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Regular points for α-harmonic functions

Dedicated to Professor Makoto Ohtsuka on the occasion of his 60th birthday

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Let D be an open set in the n-dimensional Euclidean space R^n $(n \geqq 2)$ and let $0<\alpha<2$. A boundary point $x_0 \in \partial D$ is said to be regular for D with respect to α-harmonic functions, or simply α-regular for D, if $\beta^{\alpha}_{CD}\varepsilon_{x_0}=\varepsilon_{x_0}$, where ε_{x_0} is the unit point measure at x_0 and β^{α}_{CD} denotes the balayage to the complement CD of D with respect to the α-potentials, i.e., the potentials of the kernel $|x|^{\alpha-n}$ (cf. [1; Chap. V]). Denote by D^{α}_{reg} the set of all α-regular points for D. In the problem section of [2], J. Veselý asks whether there exists a relatively compact open set D such that

$$D^{\alpha}_{reg} \neq D^{\alpha'}_{reg} \quad \text{whenever} \quad \alpha \neq \alpha' \qquad (0 < \alpha, \alpha' < 2). \tag{1}$$

One of the purposes of this note is to answer this question, that is, to construct an open set D with property (1).

Through a communication with J. Veselý, the author learned that M. Kanda of Tsukuba University indicated him another solution to this problem which is more probabilistic.

Now, let us recall Wiener's criterion for α-regularity ([1; Theorem 5.2]):

Wiener's criterion: Let D be an open set and $E=CD$. Let $0<q<1$ and

$$E_k = E \cap \{x \in R^n;\ q^{k+1} \leqq |x-x_0| < q^k\}, \quad k = 1, 2,\dots.$$

Then, $x_0 \in \partial D$ is α-regular $(0<\alpha<2)$ if and only if

$$\textstyle\sum_{k=1}^{\infty} C_\alpha(E_k)q^{k(\alpha-n)} = \infty, \tag{2}$$

where C_α denotes the α-capacity (Riesz capacity of order α; cf. [1; Chap. II]). Now, we extend the definition of α-regularity for $0<\alpha<n$ by the above equality (2). In section 1, we shall construct an open set D for which (1) holds for $0<\alpha$, $\alpha'<n$.

By the definition of the α-capacity, we see easily that if $0<\alpha<\alpha'<n$, then

$$C_\alpha(F) \leqq C_{\alpha'}(F)d(F)^{\alpha'-\alpha}$$

for any bounded Borel set F, where $d(F)$ denotes the diameter of F. Therefore, in veiw of Wiener's criterion, $0<\alpha<\alpha'<n$ implies $D^{\alpha}_{reg} \subset D^{\alpha'}_{reg}$ for any open set D.

Thus, in connection with Veselý's problem, we can ask whether there exists an open set D for which

(3) $$\cup_{\alpha<\beta} D^{\alpha}_{reg} \neq D^{\beta}_{reg}$$

and

(3)′ $$\cap_{\alpha>\beta'} D^{\alpha}_{reg} \neq D^{\beta'}_{reg}$$

hold for all $\beta, \beta' \in (0, n)$. In section 2, we shall construct an open set D for which (3) and (3)′ hold for all $\beta, \beta' \in (0, n-1]$.

§1. For $a \in R^n$ and $r>0$, let $B(a, r)$ denote the closed ball with center at a and radius r.

Let $0<q<1$, $0<\beta<n$ and for each $k=1, 2,\dots$, put

$$a_k = (2^{-1}(q^k + q^{k+1}), 0,\dots, 0) \in R^n$$
$$r_{k,\beta} = 2^{-1}(1-q)q^k k^{1/(\beta-n)}.$$

Consider the sets

$$E^{(\beta)} = \cup_{k=1}^{\infty} B(a_k, r_{k,\beta}) \cup \{0\} \quad \text{and} \quad D^{(\beta)} = CE^{(\beta)}.$$

Then, we have

LEMMA 1. *Let* $0<\alpha<n$. *Then* $0 \in (D^{(\beta)})^{\alpha}_{reg}$ *if and only if* $\alpha \geqq \beta$.

PROOF. First, note that $E^{(\beta)}_k = B(a_k, r_{k,\beta}), k=1, 2,\dots$. Since $C_\alpha(B(a, r)) = A_\alpha r^{n-\alpha}$, $A_\alpha = C_\alpha(B(0, 1))$, we have

$$\sum_{k=1}^{\infty} C_\alpha(E^{(\beta)}_k) q^{k(\alpha-n)} = A_\alpha \sum_{k=1}^{\infty} (r_{k,\beta})^{n-\alpha} q^{k(\alpha-n)}$$
$$= A_\alpha 2^{\alpha-n}(1-q)^{n-\alpha} \sum_{k=1}^{\infty} k^{(n-\alpha)/(\beta-n)}.$$

Hence, by Wiener's criterion, $0 \in (D^{(\beta)})^{\alpha}_{reg}$ if and only if

$$\sum_{k=1}^{\infty} k^{(n-\alpha)/(\beta-n)} = \infty,$$

i.e., if and only if $\alpha \geqq \beta$.

Taking $\tilde{r}_{k,\beta} = 2^{-1}(1-q)q^k[k(\log k)^2]^{1/(\beta-n)}$ in place of $r_{k,\beta}$, we can similarly construct a closed set $\tilde{E}^{(\beta)}$ such that, for $\tilde{D}^{(\beta)} = C\tilde{E}^{(\beta)}$, $0 \in (\tilde{D}^{(\beta)})^{\alpha}_{reg}$ if and only if $\alpha > \beta$.

Now, let $\{\beta_m\}_{m=1}^{\infty}$ be an enumeration of all rational numbers in the open interval $(0, n)$. For each m, let $x_m = (0,\dots, 0, 1/m) \in R^n$ and let E be the closure of

$$\cup_{m=1}^{\infty} \{x_m + x;\ x \in E^{(\beta_m)}\}.$$

Then we have

THEOREM 1. *Let $D=\{x\in R^n;\ |x|<2\}\backslash E$ for the closed set E defined above. Then, for any distinct $\alpha, \alpha' \in (0, n)$,*

$$D^{\alpha}_{reg} \neq D^{\alpha'}_{reg}.$$

PROOF. Let $0<\alpha<\alpha'<n$. Then there is m such that $\alpha<\beta_m<\alpha'$. Set $E_m=\{x_m+x;\ x\in E^{(\beta_m)}\}$. Since $B(x_m, \rho)\cap E=B(x_m, \rho)\cap E_m$ for some $\rho>0$, x_m is γ-regular for D if and only if x_m is γ-regular for CE_m. Thus, by Lemma 1, x_m is α'-regular but not α-regular. Hence $D^{\alpha}_{reg}\neq D^{\alpha'}_{reg}$.

§2. Let $n\geqq 2$ and $0<q<n^{-1/2}$. We use the notation:

$$L_k = \{(t, x_2,\dots, x_n);\ 0 \leqq t \leqq n-1,\ (\textstyle\sum_{i=2}^{n} x_i^2)^{1/2} \leqq f_k(t)\}, \quad k = 2, 3,\dots,$$

where $f_k(t)=2^{-1}(1-q)q^k k^{1/(t+1-n)}$ $(0\leqq t<n-1)$ and $f_k(n-1)=0$,

$$L_0 = \{(t, 0,\dots, 0);\ 0 \leqq t \leqq n-1\},$$

$$E = L_0 \cup \textstyle\bigcup_{k=2}^{\infty} \{(0,\dots, 0, 2^{-1}(q^k+q^{k+1})) + x;\ x\in L_k\},$$

$$Q_k = \{(x_1,\dots, x_n);\ |x_i| \leqq q^k,\ i = 1, 2,\dots, n\}, \quad k = 1, 2,\dots,$$

$$Q_{k,\alpha} = \{(\alpha, 0,\dots, 0) + x;\ x\in Q_k\},$$

$$T(s, r) = \{(x_1,\dots, x_n);\ 0 \leqq x_1 \leqq s,\ (\textstyle\sum_{i=2}^{n} x_i^2)^{1/2} \leqq r\}.$$

$T(s, r)$ is a cylinder and the α-capacity of $T(1, r)$ is estimated as

$$a_\alpha r^{n-1-\alpha} \leqq C_\alpha(T(1, r)) \leqq b_\alpha r^{n-1-\alpha} \qquad \text{if} \quad r \leqq 1,$$

where a_α and b_α are positive constants depending only on α and n (cf. [3; Theorem 5.2]). By the above estimate and the equality $C_\alpha(T(s, r))=s^{n-\alpha}C_\alpha(T(1, r/s))$, we have

$$a_\alpha s r^{n-1-\alpha} \leqq C_\alpha(T(s, r)) \leqq b_\alpha s r^{n-1-\alpha} \qquad \text{if} \quad r \leqq s.$$

To simplify the calculation, we modify Wiener's criterion in the following form: Let $P_\alpha=(\alpha, 0,\dots, 0)$ and D be the complement of E. Then, $P_\alpha\in\partial D$ is γ-irregular if and only if

$$\textstyle\sum_{k=1}^{\infty} C_\gamma(E \cap Q_{k,\alpha})q^{k(\gamma-n)} < \infty.$$

LEMMA 2. *If $0<\alpha<\beta<n-1$, then $P_\beta\notin D^{\alpha}_{reg}$ and $P_\beta\in D^{\beta}_{reg}$, i.e.,*

$$\textstyle\bigcup_{\alpha<\beta} D^{\alpha}_{reg} \neq D^{\beta}_{reg} \qquad \textit{for all} \quad \beta\in(0, n-1).$$

PROOF. First, we show $P_\beta\in D^{\beta}_{reg}$. There is an integer N such that if $k\geqq N$, then $C_\beta(E\cap Q_{k,\beta})\geqq C_\beta(L_k\cap Q_{k,\beta})\geqq C_\beta(T(q^k, f_k(\beta)))$. Hence we have

$$\begin{aligned}&\sum_{k=N}^{\infty} C_\beta(E\cap Q_{k,\beta})q^{k(\beta-n)}\\&\geqq \sum_{k=N}^{\infty} a_\beta q^{k(\beta+1-n)}(2^{-1}(1-q)q^k k^{1/(\beta+1-n)})^{n-1-\beta}\\&= a_\beta(2^{-1}(1-q))^{n-1-\beta}\sum_{k=N}^{\infty} k^{-1}\\&=\infty.\end{aligned}$$

Next we show that $P_\beta \notin D^\alpha_{reg}$, if $\alpha<\beta$. There is an integer N and a constant ε, $0<\varepsilon<1$, such that if $j\geqq k\geqq N$, then $C_\alpha(L_j\cap Q_{k,\beta})\leqq C_\alpha(T(2q^k, f_j(\beta-q^k)))$ and $j^{(n-1-\alpha)/(\beta-q^k+1-n)}\leqq j^{-1-\varepsilon}\leqq k^{-1-\varepsilon}$. Thus, by countable subadditivity of C_α (cf. [1; Chap. II]) and the fact that $C_\alpha(L_0)=0$ for $0<\alpha<n-1$, we have

$$\begin{aligned}&\sum_{k=N}^{\infty} q^{k(\alpha-n)}C_\alpha(E\cap Q_{k,\beta})\\&\leqq \sum_{k=N}^{\infty} q^{k(\alpha-n)}\sum_{j=k}^{\infty} C_\alpha(T(2q^k, f_j(\beta-q^k)))\\&\leqq \sum_{k=N}^{\infty} q^{k(\alpha-n)}\sum_{j=k}^{\infty} b_\alpha 2q^k(2^{-1}(1-q)q^j j^{1/(\beta-q^k+1-n)})^{n-1-\alpha}\\&\leqq c\sum_{K=N}^{\infty} k^{-1-\varepsilon}q^{k(\alpha+1-n)}\sum_{j=k}^{\infty} q^{j(n-1-\alpha)}\\&\leqq c'\sum_{k=N}^{\infty} k^{-1-\varepsilon}<\infty,\end{aligned}$$

where c, c' are constants depending only on q, n and α.

Now we consider the following function $\tilde{f}_k$ instead of f_k:

$$\begin{cases}\tilde{f}_k(t)=2^{-1}(1-q)q^k(k(\log k)^2)^{1/(t+1-n)} & (0\leqq t<n-1),\\ \tilde{f}_k(n-1)=0,\end{cases}$$

and construct $\tilde{L}_k$ and $\tilde{E}$ by this $\tilde{f}_k$ as before. Then, for $\tilde{D}=C\tilde{E}$, we have

LEMMA 3. *If* $0<\beta<\alpha<n-1$, *then* $P_\beta\notin\tilde{D}^\beta_{reg}$ *and* $P_\beta\in\tilde{D}^\alpha_{reg}$, *i.e.*,

$$\bigcap_{\alpha>\beta}\tilde{D}^\alpha_{reg}\neq\tilde{D}^\beta_{reg}\qquad \text{for all}\quad \beta\in(0,n-1).$$

PROOF. Let N be a sufficiently large integer. Then as in the proof of Lemma 2, we have

$$\begin{aligned}&\sum_{k=N}^{\infty} C_\beta(\tilde{E}\cap Q_{k,\beta})q^{k(\beta-n)}\\&\leqq \sum_{k=N}^{\infty} q^{k(\beta-n)}\sum_{j=k}^{\infty} C_\beta(T(2q^k, \tilde{f}_j(\beta-q^k)))\\&\leqq c\sum_{k=N}^{\infty} q^{k(\beta+1-n)}(k(\log k)^2)^{(n-1-\beta)/(\beta-q^k+1-n)}\sum_{j=k}^{\infty} q^{j(n-1-\beta)}\\&\leqq c'\sum_{k=N}^{\infty}(k(\log k)^2)^{-1/(1+dq^k)}<\infty,\end{aligned}$$

where c, c' and d are positive constants depending only on q, n and β. Thus, $P_\beta\notin\tilde{D}^\beta_{reg}$.

If $\beta<\alpha$, then

$$\begin{aligned}\sum_{k=N}^{\infty} C_\alpha(E \cap Q_{k,\beta}) q^{k(\alpha-n)} &\geqq \sum_{k=N}^{\infty} C_\alpha(T(q^k, f_k(\beta))) q^{k(\alpha-n)} \\ &\geqq \sum_{k=N}^{\infty} c(k(\log k)^2)^{(n-1-\alpha)/(\beta+1-n)} \\ &= \infty,\end{aligned}$$

which shows $P_\beta \in \tilde{D}^\alpha_{reg}$.

Now we have

THEOREM 2. *There exists an open set D with the property*

$$\bigcup_{\alpha<\beta} D^\alpha_{reg} \neq D^\beta_{reg} \quad \text{and} \quad \bigcap_{\alpha>\beta'} D^\alpha_{reg} \neq D^{\beta'}_{reg}$$

for all β, $\beta' \in (0, n-1]$.

PROOF. Using the sets E and $\tilde{E}$ constructed above and the sets $E^{(n-1)}$ and $\tilde{E}^{(n-1)}$ given in the previous section, we can easily construct a required open set D.

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