Hiroshima Math. J.
14 (1984), 407–413

Locally finite simple Lie algebras

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(Received January 14, 1984)

In the study of infinite-dimensional Lie algebras, the notions of ascendant subalgebras and serial subalgebras are fundamental. The notions generalizing these ones, weakly ascendant subalgebras and weakly serial subalgebras, were introduced and investigated in [7] and [2]. On the other hand, taking account of a result of Levič [5], a recent result of Stewart [6, Theorem 8] is expressed as follows: A locally finite Lie algebra over a field of characteristic 0 has no non-trivial ascendant subalgebras if and only if it has no non-trivial serial subalgebras.

In connection with these, we shall mainly study locally finite simple Lie algebras over a field $\mathfrak{k}$ of arbitrary characteristic. Actually there exist locally finite simple infinite-dimensional Lie algebras (Example 3).

In Section 2, we shall show that for a locally finite Lie algebra L over $\mathfrak{k}$, if H wser L then $H/\mathrm{Core}_L(H)$ is locally nilpotent (Theorem 5). We shall use this to give a simple proof and a refinement of Stewart's result stated above (Theorem 7).

In Section 3, we shall show that for a locally finite non-abelian simple Lie algebra L over $\mathfrak{k}$, if H wser L and $H \neq L$ then any finite-dimensional subalgebra of H belongs to $\mathfrak{e}^*(L)$, and

$$\begin{aligned} \cup \{H \mid H \text{ wser } L,\ H \neq L\} &= \cup \{H \mid H \text{ wasc } L,\ H \neq L\} \\ &= \cup \{H \mid H \leq^{\omega} L,\ H \neq L\} = \cup \{H \mid H \leq L,\ H \in \mathfrak{e}^*(L)\} = \mathfrak{e}(L) \end{aligned}$$

(Theorem 10). As a consequence of this we shall show that a locally finite non-abelian Lie algebra L over $\mathfrak{k}$ has no non-trivial weakly ascendant subalgebras if and only if L has no non-trivial weakly serial subalgebras, if and only if L is simple with $\mathfrak{e}^*(L) = \{0\}$, and if and only if L is simple with $\mathfrak{e}(L) = 0$ (Theorem 11).

1.

Throughout this paper, $\mathfrak{k}$ is a field of arbitrary characteristic unless otherwise specified, and L is a not necessarily finite-dimensional Lie algebra over $\mathfrak{k}$. When H is a subalgebra (resp. an ideal) of L, we denote $H \leq L$ (resp. $H \lhd L$).

Let $H \leq L$. For an ordinal ρ, H is a ρ-step weakly ascendant subalgebra (resp. a ρ-step ascendant subalgebra) of L, denoted by $H \leq^{\rho} L$ (resp. $H \lhd^{\rho} L$), if there exists an ascending chain $\{H_\sigma | \sigma \leq \rho\}$ of subspaces (resp. subalgebras) of L such that

(1) $H_0 = H$ and $H_\rho = L$,

(2) $[H_{\sigma+1}, H] \subseteq H_\sigma$ (resp. $H_\sigma \lhd H_{\sigma+1}$) for any ordianl $\sigma < \rho$,

(3) $H_\lambda = \cup_{\sigma<\lambda} H_\sigma$ for any limit ordinal $\lambda \leq \rho$.

H is a weakly ascendant subalgebra (resp. an ascendant subalgebra) of L, denoted by H wasc L. (resp. H asc L), if $H \leq^\rho L$ (resp. $H \lhd^\rho L$) for some ordinal ρ. When ρ is finite, H is a weak subideal (resp. subideal) of L and denoted by H wsi L (resp. H si L).

For a totally ordered set Σ, H is a weakly serial subalgebra (resp. a serial subalgebra) of type Σ of L, denoted by H wser L (resp. H ser L), if there exists a collection $\{\Lambda_\sigma, V_\sigma | \sigma \in \Sigma\}$ of subspaces (resp. subalgebras) of L such that

(1) $H \subseteq \Lambda_\sigma$ and $H \subseteq V_\sigma$ for all $\sigma \in \Sigma$,

(2) $\Lambda_\tau \subseteq V_\sigma \subseteq \Lambda_\sigma$ if $\tau < \sigma$,

(3) $L \backslash H = \cup_{\sigma \in \Sigma} (\Lambda_\sigma \backslash V_\sigma)$,

(4) $[\Lambda_\sigma, H] \subseteq V_\sigma$ (resp. $V_\sigma \lhd \Lambda_\sigma$) for all $\sigma \in \Sigma$.

Then any weakly ascendant (resp. ascendant) subalgebra of L is weakly serial (resp. serial).

Let Δ be any one of the relations wasc, $\leq^\omega$, asc, wser and ser. Then we call a Lie algebra L Δ-simple if $H \Delta L$ implies $H=0$ or L. By Example 1 we have the following diagram of implications:

$$\begin{array}{ccc} \text{simple} = \text{asc-simple} & \longleftarrow & \text{ser-simple} \\ \uparrow\!\!\not\downarrow & & \uparrow\!\!\not\downarrow \\ \text{wasc-simple} & \longleftarrow & \text{wser-simple}. \end{array}$$

Then there exists a Lie algebra satisfying wser-simplicity (Example 2).

An element x of L is called a left Engel element of L if for any $y \in L$ there exists an integer $n = n(x, y) > 0$ such that $[y, {}_n x] = 0$, and the set of left Engel elements of L is denoted by $\mathfrak{e}(L)$. Similarly a subset S of L is called a left Engel subset of L if for any $y \in L$ there exists an integer $n = n(S, y) > 0$ such that $[y, {}_n S] = 0$, and the collection of left Engel subsets of L is denoted by $\mathfrak{e}^*(L)$ ([7]).

The Hirsch-Plotkin radical $\rho(L)$ of L is the largest locally nilpotent ideal of L.

A class of Lie algebras is a collection of Lie algebras over $\mathfrak{k}$ together with their isomorphic copies and the 0-dimensional Lie algebra. We denote by $\mathfrak{F}$, $\mathfrak{F}_1$, $\mathfrak{N}$, $\mathfrak{E}$, L$\mathfrak{F}$ and L$\mathfrak{N}$ the classes of finite-dimensional, 0 or 1-dimensional, nilpotent, Engel, locally finite and locally nilpotent Lie algebras respectively.

Lemma 1 ([7, Lemma 1]). *Let $H \leq L$. Then $H \leq^\omega L$ if and only if for any $x \in L$ there exists an integer $n = n(x) > 0$ such that $[x, {}_n H] \subseteq H$.*

As an immediate consequence of Lemma 1 we have

LEMMA 2. (1) *For any* $x \in L$, $x \in \mathfrak{e}(L)$ *if and only if* $\langle x \rangle \leq^{\omega} L$.
(2) *For any* $H \leq L$, *if* $H \in \mathfrak{e}^*(L)$ *then* $H \leq^{\omega} L$.

LEMMA 3 ([4, Lemma 2.1]). *Let* H wasc L. *Then for a finite subset* X *of* L *and finite subsets* $Y_1, Y_2, \ldots$ *of* H, *there exists an integer* $n = n(X, Y_1, Y_2, \ldots) > 0$ *such that* $[X, Y_1, \ldots, Y_n] \subseteq H$.

LEMMA 4 ([1, Proposition 13.2.4] and [2, Corollary 2.4]). *Let* $L \in \mathrm{L}\mathfrak{F}$ *and let* $H \leq L$. *Then* H wser L (*resp.* H ser L) *if and only if* $H \cap F$ wsi F (*resp.* $H \cap F$ si F) *for any finite-dimensional subalgebra* F *of* L.

2.

In this section we shall give a simple proof and a supplement for Stewart's theorem stated in the introduction.

Our key lemma is a special case of the following theorem. For $H \leq L$, we denote by $\mathrm{Core}_L(H)$ the largest ideal of L contained in H as usual.

THEOREM 5. *Let* L *be a locally finite Lie algebra over a field* $\mathfrak{k}$. *If* H wser L, *then* $H/\mathrm{Core}_L(H)$ *is locally nilpotent.*

PROOF. We may assume $\mathrm{Core}_L(H) = 0$ without loss of generality. Let $\mathscr{S}$ be the collection of finite-dimensional subalgebras of H and put

$$M = \cup \{F^{\omega} \mid F \in \mathscr{S}\},$$

where $F^{\omega} = \bigcap_{m=1}^{\infty} F^m$. Then M is a subspace of H. We assert that $M \lhd L$. In fact, let $x \in L$ and $F \in \mathscr{S}$. Then $F(x) = \langle x, F \rangle \in \mathfrak{F}$. By Lemma 4 $H \cap F(x)$ wsi $F(x)$. Put $F_1 = H \cap F(x)$. Then $F \leq F_1 \in \mathscr{S}$ and $[x, {}_nF] \subseteq F_1$ for some n. It follows that

$$\begin{aligned} [x, F^{\omega}] &\subseteq [x, \textstyle\bigcap_{m=1}^{\infty} F^{n+m-1}] \subseteq \bigcap_{m=1}^{\infty} [x, {}_nF, {}_{m-1}F] \\ &\subseteq \textstyle\bigcap_{m=1}^{\infty} [F_1, {}_{m-1}F] \subseteq \bigcap_{m=1}^{\infty} F_1^m = F_1^{\omega} \subseteq M. \end{aligned}$$

Hence $M \lhd L$, as asserted. By our assumption we have $M = 0$ and therefore any $F \in \mathscr{S}$ is nilpotent. Thus $H \in \mathrm{L}\mathfrak{N}$.

Furthermore we need the "only if" part of the following

LEMMA 6. *Let* L *be a locally finite Lie algebra over a field of characteristic* 0. *Then* H ser L *and* $H \in \mathrm{L}\mathfrak{N}$, *if and only if* $H \leq \rho(L)$.

PROOF. If $H \leq \rho(L)$, then we use Lemma 4 to see that H ser $\rho(L)$ and therefore H ser L. The converse follows from [1, Theorem 13.3.7].

THEOREM 7. *For a locally finite non-abelian Lie algebra* L *over a field of characteristic* 0, *the following statements are equivalent*;

(1) *L is simple* (*=asc-simple*).

(2) *L is ser-simple.*

(3) *$\rho(L)=0$ and* $\mathrm{Core}_L(H)=0$ *for any serial proper subalgebra H of L.*

PROOF. (1)⇒(3). Let L be simple. If $\rho(L)=L$, then any minimal ideal of L is central by [1, Lemma 7.1.6] and therefore L is abelian. Hence $\rho(L)=0$. Clearly $\mathrm{Core}_L(H)=0$ for any serial proper subalgebra H of L.

(3)⇒(2). Assume (3) and let H ser L, $H \neq L$. Then by Theorem 5 $H \in \mathrm{L}\mathfrak{N}$. It follows from Lemma 6 that $H=0$. Hence L is ser-simple.

(2)⇒(1) is obvious.

3.

In this section we shall investigate weakly ascendant and weakly serial subalgebras of locally finite Lie algebras.

We begin with the following

PROPOSITION 8. *Let L be a Lie algebra over a field $\mathfrak{k}$.*

(1) *Let H* wasc *L. If $H \in \mathrm{L}\mathfrak{N}$ then any finite-dimensional subalgebra of H belongs to $\mathfrak{e}^*(L)$ and if $H \in \mathfrak{E}$ then any element of H belongs to $\mathfrak{e}(L)$.*

(2) *Let $\mathfrak{X}_i$ $(i=1, 2)$ be any classes of Lie algebras such that $\mathfrak{F}_1 \leq \mathfrak{X}_i \leq \mathfrak{E}$. Then the following subsets of L coincide each other*:

a) $\cup \{H | H \text{ wasc } L, H \in \mathfrak{X}_1\}$,

b) $\cup \{H | H \leq^{\omega} L, H \in \mathfrak{X}_2\}$,

c) $\cup \{H | H \leq L, H \in \mathfrak{e}^*(L)\}$,

d) $\mathfrak{e}(L)$.

PROOF. (1) In case $H \in \mathrm{L}\mathfrak{N}$, let F be any finite-dimensional subalgebra of H and let $x \in L$. Then by Lemma 3 there exists an integer $n=n(x, F) \geq 0$ such that $[x, {}_nF] \subseteq H$. Since $H \in \mathrm{L}\mathfrak{N}$, $\langle F, [x, {}_nF]\rangle \in \mathfrak{N}$. It follows that $[x, {}_{n+m}F]=0$ for some m. Hence $F \in \mathfrak{e}^*(L)$.

In case $H \in \mathfrak{E}$, let $y \in H$ and let $x \in L$. Then by Lemma 3 there exists an integer $n=n(x, y) \geq 0$ such that $[x, {}_ny] \in H$. It follows that $[x, {}_{n+m}y]=0$ for some m. Hence $y \in \mathfrak{e}(L)$.

(2) Denote by $A_{\mathfrak{X}_1}$, $B_{\mathfrak{X}_2}$ and C the sets in a), b) and c) respectively. Then by Lemma 2 we have

$$\begin{array}{ccccccc} A_{\mathfrak{E}} & \supseteq & A_{\mathfrak{X}_1} & \supseteq & A_{\mathfrak{F}_1} & & \\ \cup\!| & & & & \cup\!| & & \\ B_{\mathfrak{E}} & \supseteq & B_{\mathfrak{X}_2} & \supseteq & B_{\mathfrak{F}_1} & \supseteq & C = \mathfrak{e}(L). \end{array}$$

But by (1) we have $A_{\mathfrak{E}} \subseteq \mathfrak{e}(L)$. Hence we have the statement.

PROPOSITION 9. *Let L be a locally finite Lie algebra over a field $\mathfrak{k}$.*

(1) *Let H wser L. If $H \in \mathrm{L}\mathfrak{N}$ then any finite-dimensional subalgebra of H belongs to $\mathfrak{e}^*(L)$ and if $H \in \mathfrak{E}$ then any element of H belongs to $\mathfrak{e}(L)$.*

(2) *Let $\mathfrak{X}_i$ $(1 \leq i \leq 3)$ be any classes of Lie algebras such that $\mathfrak{F}_1 \leq \mathfrak{X}_i \leq \mathfrak{E}$. Then the following subsets of L coincide each other:*

a) $\bigcup \{H | H \text{ wser } L,\ H \in \mathfrak{X}_1\}$,

b) $\bigcup \{H | H \text{ wasc } L,\ H \in \mathfrak{X}_2\}$,

c) $\bigcup \{H | H \leq^{\omega} L,\ H \in \mathfrak{X}_3\}$,

d) $\bigcup \{H | H \leq L,\ H \in \mathfrak{e}^*(L)\}$,

e) $\mathfrak{e}(L)$.

PROOF. (1) In case $H \in \mathrm{L}\mathfrak{N}$, let F be any finite-dimensional subalgebra of H and let $x \in L$. Then $F_1 = \langle F, x \rangle \in \mathfrak{F}$. By Lemma 4, $H \cap F_1$ wsi F_1. Hence $[x, {}_nF] \subseteq H \cap F_1$ for some n. Put $G = \langle F, [x, {}_nF] \rangle \leq H$. Then $G \in \mathfrak{N}$ and therefore $[x, {}_{n+m}F] = 0$ for some m. Thus $F \in \mathfrak{e}^*(L)$. In case $H \in \mathfrak{E}$, $H \in \mathrm{L}\mathfrak{F} \cap \mathfrak{E} \leq \mathrm{L}\mathfrak{N}$ and therefore it follows that any element of H belongs to $\mathfrak{e}(L)$.

(2) Denoting by $A_{\mathfrak{X}_1}$ the set in a), we have $A_{\mathfrak{E}} \supseteq A_{\mathfrak{F}_1} \supseteq \mathfrak{e}(L)$ by Lemma 2 and $A_{\mathfrak{E}} \subseteq \mathfrak{e}(L)$ by (1). Hence $A_{\mathfrak{X}_1} = \mathfrak{e}(L)$. Owing to Proposition 8, we have the statement.

Now we can show a structure theorem of locally finite simple Lie algebras.

THEOREM 10. *Let L be a locally finite non-abelian simple Lie algebra over a field $\mathfrak{k}$.*

(1) *If H wser L and $H \neq L$, then any finite-dimensional subalgebra of H belongs to $\mathfrak{e}^*(L)$ and any element of H belongs to $\mathfrak{e}(L)$.*

(2) *Let $\mathfrak{X}_i$ $(1 \leq i \leq 3)$ be any classes of Lie algebras such that $\mathfrak{F}_1 \leq \mathfrak{X}_i \leq \mathfrak{E}$. Then the following subsets of L coincide each other:*

a) $\bigcup \{H | H \text{ wser } L,\ H \neq L\}$,

b) $\bigcup \{H | H \text{ wasc } L,\ H \neq L\}$,

c) $\bigcup \{H | H \leq^{\omega} L,\ H \neq L\}$,

d) $\bigcup \{H | H \text{ wser } L,\ H \in \mathfrak{X}_1\}$,

e) $\bigcup \{H | H \text{ wasc } L,\ H \in \mathfrak{X}_2\}$,

f) $\bigcup \{H | H \leq^{\omega} L,\ H \in \mathfrak{X}_3\}$,

g) $\bigcup \{H | H \leq L,\ H \in \mathfrak{e}^*(L)\}$,

h) $\mathfrak{e}(L)$.

PROOF. If H wser L and $H \neq L$, then $H \in \mathrm{L}\mathfrak{N} \leq \mathfrak{E}$ by Theorem 5. Hence, denoting by A, B, C and $D_{\mathfrak{X}_1}$ the sets in a), b), c) and d) respectively, we have $D_{\mathfrak{E}} \supseteq A \supseteq B \supseteq C \supseteq \mathfrak{e}(L)$. Thus the statement follows from Proposition 9.

As immediate consequences of Theorem 10 we have

THEOREM 11. *Let L be a locally finite non-abelian Lie algebra L over a* *field* $\mathfrak{k}$. *Then the following statements are equivalent*:

(1) *L is wser-simple.*

(2) *L is wasc-simple.*

(3) *L is $\leq^{\omega}$-simple.*

(4) *L is simple and $\mathfrak{e}^*(L)=\{0\}$.*

(5) *L is simple and $\mathfrak{e}(L)=0$.*

THEOREM 12. *For a locally finite Lie algebra L over a field $\mathfrak{k}$, the following statements are equivalent*:

(1) *L is wser-simple.*

(2) *L is wasc-simple.*

(3) *L is $\leq^{\omega}$-simple.*

4.

EXAMPLE 1. Let L be the Lie algebra over a field of characteristic $\neq 2$ with basis $\{x, y, z\}$ such that $[x, z]=2x$, $[y, z]=-2y$, $[x, y]=z$. Then L is asc-simple and ser-simple. Since $\langle x \rangle$ wsi L, L is neither wasc-simple nor wser-simple.

EXAMPLE 2. Let L be the Lie algebra over a formal real field with basis $\{x, y, z\}$ such that $[x, y]=z$, $[y, z]=x$, $[z, x]=y$. Then L has no non-trivial weak subideals ([3, Example 4.3]). Hence L is wasc-simple and wser-simple.

EXAMPLE 3. For any integer $n \geq 2$, a matrix in $\mathfrak{sl}(n, \mathfrak{k})$ may be regarded as a matrix in $\mathfrak{sl}(n+1, \mathfrak{k})$ with the $n+1$ th row and column consisting of 0. Thus $\mathfrak{sl}(n, \mathfrak{k}) \subseteq \mathfrak{sl}(n+1, \mathfrak{k})$ $(n \geq 2)$. Then $L=\bigcup_{n=2}^{\infty} \mathfrak{sl}(n, \mathfrak{k})$ is a locally finite simple Lie algebra over $\mathfrak{k}$. In fact, let $H \lhd L$ and $H \neq 0$. Then there is an integer $m>0$ such that $H \cap \mathfrak{sl}(n, \mathfrak{k}) \neq 0$ for any $n \geq m$. Since $\mathfrak{sl}(n, \mathfrak{k})$ is simple unless the characteristic of $\mathfrak{k}$ divides n, $H \cap \mathfrak{sl}(n, k)=\mathfrak{sl}(n, \mathfrak{k})$ and $H \supseteq \mathfrak{sl}(n, \mathfrak{k})$ for such an n. Hence $H=L$ and L is simple. Furthermore, since $\langle e_{12} \rangle \leq^2 L$, L is neither wasc-simple nor wser-simple.

References

[1] R. K. Amayo and I. Stewart: Infinite-dimensional Lie Algebras, Noordhoff, Leyden, 1974.

[2] M. Honda: Weakly serial subalgebras of Lie algebras, Hiroshima Math. J. **12** (1982), 183–201.

[3] M. Honda: Joins of weak subideals of Lie algebras, Hiroshima Math. J. **12** (1982), 657–673.

[4] M. Honda: Joins of weakly ascendant subalgebras of Lie algebras, Hiroshima Math J. **14** (1984), 333–358.

[5] E. M. Levič: On simple and strictly simple rings, Latvijas PSR Zinātņu Akad. Vēstis Fiz. Tehn. Zinātņu Sēr. **6** (1965), 53–58.
[6] I. Stewart: Subideals and serial subalgebras of Lie algebras, Hiroshima Math. J. **11** (1981), 493–498.
[7] S. Tôgô: Weakly ascendant subalgebras of Lie algebras, Hiroshima Math. J. **10** (1980), 175–184.

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