satisfies the coheight condition in B,*

(2) *$P(A) = P(B)$.*

Then we have $A = B$.

PROOF. It is sufficient to prove that $A_{(r)} = B_{(r)}$ for all r. Since $A_{(r)} \subset B_{(r)}$ and $A_{(r)}$ satisfies the coheight condition in $B_{(r)}$ by Proposition 2.10 and since $B_{(r)}$ is bounded, it follows from condition (2) and a simialr argument to that in the proof of Theorem 2.8 that $A_{(r)}$ contains a linear basis for $B_{(r)}$. Hence $A_{(r)} = B_{(r)}$.

PROPOSITION 2.12. *Let G be a subhyperalgebra of a hyperalgebra H. Then G satisfies the coheight condition in H if and only if $G \cap V^r(H) = V^r(G)$ for all r.*

PROOF. ($\Leftarrow$) Let x be a primitive element of G. Assume that $\text{coh}_H\, x \geq r$ for a nonnegative integer r. Then we have

$$x \in G \cap V^r(H) = V^r(G),$$

which implies that $\text{max-coh}_G\, x \geq r$. It follows that

$$\text{max-coh}_G\, x = \text{max-coh}_H\, x,$$

and this proves the coheight condition of G.

($\Rightarrow$) We may check that the conditions of Lemma 2.11 hold. First, it is clear that $V^r(G) \subset G \cap V^r(H)$. Take any x of $P(G \cap V^r(H))$. Then we have $\text{max-coh}_H\, x \geq r$. Since G satisfies the coheight condition in H, it follows that $\text{max-coh}_G\, x \geq r$. We see by definition that $x \in V^r(G) \cap P(G) = P(V^r(G))$. Therefore we have

$$P(G \cap V^r(H)) = P(V^r(G)).$$

Finally we show that $V^r(G)$ satisfies the coheight condition in $G \cap V^r(H)$.

Let $x \in P(V^r(G))$ have the maximal coheight s in $G \cap V^r(H)$. Then we have

$$x \in V^s(G \cap V^r(H)) \subset V^{r+s}(H).$$

Thus x has coheight $r+s$ in H, so that max-coh$_G\, x \geq r+s$ by the coheight condition of G. Therefore there exists an element u in G such that $x = V^{r+s}(u)$. Since $V^r(u) \in V^r(G)$, it follows that

$$x = V^s(V^r(u)) \in V^s(V^r(G)),$$

which implies that max-coh$_{V^r(G)}\, x \geq s$. Hence max-coh$_{V^r(G)}\, x = s$. This completes the proof.

A hyperalgebra H is said to be *stable* if ther is an integer n_0 such that $V^n(H) = V^{n_0}(H)$ for $n \geq n_0$.

LEMMA 2.13. *Let H be a stable hyperalgebra and G a subhyperalgebra of H satisfying the coheight condition. Then G is also stable and H has a Sweedler basis which contains a Sweedler basis of G.*

PROOF. Let n_0 be an integer such that $V^n(H) = V^{n_0}(H)$ for $n \geq n_0$. If $n \geq n_0$, then we have by Proposition 2.12 that

$$V^n(G) = G \cap V^n(H) = G \cap V^{n_0}(H) = V^{n_0}(G).$$

It follows that G is stable.

To construct a Sweedler basis we first take a linear basis of $P(V^{n_0}(G))$. Extend this to a basis of $P(V^{n_0}(H))$ and to a basis of $P(V^{n_0-1}(G))$ independently. The coheight condition of G implies that

$$P(V^{n_0}(H)) \cap P(V^{n_0-1}(G)) = P(V^{n_0}(G)).$$

It follows that the union of the bases of $P(V^{n_0}(H))$ and of $P(V^{n_0-1}(G))$ is linearly independent, so that it forms a basis of the subspace $P(V^{n_0}(H)) + P(V^{n_0-1}(G))$. Extend this to a basis of $P(V^{n_0-1}(H))$. Continuing this procedure, we finally obtain a basis of $P(H)$ which we desired.

Let H and G be as above. Then H and G have linear bases as in the proof of Theorem 2.8. Thus we can conclude:

THEOREM 2.14. *Let H be a stable hyperalgebra and G a subhyperalgebra of H. The G is relatively smooth in H if and only if G satisfies the coheight condition in H.*

COROLLARY 2.15. *Let G be a subhyperalgebra of a stable hyperalgebra H. If there is a coalgebra map of H into G which is identical on $P(G)$, then*

G has a right/left G-linear coalgebra retraction.

PROOF. If there is a coalgebra map of H into G which is identical on $P(G)$, then as is seen in the proof of Proposition 2.6 G satisfies the coheight condition and so G is relatively smooth. Therefore the assertion follows from Corollary 1.4.

3. Smooth hyperalgebras

In this section we consider smooth hyperalgebras in connection with relative smoothness. Following [10, 1.9.5] we say that a coalgebra is smooth if it is isomorphic to $B(U)$ for some vector space U. A hyperalgebra is said to be smooth if it is smooth as a coalgebra.

Here, we briefly recall the cofree coalgebra $B(U)$ ([7], Chapter 12). Let U be a vector space. The cofree coalgebra $B(U)$ on U is characterized by the following UMP: For any coalgebra C and a linear map $f\colon C^+ \to U$, where C^+ is the kernel of ε, there exists a unique coalgebra map $F\colon C \to B(U)$ such that the diagram

$$\begin{tikzcd} C^+ \arrow[r, hook] \arrow[d, "f"'] & C \arrow[d, "F"] \\ U & B(U) \arrow[l, "\pi_U"] \end{tikzcd}$$

is commutative, where π_U is the canonical projection.

If $\beta\colon U \to V$ is a linear map of vector spaces, we have a uniquely determined coalgebra map $B(\rho)\colon B(U) \to B(V)$ which makes the diagram

$$\begin{tikzcd} B(U) \arrow[r, "B(V)"] \arrow[d, "\pi_U"'] & B(V) \arrow[d, "\pi_V"] \\ U \arrow[r, "\rho"'] & V \end{tikzcd}$$

commutative.

Now let C, D be coalgebras. If $\rho\colon C \to D$ is a coalgebra map, then it is easy to see that the image of $P(C)$ under ρ is contained $P(D)$. We call $\rho_0\colon P(C) \to P(D)$ the restriction of ρ ot $P(C)$. The following lemma is easily shown:

LEMMA 3.1. *Let $\rho\colon C \to D$ be a surjective coalgebra map. If ρ has a coalgebra splitting, then $\rho_0\colon P(C) \to P(D)$ is surjective.*

COROLLARY 3.2. *If G is a relatively smooth subhyperalgebra of a hyperalgebra H, then we have $\rho(P(H)) = P(H//G)$.*

LEMMA 3.3. *Let V, W be vector spaces, V_0, W_0 be subspaces of V, W re-*

spectively and $\rho\colon V \to W$ *a surjective linear map with* $\rho(V_0)=W_0$. *Let* $q\colon W \to W_0$ *be a linear map such that* $q|_{W_0}=\mathrm{id}_{W_0}$. *Then there is a linear map* $p\colon V \to V_0$ *such that* $p|_{V_0}=\mathrm{id}_{V_0}$ *and* $qp=\rho_0 p$, *where* $\rho_0=\rho|_{V_0}$.

The proof is easy and omitted.

PROPOSITION 3.4. *Let*

$$k \longrightarrow A \xrightarrow{j} B \xrightarrow{\rho} C \longrightarrow k$$

be an exact sequence of coalgebras. Let $\mathfrak{a}$, $\mathfrak{b}$ *and* $\mathfrak{c}$ *be the spaces of primitive elements of* A, B *and* C *respectively. Assume that the sequence*

$$0 \longrightarrow \mathfrak{a} \xrightarrow{j_0} \mathfrak{b} \xrightarrow{\rho_0} \mathfrak{c} \longrightarrow 0$$

of vector spaces is exact. Then there are injective coalgebra maps α, β *and* γ *which make the diagram*

$$\begin{tikzcd}
k \arrow[r] \arrow[d, equal] & A \arrow[r, "j"] \arrow[d, "\alpha"] & B \arrow[r, "\rho"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & k \arrow[d, equal] \\
k \arrow[r] & B(\mathfrak{a}) \arrow[r, "B(j_0)"] \arrow[d, "\pi_{\mathfrak{a}}"] & B(\mathfrak{b}) \arrow[r, "B(\rho_0)"] \arrow[d, "\pi_{\mathfrak{b}}"] & B(\mathfrak{c}) \arrow[r] \arrow[d, "\pi_{\mathfrak{c}}"] & k \\
0 \arrow[r] & \mathfrak{a} \arrow[r, "j_0"] & \mathfrak{b} \arrow[r, "\rho_0"] & \mathfrak{c} \arrow[r] & 0
\end{tikzcd}$$

commutative.

PROOF. Let $h\colon C^+ \to \mathfrak{c}$ be any linear map with $h|_{\mathfrak{c}}=\mathrm{id}_{\mathfrak{c}}$. Then the UMP of $B(\mathfrak{c})$ implies that there is a unique coalgebra map $\gamma\colon C \to B(\mathfrak{c})$ which makes the diagram

$$\begin{tikzcd}
C^+ \arrow[r, hook] \arrow[d, "h"'] & C \arrow[d, "\gamma"] \\
\mathfrak{c} & B(\mathfrak{c}) \arrow[l, "\pi_{\mathfrak{c}}"]
\end{tikzcd}$$

commutative. Since h is injective on $\mathfrak{c}$, γ is injective.

By Lemma 3.3 there is a linear map $g\colon B^+ \to \mathfrak{b}$ such that $g|_{\mathfrak{b}}=\mathrm{id}_{\mathfrak{b}}$ and $h\rho|_{B^+}=\rho_0 g$. The UMP of $B(\mathfrak{b})$ guarantees the existence of a coalgebra map $\beta\colon B \to B(\mathfrak{b})$ such that the diagram

$$\begin{tikzcd}
B^+ \arrow[r, hook] \arrow[d, "g"'] & B \arrow[d, "\beta"] \\
\mathfrak{b} & B(\mathfrak{b}) \arrow[l, "\pi_{\mathfrak{b}}"]
\end{tikzcd}$$

is commutative. By the same reason as γ, we see that β is injective. Moreover we have $\gamma\rho = B(\rho_0)\beta$ by the UMP of $B(\mathfrak{c})$. In fact, we have

$$\pi_{\mathfrak{c}}\gamma\rho|_{B^+} = h\rho|_{B^+} = \rho_0 g = \rho_0\pi_{\mathfrak{b}}\beta|_{B^+} = \pi_{\mathfrak{c}}B(\rho_0)\beta|_{B^+}.$$

The existence of α is proved as follows. Since $\rho j = \varepsilon_A$ we have

$$\rho_0 gj(A^+) = h\rho j(A^+) = 0.$$

This implies that $gj(A^+) \subset j_0(\mathfrak{a})$. It follows that there is a linear map $f: A^+ \to \mathfrak{a}$ such that $j_0 f = gj|_{A^+}$. f satisfies an additional condition that $f|_{\mathfrak{a}} = \mathrm{id}_{\mathfrak{a}}$. In fact, let $a \in \mathfrak{a}$. Then we have $j_0 f(a) = gj(a) = gj_0(a) = j_0(a)$ because $j_0(a) \in \mathfrak{b}$. Therefore $f(a) = a$. By the UMP of $B(\mathfrak{a})$ there is a coalgebra map $\alpha: A \to B(\mathfrak{a})$ such that the diagram

$$\begin{array}{ccc} A^+ & \hookrightarrow & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathfrak{a} & \underset{\pi_{\mathfrak{a}}}{\longleftarrow} & B(\mathfrak{a}) \end{array}$$

is commutative. We see as before that α is injective. Finally we must show that $\beta j = B(j_0)\alpha$. This follows from the UMP of $B(\mathfrak{b})$. Indeed we have

$$\pi_{\mathfrak{b}}\beta j|_{A^+} = gj|_{A^+} = j_0 f = j_0\pi_{\mathfrak{a}}\alpha|_{A^+} = \pi_{\mathfrak{b}}B(j_0)\alpha|_{A^+},$$

so that we have $\beta j = B(j_0)\alpha$. This completes the proof.

PROPOSITION 3.5. *Let the diagram*

$$\begin{array}{ccccccccc} k & \longrightarrow & A & \xrightarrow{j} & B & \xrightarrow{\rho} & C & \longrightarrow & k \\ \| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \| \\ k & \longrightarrow & B(\mathfrak{a}) & \xrightarrow[B(j_0)]{} & B(\mathfrak{b}) & \xrightarrow[B(\rho_0)]{} & B(\mathfrak{c}) & \longrightarrow & k \end{array}$$

be as in Proposition 3.4. *If β is an isomorphism, then both α and γ are isomorphisms.*

PROOF. We have only to show the surjectivity of α and γ.

Since both $B(\rho_0)$ and β are surjective, it is clear that γ is surjective.

Before proving the surfectivity of α we note that the sequence $k \to B(\mathfrak{a}) \to B(\mathfrak{b}) \to B(\mathfrak{c}) \to k$ is exact.

Let x be any element in $B(\mathfrak{a})$. Then there is an element b in B such that $\beta(b) = B(j_0)(x)$. Since $B(j_0)(x) \in h\text{-ker}\, B(\rho_0)$, we have $b \in h\text{-ker}\, \rho$, and thus $b = j(a)$ for some a. It is easy to see that $\alpha(a) = x$, which proves the surjectivity of α.

THEOREM 3.6. *If H is a smooth hyperalgebra, then every relatively smooth* *subhyperalgebra of H is smooth.*

PROOF. By Corollary 3.2 the sequence

$$0 \longrightarrow P(G) \xrightarrow{j_0} P(H) \xrightarrow{\rho_0} P(J) \longrightarrow 0$$

of the spaces of primitive elements is exact. Therefore by Proposition 3.4 we obtain a commutative diagram of coalgebras

$$\begin{array}{ccccccccc} k & \longrightarrow & G & \xrightarrow{\quad j \quad} & H & \xrightarrow{\quad \rho \quad} & J & \longrightarrow & k \\ \| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \| \\ k & \longrightarrow & B(P(G)) & \xrightarrow[B(j_0)]{} & B(P(H)) & \xrightarrow[B(\rho_0)]{} & B(P(J)) & \longrightarrow & k. \end{array}$$

Since H is smooth and $\beta(P(H))=P(H)$ we have, by Theorem 12.2.6 in [7], that β is an isomorphism. It follows from Proposition 3.5 that α is an isomorphism, which implies that G is smooth. This proves the theorem.

In the above proof, we notice that the smoothness of G is deduced from only the fact that ρ_0 is surjective. Therefore we may have

COROLLARY 3.7. *Let H be a smooth hyperalgebra, G a subhyperalgebra of H and $\rho: H \to H//G$ the natural map. If $\rho_0: P(H) \to P(H//G)$ is surjective, then G is smooth.*

In comparison with Theorem 3.6, we may ask naturally whether a smooth subhyperalgebra is always relatively smooth. The following theorem answers this question:

THEOREM 3.8. *Every smooth subhyperalgebra of any hyperalgebra is relatively smooth.*

PROOF. Let G be a smooth subhyperalgebra of a hyperalgebra H. We may identify G with $B(\mathfrak{g})$, where $\mathfrak{g}=P(G)$. Let $f: H^+ \to \mathfrak{g}$ be a linear map extending $\pi_{\mathfrak{g}}|_{G^+}: G^+ \to \mathfrak{g}$. Then there is a unique coalgebra map $\eta: H \to B(\mathfrak{g})=G$ such that the diagram

$$\begin{array}{ccc} H^+ & \hookrightarrow & H \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \eta} \\ \mathfrak{g} & \xleftarrow[\pi_{\mathfrak{g}}]{} & G \end{array}$$

is commutative. It is easy to see that η is a retraction of G. Thus G is relatively smooth in H.

Combining Corollary 3.7 and Theorem 3.8 we have

COROLLARY 3.9. *Let H, G, ρ be as in Corollary* 3.7. *If ρ_0: $P(H)\rightarrow P(H//G)$ is surjective, then G is relatively smooth.*

REMARK 3.10. In the above corollary the assumption that H is smooth cannot be dropped. To see this we use an example due to Newman [5]: Let $k=\boldsymbol{Z}/(2)$. Let H be a hyperalgebra over k defined as follows:

$$H = k[X, Y, Z, W]/(X^2, Y^2, Z^2, W^2),$$

where

$$\begin{aligned}
\Delta X &= 1 \otimes X + X \otimes 1 \\
\Delta Y &= 1 \otimes Y + X \otimes X + Y \otimes 1 \\
\Delta Z &= 1 \otimes Z + X \otimes XY + Y \otimes Y + XY \otimes X + Z \otimes 1 \\
\Delta W &= 1 \otimes W + X \otimes X + W \otimes 1.
\end{aligned}$$

It is easy to see that $P(H)$ is the subspace of H spanned by x and $y-w$, where small letters represent the cosets containing the corresponding capitals. Put $G=k[x, w]$ and let ρ: $H\rightarrow H//G$ be the natural map. Then it follows that $H//G=k[\bar{y}, \bar{z}]$, where y and $\bar{z}$ are the images of y and z respectively under ρ. Thus $P(H//G)=k\bar{y}$, so that ρ maps $P(H)$ onto $P(H//G)$. G does not satisfy the coheight condition in H:

$$\text{max-coh}_H\, x = 2 \quad \text{but} \quad \text{max-coh}_G\, x = 1.$$

Therefore, by Proposition 2.6, G is not relatively smooth in H.

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