Hiroshima Math. J.
14 (1984), 265–281

Asymptotic behavior of solutions to certain nonlinear parabolic evolution equations

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(Received May 17, 1983)

1. Introduction

In this paper we are concerned with nonlinear evolution equations of the form

$$\text{(E)} \qquad (d/dt)u(t) \in -\partial\phi^t(u(t)), \qquad t > 0,$$

where ϕ^t, $0 \leqq t \leqq \infty$, are proper, *l.s.c.* (lower semi-continuous), convex functionals on a Hilbert space H, $\partial\phi^t$ is the subdifferential of ϕ^t for $t \geqq 0$ and u stands for an H-valued unknown function on $[0, \infty)$. By a (*strong*) *solution of* (E) we mean a function u in $W^{1,1}_{loc}([0, \infty); H)$ such that

$$\phi^{(\cdot)}(u(\cdot)) \in L^1_{loc}(0, \infty)$$

and

$$-(d/dt)u(t) \in \partial\phi^t(u(y)) \qquad \text{for a.e.} \quad t \geqq 0.$$

So far the existence and uniqueness of solutions to (E) have been discussed under various smoothness assumptions on the mapping $t \to \phi^t(\cdot)$. For instance, see [7] and [11]. Our objective here is to discuss the asymptotic behavior of solutions to (E). In fact we shall show the strong limit

$$\text{(1.1)} \qquad \mathrm{s} - \lim_{t\to\infty} u(t) \qquad \text{in} \quad H$$

exists for each solution u of (E) under suitable assumptions and investigate geometric properties of the limits. The assumptions which will be imposed on the family $\{\phi^t\}$ are stated as follows:

(i) $F(\phi^t)(=\{x \in H;\ \phi^t(x) = \min \phi^t\})$ is non-empty and independent of $t \geqq 0$.

(ii) For each $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such that *dist* $(x, F) \leqq \varepsilon$ whenever $\phi^t(x) \leqq \delta(\varepsilon)$ for $t \geqq 0$, where $F \equiv F(\phi^0)$; or else

(ii)$'$ there exist a number $c \geqq 1$ and a proper, *l.s.c.*, even, convex functional ψ on H such that $\psi(x) \leqq \phi^t(x) \leqq c\psi(x)$ for $x \in H$ and $t \geqq 0$.

Suppose for the moment that a family $\{\Psi^t\} = \{\Psi^t;\ 0 \leqq t \leqq \infty\}$ of proper *l.s.c.* convex functionfils on H is given, and that the associated evolution equation

$$(d/dt)v(t) \in -\partial\Psi^t(v(t)), \quad t>0, \tag{1.2}$$

has at least one solution on $[0, \infty)$. Taking any but a fixed solution ω of (1.2) we may define a new family $\{\phi^t\}$ of proper *l.s.c.* conves functionals on H by

$$\phi^t(x) = \Psi^t(x+\omega(t)) + (\omega'(t), x) - \Psi^t(\omega(t)) \quad \text{for} \quad x \in H \quad \text{and} \quad t \geqq 0, \tag{1.3}$$

where $\omega'(t) = (d/dt)\omega(t)$ and $(\cdot, \cdot)$ stands for the inner product of H. Then $\partial\phi^t(x) = \partial\Psi^t(x+\omega(t)) + \omega'(t)$ for $t \geqq 0$ and it is seen that v is a solution of (1.2) if and only if $u = v - \omega$ is a solution of (E). Therefore, if the family $\{\phi^t\}$ defined by (1.3) satisfies our conditions (i), (ii) (or (ii)') then the strong limit (1.1) of $u = v - \omega$ exists, and it will be concluded that v is asymptotically equal to $\omega +$ *const.* This situation is particularly interesting in the case where ω is a periodic solution of (1.2), and the present work was motivated by this observation.

Our results are applicable to the investigation of the asymptotic stability of solutions of nonlinear parabolic partial differential equations, and some of the applications will be discussed in the forthcoming paper [10].

2. Main results

Let H be a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. For a point x in H and a set C in H we denote by *dist* (x, C) the distance between x and C.

Let $\{\phi^t\} = \{\varphi^t; 0 \leq t < \infty\}$ be a family of proper *l.s.c.* convex functions φ^t on H. We denote by $\partial\varphi^t$ the subdifferential of φ^t and refer to the book of Brézis [3] for basic properties of proper *l.s.c.* convex functionals and their subdifferentials. Also, by $F(\varphi^t)$ is mean the set of all points at which φ^t attains the minimum, i.e.

$$F(\varphi^t) = \{x \in H;\ \varphi^t(x) = \min \varphi^t\}.$$

We now consider the evolution equation

$$u'(t) \in -\partial\varphi^t(u(t)), \qquad t > 0, \tag{E}$$

where $u'(t) = (d/dt)u(t)$ and the definition of solution of (E) was given in Section 1.

The main results of this paper are then stated as follows:

THEOREM 2.1. *Suppose the following conditions hold*:

(A1) $F(\varphi^t) \equiv F \neq \emptyset$ (*i.e.*, $F(\varphi^t)$ *is a non-empty set* F *independent of* t) *and* $\min \varphi^t = 0$ *for any* $t \geq 0$.

(A2) *There exists a non-negative measurable function* $a(\cdot)$ *defined everywhere on* $[0, \infty)$ *such that*

(2.1) $$\int_0^\infty a(t)dt = \infty$$

and such that for each $\varepsilon>0$ *there is a number* $\delta=\delta(\varepsilon)>0$ *satisfying*

(2.2) $$\mathit{dist}\,(x, F) \leq \varepsilon \quad \mathit{for} \quad x \in \cup_{t\geq 0}\{x \in D(\varphi^t);\ \varphi^t(x) \leq \delta a(t)\}.$$

Then, for each solution u of (E), we have

$$s - \lim_{t\to\infty} u(t) \in F,$$

that is, u converges strongly in H to some point of F as $t\to\infty$.

It is well-known (cf. [4]) that any solution of (E) converges strongly in H as $t\to\infty$ if φ^t is independent of t in the sense that $\varphi^t\equiv\varphi$ for $t\geqq 0$ and if the set $\{x\in H;\ \varphi(x)\leq\lambda\}$ is compact in H for some $\lambda>\min\varphi$. The above theorem is a generalization of this useful result.

REMARK 2.1. Condition (A2) follows from the following simpler condition:

(A2)′ There are a non-negative measurable function $a(\cdot)$ everywhere defined on $[0, \infty)$ and a continuous function f on $[0, \infty)$ such that

$$\int_0^\infty a(t)dt = \infty$$

$$f(0) = 0, \quad f(r) > 0 \qquad \text{for} \quad r > 0,$$

and

$$a(t)f(\mathit{dist}\,(x, F)) \leq \varphi^t(x) \qquad \text{for} \quad x \in D(\varphi^t) \quad \text{and} \quad t \geq 0.$$

THEOREM 2.2. *Suppose the following conditions hold:*

(B1) $D(\varphi^s)\supset D(\varphi^t)$ *for* $s, t\geqq 0$ *with* $s\leqq t$, *and there exists a proper l.s.c. convex functional* ψ *on H such that*

$$F(\varphi^t) = F(\psi)\,(\equiv F) \neq \emptyset \quad \mathit{and} \quad \min\varphi^t = \min\psi = 0 \quad \mathit{for} \quad t \geqq 0.$$

(B2) *There exist a number* $c\geq 1$ *and a positive and measurable function* $b(\cdot)$ *defined everywhere on* $[0, \infty)$ *such that*

$$\int_0^\infty b(t)dt = \infty,$$

(2.3) $$-c^{-1}D(\varphi^t) \subset D(\varphi^t) \qquad \mathit{for} \quad t \geq 0$$

and

(2.4) $$b(t)\psi(x) \leq \varphi^t(x) \leq b(t)\psi(-cx) \qquad \mathit{for} \quad t \geq 0 \quad \mathit{and} \quad x \in D(\varphi^t).$$

Then, for each solution u of (E), we have

$$s - \lim_{t\to\infty} u(t) \in F.$$

REMARK 2.2. If in Theorem 2.2, ψ is even (i.e. $\psi(x)=\psi(-x)$ for $x \in H$) and satisfies

$$b(t)\psi(x) \le \varphi^t(x) \le cb(t)\psi(x) \quad \text{for} \quad t \geqq 0 \quad \text{and} \quad x \in D(\varphi^t), \tag{2.5}$$

then (2.4) holds. In fact, since $\min \psi = \psi(0) = 0$, it follows that

$$c\psi(x) = c\psi(c^{-1}cx + c^{-1}(c-1)0) \le \psi(cx) = \psi(-cx),$$

so that (2.4) follows from (2.5).

In [4] it was proved that if φ is a proer, *l.s.c.*, even, convex functional on H, then every solution of $u'(t) \in -\partial\varphi(u(t))$, $t>0$, converges strongly in H to some point of $F(\varphi)$ as $t\to\infty$. This result is a special case of the following corollary to Therem 2.2.

COROLLARY to THEOREM 2.2 (cf. [6], [9]). *Let φ be a proper l.s.c. convex functional on H such that there are a number $c \ge 1$ and a proper l.s.c. even convex functional ψ on H satisfying*

$$F(\varphi) = F(\psi) \equiv F, \quad \min \varphi = \min \psi = 0, \quad -c^{-1}D(\varphi) \subset D(\varphi)$$

and

$$\psi(x) \le \varphi(x) \le c\psi(x) \qquad \textit{for} \quad x \in D(\psi).$$

Then, for each solution u of $u'(t) \in -\partial\varphi(u(t))$, $t>0$, we have

$$s - \lim_{t\to\infty} u(t) \in F.$$

REMARK 2.3. We note that a solution u of $u'(t) \in -\partial\varphi(u(t))$, $t>0$, does not necessarily converges in the strong topology of H as $t\to\infty$ under the following assumption which is more general than that imposed in the above corollary: There are proper *l.s.c.* convex even functionals ψ_1, ψ_2 on H such that

$$F(\varphi) = F(\psi_1) = F(\psi_2)$$

and

$$\psi_1(x) \le \varphi(x) \le \psi_2(x) \qquad \text{for any} \quad x \in H.$$

In fact we shall give a counterexample in Section 5. Thus the relation $\psi_2(\cdot) = c\psi_1(\cdot)$ with $c \geqq 1$ is essential for solutions of (E) to converge strongly in H as $t\to\infty$, and this fact is also seen from the proofs of Theorem 2.2 and its corollary.

3. Proof of Theorem 2.1

We first prepare two lemmas which will be used both in the proof of Theorem 2.1 and in that of Theorem 2.2.

LEMMA 3.1. Assume $\min \varphi^t = 0$ for each $t \geq 0$ and $\bigcap_{t\geq 0} F(\varphi^t) \neq \emptyset$. Let u be a solution of (E). Then $\|u(t)-y\|$ is decreasing in t and converges as $t \to \infty$ for each $y \in \bigcap_{t\geq 0} F(\varphi^t)$, and $\varphi^{(\cdot)}(u(\cdot)) \in L^1(0, \infty)$.

PROOF. Let $y \in \bigcap_{t\geq 0} F(\varphi^t)$. Then by the definition of subdifferential

$$(u'(s),\, u(s)-y) \leq \varphi^s(y) - \varphi^s(u(s)) = -\varphi^s(u(s))$$

for a.e. $s \geq 0$. Hence we see that

$$2^{-1}\|u(T)-y\|^2 + \int_t^T \varphi^s(u(s))ds \leq 2^{-1}\|u(t)-y\|^2$$

holds for each pair t, T with $0 \leqq t \leqq T$, and this completes the proof.

LEMMA 3.2. Let C be a non-empty closed convex subset of $\bigcap_{t\geq 0} F(\varphi^t)$, and let P_C be the projection from H onto C. Let u be a solution of (E). Then $v(t) \equiv P_C u(t)$ converges strongly in H to some point of C as $t \to \infty$, and moreover, $\|u(t)-v(t)\|$ is decreasing in $t \geqq 0$.

PROOF. Let $t \geq 0$ and $h > 0$. Then by the definition of v and Lemma 3.1 we have

$$\|u(t+h) - v(t+h)\| \leq \|u(t+h) - v(t)\| \leq \|u(t) - v(t)\|\,. \tag{3.1}$$

This implies that $\|u(t)-v(t)\|$ is decreasing in t and converges as $t \to \infty$. On the other hand, the parallelogram law yields

$$\begin{aligned}\|v(t+h) - v(t)\|^2 &+ 4\|2^{-1}(v(t+h) + v(t)) - u(t+h)\|^2 \\ &= 2\|v(t+h) - u(t+h)\|^2 + 2\|v(t) - u(t+h)\|^2.\end{aligned} \tag{3.2}$$

Since $2^{-1}(v(t+h)+v(t)) \in C$, we have

$$\|v(t+h) - u(t+h)\| \leq \|2^{-1}(v(t+h) + v(t)) - u(t+h)\|\,. \tag{3.3}$$

Combining (3.1), (3.2) and (3.3), we obtain

$$\|v(t+h) - v(t)\|^2 \leq 2\|v(t) - u(t)\|^2 - 2\|v(t+h) - u(t+h)\|^2.$$

Letting $t \to \infty$ in this inequality, we can conclude that $v(t)$ converges strongly in H to a point of C as $t \to \infty$. Q.E.D.

PROOF OF THEOREM 2.1. Let u be a solution of (E). First Lemma 3.1 implies that $\varphi^{(\cdot)}(u(\cdot)) \in L^1(0, \infty)$. Next, given $\varepsilon > 0$, take a positive number $\delta = \delta(\varepsilon)$ satisfying (2.2) in (A2). Then we see from (2.1) in (A2) that the set

$$\{t \geq 0;\ \varphi^t(u(t)) \leq \delta a(t)\}$$

is non-empty. Hence $\inf_{t \geq 0} dist\,(u(t), F) \leq \varepsilon$, and so

$$\inf_{t \geq 0} dist\,(u(t), F) = 0. \tag{3.4}$$

Also, according to Lemma 3.2, $v(t) = P_F u(t)$ converges strongly in H to some $z \in F$ as $t \to \infty$ and $\|u(t) - v(t)\|$ is decreasing in t. Therefore, noting (cf. Lemma 3.1) that $\|u(t) - z\|$ is decreasing in t, we infer from (3.4) that

$$\begin{aligned}
\lim_{t \to \infty} \|u(t) - z\| &\leq \lim_{t \to \infty} (\|u(t) - v(t)\| + \|v(t) - z\|) \\
&= \lim_{t \to \infty} \|u(t) - v(t)\| = \inf_{t \geq 0} \|u(t) - v(t)\| \\
&= \inf_{t \geq 0} dist\,(u(t), F) = 0.
\end{aligned}$$

Thus $\mathrm{s} - \lim_{t \to \infty} u(t) = z \in F$.

4. Proof of Theorem 2.2

Throughout this section suppose all of the assumptions of Theorem 2.2 hold.

Let u be a solution of (E). Then $\varphi^{(\cdot)}(u(\cdot)) \in L^1(0, \infty)$ by Lemma 3.1 and $\int_0^\infty b(t)dt = \infty$ by assumption. Hence, for each integer $n = 1, 2, \ldots$, the inequality $\varphi^t(u(t)) \leq n^{-1} b(t)$ holds for t in a subset of $[0, \infty)$ with positive measure. Put

$$t_n = \inf\{t \geq 0;\ \varphi^t(u(t)) \leq n^{-1} b(t)\} \qquad \text{for} \quad n = 1, 2, \ldots .$$

Then

$$0 \leq t_n < \infty \quad \text{and} \quad t_n \leq t_{n+1} \qquad \text{for} \quad n = 1, 2, \ldots .$$

With regard to the sequence $\{t_n\}$ there are two cases listed below:

(a) $\{t_n\}$ is bounded, i.e. $t_n \uparrow T$ for some $T \geqq 0$ as $n \to \infty$;

(b) $\{t_n\}$ is unbounded, i.e. $t_n \uparrow \infty$ as $n \to \infty$.

We then show the assertion of Theorem 2.2 in each of the above cases.

Case (a). In this case there is a sequence $\{s(n)\}$ such that

$$t_n \leq s(n), \quad u(s(n)) \in D(\varphi^{s(n)}), \quad s(n) \uparrow T \quad \text{and} \quad \varphi^{s(n)}(u(s(n))) \leq n^{-1} b(s(n)).$$

From the lower semi-continuity of ψ and (2.4) in (B2) it follows that

$$\begin{aligned}
\psi(u(T)) &\leq \liminf_{t \to T} \psi(u(t)) \leq \liminf_{n \to \infty} \psi(u(s(n))) \\
&\leq \liminf_{n \to \infty} b(s(n))^{-1} \varphi^{s(n)}(u(s(n))) \leq \lim_{n \to \infty} n^{-1} = 0.
\end{aligned}$$

This shows $u(T)\in F$ since $\min\psi=0$ by (B1). Hence $u(t)=u(T)$ for all $t\geq T$ by Lemma 3.1 and $\mathrm{s}-\lim_{t\to\infty}u(t)=u(T)\in F$.

Before proceeding to the second case (b) we prove the following lemma.

LEMMA 4.1. $0\in F$.

PROOF. Let t be any non-negative number. Then from (2.3), (2.4) in (B2) it follows that

$$0 = c(1+c)^{-1}(-c^{-1}x) + (1+c)^{-1}x\in D(\varphi^t) \subset D(\psi)$$

and

$$\psi(-c^{-1}x) \leq \psi(x)$$

for any $x\in D(\varphi^t)$. Therefore, for any $x\in D(\varphi^t)$,

$$\begin{aligned}\psi(0) &= \psi(c(1+c)^{-1}(-c^{-1}x) + (1+c)^{-1}x)\\ &\leq c(1+c)^{-1}\psi(-c^{-1}x) + (1+c)^{-1}\psi(x) \leq \psi(x),\end{aligned}$$

so that the application of (2.4) again gives

$$\varphi^t(0) = b(t)\psi(0) \leq b(t)\psi(x) \leq \varphi^t(x) \qquad \text{for any} \quad x\in D(\varphi^t).$$

This means that $0\in F(\varphi^t)=F$.

We next consider

Case (b). In this case we choose a sequence $\{s(n)\}$ so that

$$t_n \leq s(n), \quad u(s(n))\in D(\varphi^{s(n)}), \quad \varphi^{s(n)}(u(s(n))) \leq n^{-1}b(s(n))$$

and

$$\|u(t_n) - u(s(n))\|^2 \leq 1/n \qquad \text{for} \quad n=1, 2, 3,\dots.$$

Let $n\geqq 1$. Then assumption (B1) implies that

$$D(\varphi^t) \supset D(\varphi^{s(n)}) \supset -c^{-1}D(\varphi^{s(n)}) \qquad \text{for} \quad 0\leq t\leq s(n).$$

Hence it follows from (2.4) in (B2) and the definition of $\{t_n\}$ that

$$\begin{aligned}\varphi^t(-c^{-1}u(s(n))) &\leq b(t)\psi(u(s(n))) \leq b(t)b(s(n))^{-1}\varphi^{s(n)}(u(s(n)))\\ &\leq n^{-1}b(t) \leq \varphi^t(u(t)) \qquad \text{for a.e.} \quad t\in[0, t_n).\end{aligned}$$

Therefore, by the definition of subdifferential, we have

$$(-u'(t), -c^{-1}u(s(n))-u(t)) \leq \varphi^t(-c^{-1}u(s(n))) - \varphi^t(u(t)) \leq 0 \qquad \text{for} \quad \text{a.e.} \quad t\in[0, t_n),$$

so that

$$(4.1) \qquad (-u'(t),\ -u(s(n))) \le c(-u'(t),\ u(t)) \qquad \text{for} \quad \text{a.e. } t\in[0,\ t_n).$$

It follows from (4.1) that

$$\begin{aligned}(4.2) \quad \|u(T) - u(s(n))\|^2 &= 2\int_T^{s(n)} (-u'(t),\ u(t) - u(s(n)))dt \\ &= 2\int_T^{t_n} (-u'(t),\ u(t) - u(s(n)))dt + 2\int_{t_n}^{s(n)} (-u'(t),\ u(t) - u(s(n)))dt \\ &\leqq 2(1+c)\int_T^{t_n} (-u'(t),\ u(t))dt + 2\|u(t_n) - u(s(n))\|^2 \\ &\leqq (1+c)|\ \|u(t_n)\|^2 - \|u(T)\|^2| + 2n^{-1}\end{aligned}$$

for $0\leqq T\leqq t_n$. Since $0\in F$ by Lemma 4.1, we see from Lemma 3.1 that $\|u(s)\|$ is decreasing in s and converges as $s\to\infty$, and so we can deduce from (4.2) that the strong limit

$$z_0 = \mathrm{s} - \lim_{T\to\infty} u(T) \quad \text{exists in} \quad H.$$

Finally we show $z_0\in F$. In fact, we have

$$\begin{aligned}\psi(z_0) &\leqq \liminf_{T\to\infty} \psi(u(T)) \leqq \liminf_{n\to\infty} \psi(u(s(n))) \\ &\leqq \liminf_{n\to\infty} b(s(n))^{-1}\varphi^{s(n)}(u(s(n))) \leqq \lim_{n\to\infty} n^{-1} = 0.\end{aligned}$$

which implies $z_0\in F$. Thus the proof of Theorem 2.2 is complete.

5. Counterexamples

In this section we give two counterexamples which are related to the assumptions of Theorem 2.2.

The next example says that the restrictions $D(\varphi^t)\supset D(\varphi^s)$ for $0\leqq s\leqq t$ can not be dropped in Theorem 2.2.

Example 5.1. Let H be the space l^2. Let $\{e_i;\ i\geqq 1\}$ be an orthonormal base of H. Then we put

$$e(t) = \cos(t-2^{-1}n\pi)e_n + \sin(t-2^{-1}n\pi)\, e_{n+1}$$

and

$$L(t) = \{re(t);\ r\in \boldsymbol{R}\}$$

for $t\in[2^{-1}(n-1)\pi,\ 2^{-1}n\pi)$, $n=1, 2, 3,\dots$. Now, given $\varepsilon>0$, we define non-negative proper *l.s.c.* convex functionals ψ and φ^t, $0\le t<\infty$, by putting

$$\psi(x) = 2^{-1}\textstyle\sum_{n=1}^{\infty} \varepsilon^n(e_n,\ x)^2 \qquad \text{for} \quad x\in H,$$

and

$$\varphi^t(x) = \begin{cases} \psi(x) & \text{if} \quad x \in L(t), \\ \infty & \text{otherwise.} \end{cases}$$

It is not difficult to verify that

(i) ψ and φ^t, $0 \leqq t < \infty$, are *even* on H, and

$$F(\varphi^t) = F(\psi) = \{0\} \quad \text{for any} \quad t \geq 0;$$

(ii) $\psi(x) = \varphi^t(x)$ for any $x \in D(\varphi^t)$ $(= L(t))$ and $t \geq 0$.

Therefore the family $\{\varphi^t\}$ satisfies all of the assumptions of Theorem 2.2 except that $D(\varphi^s) \supset D(\varphi^t)$ for $0 \leqq s \leqq t$. By virtue of the existence theorem in [7] (or [11]), there exists a unique solution u of (E) associated with initial condition $u(0) = e_1$. Also, it is shown after elementary computations that $\|u(t)\| \geq 1 - 2\varepsilon\pi$ for every $t \geq 0$ if $0 < \varepsilon < 4^{-1}\pi$, which means that $u(t)$ does not converge strongly in H to 0 as $t \to \infty$.

Next we give a counterexample which is related to Remark 2.3.

EXAMPLE 5.2. We take the space $H = l^2$ and consider the functionals φ, ψ_1 and ψ_2 on H which are defined respectively by

$$\varphi(x) = \textstyle\sum_{i=1}^{\infty} \alpha_i f_i(x_i, x_{i+1}),$$

$$\psi_1(x) = \textstyle\sum_{i=1}^{\infty} \alpha_i g_i(x_i, x_{i+1})$$

and

$$\psi_2(x) = \textstyle\sum_{i=1}^{\infty} \alpha_i h_i(x_i, x_{i+1})$$

for $x = (x_i)_{i \geq 1} \in l^2$. Here a sequence $\{\alpha_i\}_{i \geq 1}$ in $\boldsymbol{R}$ and functions f_i, g_i, h_i on $\boldsymbol{R}^2$ are defined as follows:

$$f_i(\xi, \eta) = \begin{cases} \{\mathrm{Tan}^{-1}(|\xi|/\eta)\}^{\lambda(i)}(\xi^2 + \eta^2)^{1/2} & \text{if} \quad \eta > 0 \quad \text{and} \quad \xi^2 + \eta^2 \leq 1, \\ (\pi/2)^{\lambda(i)}|\xi| + \lambda(i)(\pi/2)^{\lambda(i)-1}|\eta| & \text{if} \quad \eta \leq 0 \quad \text{and} \quad \xi^2 + \eta^2 \leq 1, \\ \infty & \text{otherwise,} \end{cases}$$

where

$$\lambda(i) = \pi^2 b^{i+1}(b-1)/8, \quad b = (\log 2)^{-1};$$

$$g_i(\xi, \eta) = \begin{cases} f_i(\xi, (1-\eta^2)^{1/2}) & \text{if} \quad \xi^2 + \eta^2 \leq 1, \\ \infty & \text{otherwise;} \end{cases}$$

$$h_i(\xi, \eta) = \begin{cases} (\pi/2)^{\lambda(i)}|\xi| + \lambda(i)(\pi/2)^{\lambda(i)-1}|\eta| & \text{if} \quad \xi^2 + \eta^2 \leq 1, \\ \infty & \text{otherwise;} \end{cases}$$

and $\{\alpha_i\}_{i\geq 1}$ is a sequence of positive numbers satisfying

$$\sum_{i=1}^{\infty} \alpha_i\{(\pi/2)^{\lambda(i)} + \lambda(i)(\pi/2)^{\lambda(i)-1}\} < \infty. \tag{5.1}$$

First we verify the convexity of f_i on $\boldsymbol{R}^2$. As was shown in [1], one has

$$(\partial/\partial\xi)f_i(\xi, \eta) = \{\mathrm{Tan}^{-1}(\xi/\eta)\}^{\lambda(i)-1}(\xi^2+\eta^2)^{-1/2}[\lambda(i)\eta+\{\mathrm{Tan}^{-1}(\xi/\eta)\}\xi],$$

$$(\partial/\partial\eta)f_i(\xi, \eta) = \{\mathrm{Tan}^{-1}(\xi/\eta)\}^{\lambda(i)-1}(\xi^2+\eta^2)^{-1/2}[-\lambda(i)\xi+\{\mathrm{Tan}^{-1}(\xi/\eta)\}\eta]$$

and

$$(\partial^2/\partial\xi^2)f_i(\xi, \eta)t^2 + 2(\partial^2/\partial\xi\partial\eta)f_i(\xi, \eta)ts + (\partial^2/\partial\eta^2)f_i(\xi, \eta)s^2 > 0$$

for $\xi>0$, $\eta>0$ with $\xi^2+\eta^2\leq 1$, t, $s\in\boldsymbol{R}^2$ and $i\geq 1$. Hence f_i is convex on $\{\xi, \eta)$; $\xi\geq 0$, $\eta\geq 0\}$. Moreover

$$\mathrm{grad}\, f_i(\xi, 0) = ((\pi/2)^{\lambda(i)},\ -\lambda(i)\,(\pi/2)^{\lambda(i)-1})$$

and

$$\mathrm{grad}\, f_i(0, \eta) = (0, 0)$$

for ξ, $\eta\in(0, 1)$. Thus f_i is convex on $\boldsymbol{R}^2$.

Next, by the definitions of f_i, g_i, h_i, we have

$$g_i(\xi, \eta) = g_i(-\xi, -\eta) \leq f_i(\xi, \eta) \leq h_i(\xi, \eta) = h_i(-\xi, -\eta) \qquad \text{for} \quad (\xi, \eta)\in\boldsymbol{R}^2$$

and

$$\min g_i = g_i(0, \eta) = 0 \qquad \text{for} \quad 0\leq\eta\leq 1,$$

$$\min f_i = f_i(0, 0) = 0,$$

$$\min h_i = h_i(0, 0) = 0.$$

Hence, noting (5.1), we see that φ, ψ_1, ψ_2 are proper *l.s.c.* convex functionals on l^2, and that

$$D(\varphi) = D(\psi_1) = D(\psi_2) = \{(x_i)_{i\geq 1}\in l^2;\ x_i^2 + x_{i+1}^2 \leq 1 \qquad \text{for} \quad i\geq 1\},$$

$$F(\varphi) = F(\psi_1) = F(\psi_2) = \{0\},$$

$$\psi_1(x) = \psi_1(-x) \leq \varphi(x) \leq \psi_2(x) = \psi_2(-x) \qquad \text{for} \quad x\in l^2.$$

However, it is known (cf. [1; Lemma 6]) that there exists a sequence $\{\alpha_i\}$ of positive numbers satisfying (5.1) such that the solution u of $u'(t)\in-\partial\varphi(u(t))$, $t>0$, with initial condition $u(0)=(1, 0, 0,\ldots)$ does not converge strongly in H as $t\to\infty$.

6. Further results on the strong convergence

The assumption (A2) of Theorem 2.1 is related to the compactness property of φ^t, while (B2) of Theorem 2.2 is closely related to the evenness property of φ^t. They are quite different from each other. However the asymptotic strong convergence of solutions of (E) can also be obtained under combination of these two types of assumtions.

THEOREM 6.1. *Suppose that there are a proper l.s.c. convex functional ψ on H, a closed subspace X of H, a positive measurable function $b(\cdot)$ everywhere defined on $[0, \infty)$ and a number $c \geqq 1$ satisfying the following conditions:*

(C1) $D(\varphi^s) \supset D(\varphi^t)$ *for* $s, t \in [0, \infty)$ *with* $s \leq t$, *and*

$$F(\varphi^t) = F(\psi) \, (\equiv F) \neq \emptyset \quad \text{and} \quad \min \varphi^t = \min \psi = 0 \quad \text{for} \quad t \geqq 0.$$

(C2) $\int_0^\infty b(t)dt = \infty$ *and for each* $\varepsilon > 0$ *there is* $\delta = \delta(\varepsilon) > 0$ *such that*

$$\text{(6.1)} \qquad \operatorname{dist}(x, F \cap X) \leq \varepsilon \quad \text{for} \quad x \in \cup_{t \geq 0} \{x \in D(\varphi^t) \cap X;\ \varphi^t(x) \leq \delta b(t)\}.$$

(C3) *For the projection P_X from H onto X we have*

$$\text{(6.2)} \qquad b(t)\psi(x) \leq \varphi^t(x) \leq b(t)\psi(-cx + (1+c)P_X x) \quad \text{for} \quad x \in H \quad \text{and} \quad t \geq 0.$$

Then, for each solution u of (E), we have

$$s - \lim_{t \to \infty} u(t) \in F.$$

In the rest of this section suppose that all the assumptions of Theorem 6.1 hold.

LEMMA 6.1. (1) $\psi(P_X x) \leq \psi(x)$ *for any* $x \in H$.

(2) $\varphi^t(P_X x) \leq \varphi^t(x)$ *for any* $x \in H$ *and* $t \geq 0$.

(3) $F \cap X \neq \emptyset$ *and* $P_{F \cap X} x = P_{F \cap X}(P_X x)$ *for any* $x \in H$, *where* $P_{F \cap X}$ *is the projection from* H *onto* $F \cap X$.

PROOF. By (6.2) in (C3) we have

$$\text{(6.3)} \qquad \psi(y) \leq \psi(-cy + (1+c)P_X y) \qquad \text{for} \quad y \in H.$$

Let x be any point of H and put

$$y = c^{-1}(1+c)P_X x - c^{-1}x.$$

Then it is easy to see that $P_X y = P_X x$.

Hence we infer from (6.3) that

$$\psi(c^{-1}(1+c)P_X x - c^{-1}x) \leq \psi(x). \tag{6.4}$$

Moreover, by using (6.4), we have

$$\begin{aligned}\psi(P_X x) &= \psi(c(1+c)^{-1}\{c^{-1}(1+c)P_X x - c^{-1}x\} + (1+c)^{-1}x) \\ &\leq c(1+c)^{-1}\psi(c^{-1}(1+c)P_X x - c^{-1}x) + (1+c)^{-1}\psi(x) \\ &\leq \psi(x).\end{aligned}$$

Thus (1) holds. Next, since $\varphi^t(y) \leq b(t)\psi(-cy+(1+c)P_X y)$ for $y \in H$ by (6.2) in (C3), we see by taking $P_X x$ as y in this inequality that

$$\varphi^t(P_X x) \leq b(t)\psi(P_X x). \tag{6.5}$$

Also, by (6.2) in (C3) and (1),

$$b(t)\psi(P_X x) \leq b(t)\psi(x) \leq \varphi^t(x). \tag{6.6}$$

The above inequalities (6.5), (6.6) yield (2). Finally we show (3). For this purpose, we first note that if $x \in F(\varphi^t) = F$, then $P_X x \in F$ by (2). This shows $F \cap X \neq \emptyset$. Moreover, from the properties of projections we see that

$$\begin{aligned}\|P_{F\cap X}x - P_{F\cap X}(P_X x)\|^2 &\leq (P_{F\cap X}x - P_{F\cap X}(P_X x),\, x - P_X x) \\ &= (P_X(P_{F\cap X}x - P_{F\cap X}(P_X x)),\, x - P_X x) = 0 \qquad \text{for any} \quad x \in H,\end{aligned}$$

which implies $P_{F\cap X} = P_{F\cap X}P_X$.

Now, let u be any solution of (E). Also, we put

$$t_n = \inf\{t \geq 0;\, \varphi^t(u(t)) \leq n^{-1}b(t)\} \qquad \text{for} \quad n = 1, 2, \ldots$$

Then, just as in the proof of Theorem 2.2, we see that

$$0 \leq t_n < \infty \quad \text{and} \quad t_n \leq t_{n+1},$$

and that $\{t_n\}$ satisfies

(a) $\{t_n\}$ is bounded,

or else

(b) $\{t_n\}$ is unbounded.

In case (a) holds, we can show in a way similar to the proof of Theorem 2.2 that $s-\lim_{t\to\infty} u(t) = u(T) \in F$, where $T = \lim_{n\to\infty} t_n$.

We then suppose that (b) holds throughout the remainder of this section. First we need the following lemma.

LEMMA 6.2. *For* $n \geqq 1$, $y \in H$ *and* $T \leq t_n$,

$$\|u(T) - u(t_n)\|^2 \leq (1+c)\{\|u(T)-y\|^2 - \|u(t_n)-y\|^2\} + 2(1+c)(u(t_n) - u(T), P_X u(t_n) - y). \tag{6.7}$$

PROOF. By the definition of t_n we have

$$\varphi^s(u(s)) > n^{-1}b(s) \qquad \text{for} \quad 0 \leq s < t_n.$$

Hence, using (6.2) in (C3), we get

$$\begin{aligned}\varphi^s(c^{-1}(1+c)P_X u(t_n) - c^{-1}u(t_n)) &\leq b(s)\psi(u(t_n)) \\ &\leq b(s)\liminf_{t\to t_n}\psi(u(t)) \leq b(s)\liminf_{t\to t_n} b(t)^{-1}\varphi^t(u(t)) \\ &\leq b(s)n^{-1} \leq \varphi^s(u(s)) \qquad \text{for any} \quad s \in [0, t_n),\end{aligned}$$

that is,

$$\varphi^s(c^{-1}(1+c)P_X u(t_n) - c^{-1}u(t_n)) \leq \varphi^s(u(s)) \qquad \text{for any} \quad s \in [0, t_n). \tag{6.8}$$

From the definition of subdifferential and (6.8) it follows that

$$\begin{aligned}&(-u'(s),\ c^{-1}(1+c)P_X u(t_n) - c^{-1}u(t_n) - u(s)) \\ &\quad \leq \varphi^s(c^{-1}(1+c)P_X u(t_n) - c^{-1}u(t_n)) - \varphi^s(u(s)) \leq 0 \qquad \text{for a.e.} \quad s \in [0, t_n),\end{aligned}$$

so that

$$(-u'(s),\ -u(t_n)) \leq c(-u'(s),\ u(s)) + (c+1)(u'(s),\ P_X u(t_n)) \quad \text{for a.e.} \quad s \in [0, t_n).$$

This implies

$$\begin{aligned}&(-u'(s),\ u(s) - u(t_n)) \\ &\quad \leq (1+c)(-u'(s),\ u(s) - y) + (1+c)(u'(s),\ P_X u(t_n) - y) \\ &\qquad\qquad \text{for a.e.} \quad s \in [0, t_n) \quad \text{and} \quad y \in H.\end{aligned} \tag{6.9}$$

Integrating (6.9) over $[T, t_n]$, we obtain (6.7).

LEMMA 6.3. $s-\lim_{n\to\infty} P_X u(t_n) \in F \cap X$.

PROOF First we show that

$$dist.(P_X u(t_n),\ F \cap X) \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{6.10}$$

In fact, on account of the definition of $\{t_n\}$, we can find a sequence $\{T(n)\}$ such that

$$T(n) \geq t_n,\ \varphi^{T(n)}(u(T(n))) \leq n^{-1}b(T(n)) \quad \text{and} \quad \|u(t_n) - u(T(n))\| \leq 1/n.$$

Given $\varepsilon > 0$, take a number $\delta = \delta(\varepsilon) > 0$ as specified in (6.1). Then, using (2) of Lemma 6.1, we see that

$$\varphi^{T(n)}(P_X u(T(n))) \le \varphi^{T(n)}(u(T(n))) \le n^{-1}b(T(n)) \qquad \text{for} \quad n = 1, 2, \dots .$$

so that we infer from (6.1) of (C2) that

$$dist\,(P_X u(T(n)), F \cap X) \le \varepsilon \qquad \text{for any} \quad n \ge \delta^{-1}.$$

Therefore, by (3) of Lemma 6.1, we have

$$\begin{aligned} & dist\,(P_X u(t_n), F \cap X) \\ & \quad \le dist\,(P_X u(T(n)), F \cap X) + \|P_X u(T(n)) - P_X u(t_n)\| \\ & \qquad + \|P_{F\cap X}(P_X u(T(n))) - P_{F\cap X}(P_X u(T(n)))\| \\ & \quad \le dist\,(P_X u(T(n)), F \cap X) + 2\|u(T(n)) - u(t_n)\| \\ & \quad \le \varepsilon + 2n^{-1} \qquad \text{for any} \quad n \ge \delta^{-1}, \end{aligned}$$

which shows (6.10). Next, we note that $P_{F\cap X}u(t_n)$ converges strongly in H to some $z \in F \cap X$ as $n \to \infty$, which follows from Lemma 3.2. Since

$$\begin{aligned} \|P_X u(t_n) - z\| & \le \|P_X u(t) - P_{F\cap X}u(t_n)\| + \|P_{F\cap X}u(t_n) - z\| \\ & = \|P_X u(t) - P_{F\cap X}(P_X u(t_n))\| + \|P_{F\cap X}u(t_n) - z\| \\ & = dist.\,(P_X u(t_n), F \cap X) + \|P_{F\cap X}u(t_n) - z\|, \end{aligned}$$

we obtain $s-\lim_{n\to\infty} P_X u(t_n) = z \in F \cap X$.

In order to complete the proof of Theorem 6.1 we use the inequality (6.7) in Lemma 6.2. In fact, let $T = t_m$ with $m < n$ and $y = s-\lim_{n\to\infty} P_X u(t_n) \equiv z_0$ in (6.7). Then Lemma 3.1 implies that

$$\lim_{m,n\to\infty} \|u(t_m) - u(t_n)\| = 0,$$

and so $u(t_n)$ converges strongly in H to some u_0 as $n \to \infty$. Therefore $s-\lim_{n\to\infty} u(T(n)) = u_0$ for the same sequence $\{T(n)\}$ as in the proof of Lemma 6.3. Moreover, by (6.2) in (C3),

$$\begin{aligned} \psi(u_0) & \le \liminf_{n\to\infty} \psi(u(T(n))) \le \liminf_{n\to\infty} b(T(n))^{-1}\varphi^{T(n)}(u(T(n))) \\ & \le \lim_{n\to\infty} n^{-1} = 0, \end{aligned}$$

that is, $u_0 \in F$. Thus the proof of Theorem 6.1 is complete.

7. On the weak convergence

We here give a result on the weak convergence of the solutions of (E).

THEOREM 7.1. *Suppose that there exists a proper l.s.c. convex functional ψ on H satisfying*

(b1) $F(\varphi^t) = F(\psi)\ (\equiv F) \neq \emptyset$ *for* $t \geqq 0$,
(b2) $\min \varphi^t = \min \psi = 0$ *for* $t \geqq 0$

and

(b3) $\psi(x) \leq \varphi^t(x)$ *for* $x \in D(\varphi^t)$ *and* $t \geqq 0$.

Then, for each solution u of (E), there exists a measurable set Λ in $[0, \infty)$ such that

$$\lim_{t\to\infty} meas\,(\Lambda \cap [t, t+1)) = 1, \tag{7.1}$$

$$\mathrm{w}-\lim_{\lambda\to\infty,\lambda\in\Lambda} u(\lambda) \in F \tag{7.2}$$

and

$$\lim_{\lambda\to\infty,\lambda\in\Lambda} \varphi^\lambda(u(\lambda)) = 0. \tag{7.3}$$

PROOF. Let u be a solution of (E). For each integer $n \geq 1$, put

$$\Lambda(n) = \{t \geq 0;\ \varphi^t(u(t)) \leq n^{-1}\}. \tag{7.4}$$

Then, since $\varphi^{(\cdot)}(u(\cdot)) \in L^1(0, \infty)$ by Lemma 3.1, $\Lambda(n)$ is a measurable set in $[0, \infty)$ and $meas\,([0, \infty)\backslash\Lambda(n)) < \infty$. Also, we have $\Lambda(n) \supset \Lambda(n+1)$ for $n = 1, 2, \ldots$ Hence we can easily find a strictly increasing sequence $\{T_n\}$ in $[0, \infty)$ such that

$$meas\,(\Lambda(n) \cap [t, t+1)) \geq 1 - 1/n \qquad \text{for all} \quad t \geq T_n. \tag{7.5}$$

Now put

$$\Lambda = \cup_{n=1}^{\infty} \Lambda(n) \cap [T_n, T_{n+1}).$$

Then, it follows immediately from (7.4) and (7.5) that the set Λ satisfies (7.1) and (7.3). It remains to prove (7.2). Since u is bounded on $[0, \infty)$ by Lemma 3.1, (7.2) is equivalent to the assertion that the set

$$U = \left\{ x \in H;\ \begin{array}{l} u(t_n) \to x \text{ weakly in } H \text{ as } n\to\infty \text{ for some} \\ \{t_n\} \subset \Lambda \text{ with } t_n \to \infty \text{ as } n \to \infty \end{array} \right\}$$

is contained in F and is a singleton. In order to prove this, we first note that

$$\lim_{\lambda\to\infty,\lambda\in\Lambda} \psi(u(\lambda)) = 0, \tag{7.6}$$

which follows from the fact (cf. condition (b3)) that

$$\psi(u(\lambda)) \leq \varphi^\lambda(u(\lambda)) \leq n^{-1} \qquad \text{for} \quad \lambda \in \Lambda(n).$$

Moreover, by the lower semi-continuity of ψ and (7.6), we have

$$\psi(x) = 0 \qquad \text{for any} \quad x \in U,$$

that is

$$U \subset F.$$

At this point we recall the following fact:

$$(7.7)\qquad \begin{cases} \text{Given a weakly convergent sequence } \{x_n\} \text{ in } H \\ \text{with } x_\infty = \mathrm{w}-\lim_{n\to\infty} x_n, \text{ it holds that} \\ \text{lim-inf}_{n\to\infty} \|x_n - x_\infty\| \to \liminf_{n\to\infty} \|x_n - y\| \quad \text{if} \quad y \neq x_\infty. \end{cases}$$

We now prove that U is a singleton set. Let y, z be any elements of U, and let $\{t_n\}$, $\{s_n\}$ be sequences in Λ such that

$$\mathrm{w}-\lim_{n\to\infty} u(s_n) = y \quad \text{and } \mathrm{w}-\lim_{n\to\infty} u(t_n) = z.$$

We may assume without loss of generality that

$$s_n < t_n < s_{n+1} \qquad \text{for} \quad n = 1, 2, \dots .$$

Then, by Lemma 3.1,

$$(7.8)\quad \|u(s_n) - y\| \geq \|u(t_n) - y\| \text{ and } \|u(t_n) - z\| \geq \|u(s_{n+1}) - z\| \text{ for } n = 1, 2, \dots .$$

Moreover, it follows from (7.7) and (7.8) that

$$\begin{aligned} \liminf_{n\to\infty} \|u(s_{n+1}) - z\| &\leq \liminf_{n\to\infty} \|u(t_n) - z\| \\ &\leq \liminf_{n\to\infty} \|u(t_n) - y\| \leq \liminf_{n\to\infty} \|u(s_n) - y\|, \end{aligned}$$

and hence

$$\liminf_{n\to\infty} \|u(s_n) - z\| \leq \liminf_{n\to\infty} \|u(s_n) - y\|.$$

The last inequality together with (7.7) implies that $z = y$. Thus U is a singleton set, and the proof of Theorem 7.1 is complete.

Acknowledgements

The author would like to thank Professors Y. Kōmura and N. Kenmochi for their valuable comments and constant encouragement.

References

[1] J. B. Baillon, Un exemple concernant le comportement asymptotique de la solution du problème $du/dt + \partial\varphi(u) \ni 0$, J. Funct. Anal., **28** (1978), 369–376.

[2] J. B. Baillon and A. Haraux, Comportement a l'infini pour les equations d'evolution avec forcing periodique, Archive Rat. Mech. Anal., **67** (1977), 101–109.

[3] H. Brézis, Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, Math. Studies 5, North Holland, Amsterdam-London, 1973.

[4] R. E. Bruck, Asymptotic convergence of nonlinear contraction semigroups in Hilbert spaces, J. Funct. Anal., **18** (1975), 15–26.

[5] H. Furuya, N. Kenmochi and K. Miyashiba, Asymptotic behavior to a class of nonlinear evolution equations, to appear.

[6] G. Gripenberg, On the asymptotic behavior of nonlinear contraction semigroups, Math. Scand., **44** (1979), 385–397.

[7] N. Kenmochi, Solvability of nonlinear evolution equations with time-dependent contractions and applications, Bull. Fac. Education, Chiba Univ. **31** (1980), 1–87.

[8] H. Okochi, A note on asymptotic strong convergence of nonlinear contraction semigroups, Proc. Japan Acad., **56** (1980), 83–84.

[9] H. Okochi, Asymptotic strong convergence of nonlinear contraction semigroups, Tokyo J. Math., **5** (1982), 171–182.

[10] H. Okochi, Asymptotic behavior of solutions to certain nonlinear parabolic evolution equations II, in preparation.

[11] Y. Yamada, On nonlinear evolution equations generated by the subdifferntials, J. Fac. Sci. Univ. Tokyo Sect. IA **23** (1976), 491–515.

[12] Y. Yamada, Periodic solutions of certain nonlinear parabolic differential equations in domains with periodically moving boundaries, Nagoya Math. J., **70** (1978), 111–125.

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