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Transient and large time behaviors of solutions to heterogeneous reaction-diffusion equations

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Abstract: We consider initial-boundary value problems for heterogeneous reaction-diffusion equations $\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}(d(x)\frac{\partial u}{\partial x}+e(x)u)+\varepsilon f(x, u)$, and study transient and large time behaviors of solutions. Our method is to explicitly construct a two-timing function $u(t, \varepsilon t, x)$ that converges to the exact solution as $\varepsilon \downarrow 0$ uniformly in $t \in [0, \infty)$. Such an explicit expression of approximate solutions in terms of two-timing functions can be applied to a fairly general class of equations of the above form as well as weakly-coupled systems of such equations.

1. Introduction

We consider the initial value problem with a small parameter ε,

$$(1.1) \qquad \begin{cases} u_t + Au = \varepsilon F(u) \\ u(0) = u_0 \end{cases} \quad \text{in} \quad B,$$

where B is a Banach space and A is a sectorial operator in B and $u(t) \in D(A) \cap C^1((0, \infty); B)$. We impose the following conditions on A and F: for simplicity, we denote the norm by $\|\cdot\|_B$ and also the operator norm by the same symbol, if there is no ambiguity.

1) $\sigma(A)$, the spectral set of A, consists of $\sigma_1=\{0\}$, $\sigma_2 \subset \{\lambda \in \mathbf{C} \mid \mathrm{Re}\,\lambda > \alpha > 0$ for some constant $\alpha > 0\}$.
2) There exists $M_1 > 0$ such that
$$\|e^{-tA}\|_B \leqq M_1,$$
where e^{-tA} is a semigroup generated by A.

Let Q, P be projections corresponding to σ_1, σ_2 respectively.

3) $QB = \mathrm{Ker}\,A$ and it is a finite dimensional space.
4) There exist $M_2 > 0$, $\lambda_1 > 0$ such that
$$\|e^{-tA}P\|_B \leqq M_2 e^{-\lambda_1 t} \qquad \text{for} \quad t \geqq 0.$$
5) $F(u)$ is a twice Fréchet differentiable mapping from B into itself and for each bounded set B_0 in B there exists $M_3 > 0$ depending on B_0 such that

$$\|F(u)\|_B,\ \|F'(u)\|_B,\ \|F''(u)\|_B \leqq M_3 \quad \text{on} \quad B_0,$$

where $'$ represents the Fréchet derivative.

We are concerned with the study of transient and asymptotic behaviors of the solution $u(t;\varepsilon)$. This study is motivated by ecological problems proposed by Shigesada [8]. First let us briefly state the ecological background of the problem (1.1).

Consider a bounded heterogeneous habitat where N-species are interacting one another and are migrating by both random motion and direct movement toward favoured states. Then the population density of the i-th species u_i in a one dimensional habitat $I\equiv(0, L)$ is described by the heterogeneous reaction-diffusion-advection system

$$(1.2)\qquad \frac{\partial}{\partial t}u_i + \frac{\partial}{\partial x}J_i(x, u_i) = \varepsilon f_i(x, u), \quad x\in I,\ t>0 \qquad (i=1, 2,\dots, N),$$

where $u=(u_1, u_2,\dots, u_N)$ and the flux $J_i(x, u_i)$ of u_i takes the form

$$(1.3)\qquad J_i(x, u_i) = -\, d_i(x)\frac{\partial}{\partial x}u_i - e_i(x)u_i \qquad (i=1, 2,\dots, N),$$

where the first and second terms represent the diffusion process with $d_i(x)>0$ and the advection one, respectively. If $e_i(x)$ is written as $e_i(x)=\frac{d}{dx}E_i(x)$, the function $E_i(x)$ is called the environmental potential in the sense that individuals of the i-th species have the tendency to migrate toward the minimum points of $E_i(x)$ in I. $f_i(x, u)$ is the spatially inhomogeneous growth rate of u_i due to ecological interactions among N-species. In many ecological systems, the dispersal processes take place daily but the growth processes do only once or twice a year; that is, the processes proceed on totally different time scales. It therefore seems natural to assume ε in (1.2) to be very small.

A simple but motive example of (1.2) for a single species is

$$(1.4)\qquad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}J(x, u) = \varepsilon\{a(x)-b(x)u\}u,$$

where $J(x, u)$ is the flux of u as in (1.3) with $N=1$ and the growth rate $\varepsilon\{a(x)-b(x)u\}$ is a heterogeneous version of the Pearl-Verhulst logistic law. It is assumed here that $b(x)$ is positive, while the sign of $a(x)$ may vary.

The initial and boundary conditions for (1.2) are given by

$$\begin{aligned} &(1.5)\qquad u_i(0, x) = u_{0i}(x), \quad x\in I \\ &(1.6)\qquad J_i(x, u_i) = 0, \quad x\in \partial I, \quad t>0 \end{aligned} \qquad (i=1, 2,\dots, N),$$

respectively. We are interested in the study of the effect of the heterogeneities of

$d_i(x)$, $e_i(x)$, $f_i(x, u)$ and $u_{0i}(x)$ on the behavior of solutions to the problem (1.2), (1.5), (1.6). In ecological terms, we are concerned with the existence or extinction of the species; in other words, which species can survive and which species become extinct. To our knowledge, it is rather hard to study the transient and asymptotic behaviors of solutions of heterogeneous reaction-diffusion-advection systems such as (1.2). Although there is an extensive literature on heterogeneous reaction-diffusion systems (Fleming [2], Fife and Peletier [1], Kurland [4], Mimura and Nishiura [6] etc), only a few of those deal with the transient or the asymptotic behavior of solutions.

Recently, assuming that ε is sufficiently small, Shigesada [8] has applied the two-timing method (see, for instance, Nayfeh [7]) to the problem (1.4), (1.5), (1.6) ($N=1$) and has then constructed a lowest order approximate solution of the form

$$\tilde{u}(t, x; \varepsilon) = w(t, x)n(\varepsilon t). \tag{1.7}$$

Here $w(t, x)$ is a solution of (1.4), (1.5), (1.6) in the limit $\varepsilon \downarrow 0$; that is, w satisfies

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}J(x, w) = 0, \quad x \in I, \quad t > 0$$

together with (1.6) and

$$w(0, x) = \frac{u_0(x)}{\int_I u_0(x)dx};$$

and $n(\tau)$ is a solution of

$$\begin{cases} \dfrac{dn}{d\tau} = \displaystyle\int_I \{a(x) - b(x)n\tilde{w}(x)\}n\tilde{w}(x)dx, & \tau > 0, \\ n(0) = \displaystyle\int_I u_0(x)dx, \end{cases} \tag{1.8}$$

where $\tilde{w}(x) = \lim_{t\to\infty} w(t, x)$. The first equation of (1.8) is reduced to

$$\frac{dn}{d\tau} = (a - bn)n, \qquad \tau > 0,$$

where

$$a = \int_I a(x)\tilde{w}(x)dx, \qquad b = \int_I b(x)\tilde{w}^2(x)dx.$$

Shigesada [8] numerically showed that the approximate solution $\tilde{u}(t, x; \varepsilon)$, which is formally valid for time up to $O(1/\varepsilon)$, agrees fairly well with the exact solution even for a longer time range. On these observations, she used the O. D. E. (1.8) to study whether the species survives or becomes extinct. More precisely, observing that $\lim_{\tau\to\infty} n(\tau)=a/b>0$ if $a>0$ and $\lim_{\tau\to\infty} n(\tau)=0$ if $a<0$, she concluded from the representation (1.7) that $\lim_{t\to\infty} u(t, x; \varepsilon)>0$ if $a>0$ and $\lim_{t\to\infty} u(t, x; \varepsilon)=0$ if $a<0$, where $u(t, x; \varepsilon)$ is the solution of (1.4), (1.5), (1.6) with $N=1$; in other words, the population survives if $a>0$ and becomes extinct if $a<0$. When $e(x)$ is neglected in (1.4), this conclusion can be justified by the results of Fleming [2].

Shigesada's approach motivates us to construct a "two-timing" function of the form (1.7) that approximates the solution of (1.2), (1.5), (1.6) fairly accurately *uniformly* in time. The results will be stated in an abstract form in the next section (Theorems 1–3).

In Section 3, we give some examples to illustrate how our abstract results apply to specific equations. In particular, Shigesada's approach to the equation (1.4) will be completely justified in the following sense (see Example 3.1 for detail): let $u(t, x; \varepsilon)$ and $\tilde{u}(t, x; \varepsilon)$ be a solution of (1.4), (1.5), (1.6) ($N=1$) and a "two-timing" function of the form (1.7) respectively; then, if $a>0$,

$$\|u(t, \cdot, \varepsilon)-\tilde{u}(t, \cdot, \varepsilon)\|_{L^\infty(I)} \leqq C_1\varepsilon \qquad (0\leqq t<+\infty)$$

for some positive constant C_1 and

$$\left|\int_I u(t, x; \varepsilon)dx-n(\varepsilon t)\right| \leqq C_2\varepsilon \qquad (0\leqq t<+\infty)$$

for some positive constant C_2. On the other hand, if $a<0$, then

$$\|u(t, \cdot\,; \varepsilon)-\tilde{u}(t, \cdot\,; \varepsilon)\|_{L^\infty(I)} \leqq C_3\varepsilon e^{-\beta\varepsilon t} \qquad (0\leqq t<+\infty)$$

for some positive constants C_3 and β. Therefore, it follows from $\lim_{t\to\infty} \tilde{u}(t, x; \varepsilon)=a/b\cdot\tilde{w}(x)$ for $a>0$ and $\lim_{t\to\infty} \tilde{u}(t, x; \varepsilon)=0$ for $a<0$, that the sign of a determines the existence or extinction of the population.

2. Main results

In this section we consider the abstract equation (1.1). The results will then be applied to specific equations of the form (1.2) in the next section.

Under the conditions 1)–5) stated in Section 1, we consider the following equations:

$$(2.1) \qquad \begin{cases} \dfrac{dy}{d\tau} = QF(y) \\ \\ y(0) = Qu_0 \end{cases} \qquad \text{in} \quad B_1\equiv QB.$$

Denoting by $y(\tau; Qu_0)$ the solution of (2.1), we define $v(\tau)$ by

$$v(\tau; u_0) = y(\tau; Qu_0) + Pu_0.$$

Then we have

THEOREM 1. *Suppose that the solution* $Qv(\tau; u_0)$ $(=y(\tau; Qu_0))$ *converges as* $\tau \to \infty$ *to some* $\xi \in B_1 = QB$ *satisfying* $QF(\xi)=0$ *and that all eigenvalues of the Jacobian* $QF'(\xi)|_{B_1}$ *have negative real parts. Let* $u(t; u_0, \varepsilon)$ *be the solution of* (1.1). *Then there exist positive constants* C *and* ε_0 *such that*

$$\|u(t; u_0, \varepsilon) - e^{-tA} v(\varepsilon t; u_0)\|_B \leqq C\varepsilon$$

for all $\varepsilon \in (0, \varepsilon_0]$ *and all* $t \in [0, \infty)$.

COROLLARY TO THEOREM 1. *In addition to the assumptions of* Theorem 1, *suppose that* ξ *satisfies* $F(\xi)=0$. *Then there exist positive constants* β, C *and* ε_0 *such that*

$$\|u(t; u_0, \varepsilon) - e^{-tA} v(\varepsilon t; u_0)\|_B \leqq C\varepsilon e^{-\beta\varepsilon t}$$

for all $\varepsilon \in (0, \varepsilon_0]$ *and all* $t \in [0, \infty)$.

THEOREM 2. *Suppose that the solution* $Qv(\tau; u_0)$ *exists for* $\tau \in [0, T]$ *for some* $T < \infty$. *Then there exist positive constants* C_T *and* ε_T *depending on* T *such that*

$$\|u(t; u_0, \varepsilon) - e^{-tA} v(\varepsilon t; u_0)\|_B \leqq C_T\varepsilon$$

for all $\varepsilon \in (0, \varepsilon_T]$ *and all* $t \in [0, T/\varepsilon]$.

We next consider the stationary equation of (1.1)

$$Aw = \varepsilon F(w) \quad \text{in} \quad B. \tag{2.2}$$

THEOREM 3. *Suppose that there exists* $\xi \in B_1$ *satisfying* $QF(\xi)=0$ *and* $\det(QF'(\xi)|_{B_1}) \neq 0$. *Then there exists a positive constant* ε_0 *such that* (2.2) *has a unique solution* $w(\varepsilon)$ *satisfying* $w(\varepsilon) \in C^2((-\varepsilon_0, \varepsilon_0); B)$ *and* $w(0)=\xi$.

The proof will be given in Section 4.

3. Applications

We apply the results in Section 2 to specific models such as (1.2). We first consider the case $N=1$. Define $X = L^2(I)$ and the inner products (u, v) in X by $(u, v) = \int_I u(x)\overline{v(x)}k(x)dx$, where

$$k(x) = \int_I \exp(-U(s))ds \cdot \exp(U(x)) \quad \text{with} \quad U(x) = \int^x e(s)/d(s)ds.$$

Here we assume that each coefficient in (1.2) is real-valued and in $H^1(I)$. Let the operator A in X be $Au=\frac{\partial}{\partial x}J(x, u)$, the domain of A be $D(A)=\{u \in H^2(I) \mid J(x, u)=0$ on $x \in \partial I\}$ respectively. Then

$$(Au, v) = \int_I d(x)\left(u_x + \frac{e(x)}{d(x)}u\right)\overline{\left(v_x + \frac{e(x)}{d(x)}v\right)}\,k(x)dx$$

for $u, v \in D(A)$ and A is found to be a non-negative and self-adjoint operator in X. Thus, $\sigma(A)$ consists of $\{0=\lambda_0<\lambda_1<,\dots\}$, where $\lambda_i\,(i=0, 1, 2,\dots)$ are the eigenvalues of A and $\operatorname{Ker} A=\langle\phi_0\rangle$ with $\phi_0(x)=1/k(x)$. We now find that the projections Q and P are given by

$$Qu = (u, \phi_0)\phi_0 = \int_I u(x)\phi_0(x)k(x)dx\cdot\phi_0 = \int_I u(x)dx\cdot\phi_0$$

and $P=I-Q$, respectively.

Suppose that $f(x, u)$ takes the form $f(x, u)=\sum_{n=0}^m a_n(x)u^n$, where m is a non-negative integer and $a_n(x)\in H^1(I)$. Then $F(u)(x)=f(x, u)$ is a polynomial mapping on $X^{1/2}=D(A^{1/2})$ with the graph norm.

THEOREM 4. *$X^{1/2}=H^1(I)$ with equivalent norms.*

The proof will be given in Section 4. Thus, if we set $B\equiv H^1(I)$ and $Au=\frac{\partial}{\partial x}J(x, u)$ with $D(A)=\{u\in H^2(I)\mid J(x, u)=0$ on $x\in\partial I$ and $Au\in B\}$, then conditions 1)–5) in Section 1 are satisfied. When $N\geqq 2$, we may take $B\equiv\{H^1(I)\}^N$.

Now we consider two typical examples.

EXAMPLE 1. Consider a single species model described by

$$(3.1)\quad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left\{d(x)\dfrac{\partial u}{\partial x} + e(x)u\right\} + \varepsilon\{a(x) - b(x)u\}u, \ x\in I,\ u>0, \\ d(x)\dfrac{\partial u}{\partial x} + e(x)u = 0, \quad x\in\partial I, \quad t>0, \\ u(0, x) = u_0(x) \geqq 0, \quad u_0(x) \not\equiv 0, \quad x\in I, \end{cases}$$

where $d(x)$ is positive on $\bar{I}$. Since $Qu=\int_I u(x)dx\cdot\phi_0$, we obtain the following equation with respect to $y(\tau)=n(\tau)\phi_0$:

$$(3.2)\quad \begin{cases} \dfrac{d}{d\tau}y = \displaystyle\int_I \{a(x)-b(x)y\}y\,dx\cdot\phi_0, \quad \tau>0, \\ y(0) = \displaystyle\int_I u_0(x)dx\cdot\phi_0. \end{cases}$$

The above problem reduces to

$$(3.3)\qquad \begin{cases} \dfrac{d}{d\tau}n = (a-bn)n, & \tau > 0, \\ n(0) = \displaystyle\int_I u_0(x)dx > 0, \end{cases}$$

where $a=\int_I a(x)\phi_0(x)dx$ and $b=\int_I b(x)\phi_0^2(x)dx$. It follows from (3.3) that if $b>0$,

(i) $\lim_{\tau\to\infty} n(\tau) = a/b$ for $a > 0$,

(ii) $\lim_{\tau\to\infty} n(\tau) = 0$ for $a < 0$

and if $b<0$,

(iii) $\lim_{\tau\to\tau_0} n(\tau) = \infty$ for $a > 0$,

(iv–1) $\lim_{\tau\to\infty} n(\tau) = 0\ (0<n(0)<a/b)$ for $a < 0$,

(iv–2) $\lim_{\tau\to\tau_1} n(\tau) = \infty\ (a/b<n(0))$,

where τ_0, τ_1 are some finite numbers. For the case (i), Theorem 1 shows

$$\|u(t,\cdot\,;\varepsilon)-e^{-tA}(n(\varepsilon t)\phi_0+Pu_0)\|_{H^1} \leqq C\varepsilon, \quad 0\leqq t < +\infty$$

for some C. Hence,

$$\left|\int_I u(t, x; \varepsilon)dx - n(\varepsilon t)\right| \leqq C\varepsilon, \quad 0\leqq t < +\infty,$$

which indicates that the species will survive. For the cases (ii) and (iv–1), we note that $u=0$ is a solution of $F(u)=0$ and that $QF'(0)|_{B_1}=a<0$. Thus, Corollary to Theorem 1 shows

$$\|u(t,\cdot\,;\varepsilon)-e^{-tA}(n(\varepsilon t)\phi_0+Pu_0)\|_{H^1} \leqq C\varepsilon e^{-\beta\varepsilon t}, \quad 0\leqq t < +\infty$$

for some C and β, which indicates the extinction of the species as $t\to\infty$. Finally, consider the cases (iii) and (iv–2), where the solution $n(\tau)$ blows up in a finite time. We expect that, in these cases, the original solution $u(t, x; \varepsilon)$ also blows up in a finite time, yet we have no rigorous results.

The above observations illustrate, in a very explicit manner, the effect of the functions $a(x)$, $b(x)$, $e(x)$ and $u_0(x)$ on the transient and large time behaviors of solutions.

EXAMPLE 2. Consider a two competing species model described by the equations

$$(3.4)\quad \begin{cases} \dfrac{\partial}{\partial t}u_1 + \dfrac{\partial}{\partial x}J_1 = \varepsilon\{r_1(x)-u_1-cu_2\}u_1 & \\ & t>0,\ x\in I=(0,2), \\ \dfrac{\partial}{\partial t}u_2 + \dfrac{\partial}{\partial x}J_2 = \varepsilon\{r_2(x)-cu_1-u_2\}u_2 & \\ u_1(0,x)=u_{01}(x), \quad u_2(0,x)=u_{02}(x) & \end{cases}$$

where J_i takes the form of (1.3), $r_i(x)$ (>0) is the intrinsic growth rate of u_i $(i=1, 2)$ and $c(>0)$ is the interspecific competition rate between the two species. An ecological interest in (3.4) is to study whether or not the two species can coexist under the competitive interaction.

In order to investigate quantitatively the effect of $r_i(x)$ and $e_i(x)$ $(i=1, 2)$ on the behavior of solutions, let us specify the coefficients as follows:

$$d_1(x) = d_2(x) = 1, \quad r_1(x) = 1 - 0.1x, \quad r_2(x) = 1 + 0.5x.$$

For simplicity, let us first consider the special case where $e_i(x)\equiv 0$ $(i=1, 2)$. Ecologically, this means that the environmental potentials are spatially homogeneous (see Introduction). This special case was first studied by Su Yu [10] (he also considered non-autonomous equations). In this case, a simple calculation shows that the corresponding O. D. E. system to (2.1) takes the form

$$(3.5)\quad \begin{cases} \dfrac{d}{d\tau}n_1 = (r_1-n_1-cn_2)n_1 & \\ & \tau>0, \\ \dfrac{d}{d\tau}n_2 = (r_2-cn_1-n_2)n_2 & \\ n_1(0) = \displaystyle\int_0^2 u_{01}(x)dx, \quad n_2(0) = \int_0^2 u_{02}(x)dx, & \end{cases}$$

where $r_i=1/2\int_I r_i(x)dx$ $(i=1, 2)$; and it follows from Theorem 1 (and also from [10]) that the large time behavior of (3.4) is essentially dominated by that of (3.5). More precisely, if the solution of (3.5) approaches an asymptotically stable equilibrium point $(\tilde{n}_1, \tilde{n}_2)$ as $\tau\to+\infty$, then the original solution $(u_1(t, x; \varepsilon), u_2(t, x; \varepsilon))$ of (3.4) asymptotically enters the ε-neighborhood of the homogeneous state $(\tilde{n}_1, \tilde{n}_2)$. One easily finds that the asymptotically stable equilibrium points of (3.5) are

i) $\left(\dfrac{r_1-cr_2}{1-c^2}, \dfrac{r_2-cr_1}{1-c^2}\right)$ if $0<c<3/5$;

ii) $(0, r_2)$ if $3/5<c<5/3$;

iii) $(r_1, 0)$ and $(0, r_2)$ if $c>5/3$. Which species can survive depends on initial data.

We next consider the general case where $e_i(x)\not\equiv 0$ (or possibly $e_i(x)\equiv 0$). The functions $d_i(x)$ and $r_i(x)$ are the same as before. Put

$$Y(\tau) = {}^t(y_1(\tau),\, y_2(\tau)) = {}^t(n_1(\tau)\phi_0^1,\, n_2(\tau)\phi_0^2),$$

where $\operatorname{Ker} A_i=\langle\phi_0^i\rangle$ with $A_i=\frac{\partial}{\partial x}J_i$ and $\int_0^2 \phi_0^i dx=1$ $(i=1, 2)$. A simple calculation shows that

$$\phi_0^i = \exp(-U_i(x))/\int_I \exp(-U_i(x))dx,$$

where $U_i(x)=\int^x e_i(s)ds$ $(i=1, 2)$. The corresponding O. D. E. system to (2.1) now takes the form

$$(3.6)\qquad \begin{cases} \dfrac{d}{d\tau}n_1 = (R_1-B_{11}n_1-B_{12}cn_2)n_1 & \\ & \tau>0, \\ \dfrac{d}{d\tau}n_2 = (R_2-B_{21}cn_1-B_{22}n_2)n_2 & \\ n_1(0) = \displaystyle\int_I u_{01}(x)dx, \quad n_2(0) = \int_I u_{02}(x)dx, & \end{cases}$$

where $R_i=\int_I r_i(x)\phi_0^i(x)dx$, $B_{ij}=\int_I \phi_0^i(x)\cdot\phi_0^j(x)dx$ $(i, j=1, 2)$. Now, in order to give a more explicit quantitative analysis of the effect of $e_i(x)$ (and $d_i(x)$, $r_i(x)$ as well) on the behavior of solutions, let us specify $e_i(x)$ as

$$e_i(x) = -2\frac{d}{dx}r_i(x) \qquad (i=1, 2).$$

In terms of ecology, the above equalities mean that the intrinsic growth rates $r_i(x)$ $(i=1, 2)$ coincide with the environmental potentials multiplied by (-2) (see Introduction); in other words, the growth rates are higher wherever the environment is favorable to the species. In this case, we have

$$R_1 = \frac{5-3e^{-0.4}}{10(1-e^{-0.4})} \approx 0.907, \quad B_{11} = \frac{1+e^{-0.4}}{10(1-e^{-0.4})} \approx 0.507,$$

$$B_{12} = B_{21} = \frac{1-e^{1.6}}{4(1-e^{-0.4})(1-e^2)} \approx 0.469,$$

$$B_2 = \frac{3e^2-1}{2(e^2-1)} \approx 1.66, \quad B_{22} = \frac{e^2+1}{2(e^2-1)} \approx 0.657.$$

Putting

$$c_1 = \frac{2(5-3e^{-0.4})(e^4-1)}{5(3e^2-1)(e^{1.6}-1)} \approx 0.766, \quad c_2 = \frac{2(1-e^{-0.8})(3e^2-1)}{(e^{-1.6}-1)(5-3e^{-0.4})} \approx 1.973,$$

we find that

i) If $c<c_1$,

$$(\tilde{n}_1, \tilde{n}_2) = \frac{(1-e^{-0.4})(e^2-1)}{4(1-e^{-0.8})(e^4-1)-5(e^{1.6}-1)^2 c} \times$$

$$\left(4(5-3e^{-0.4})(e^2+1) - \frac{10(3e^2-1)\,(e^{1.6}-1)c}{e^2-1},\right.$$

$$\left.4(3e^2-1)(1+e^{-0.4}) - \frac{2(5-3e^{-0.4})(e^{1.6}-1)c}{1-e^{-0.4}}\right)$$

$$\approx \frac{1}{0.333-0.22c^2}(0.595-0.777c,\ 0.839-0.425c)$$

is the only one stable equilibrium point;

ii) if $c_1<c<c_2$, $(\tilde{n}_1, \tilde{n}_2)=\left(0, \frac{3e^2-1}{e^2+1}\right) \approx (0, 2.527)$ is the only one stable equilibrium point;

iii) if $c_2<c$, $(\tilde{n}_1, \tilde{n}_2)=\left(0, \frac{3e^2-1}{e^2+1}\right)$ and $\left(\frac{5-3e^{-0.4}}{1+e^{-0.4}}, 0\right) \approx (1.789, 0)$ are both stable equilibrium points.

Theorem 1 indicates that, if the solution of (3.6) converges to the asymptotically stable equilibrium point $(\tilde{n}_1, \tilde{n}_2)$ as $\tau \to +\infty$, then the original solution $(u_1(t, x; \varepsilon), u_2(t, x; \varepsilon))$ eventually enters an ε-neighborhood of $(\tilde{n}_1\phi_0^1(x), \tilde{n}_2\phi_0^2(x))$. (As a matter of fact, by using the result of Matano [5], it can also be proved that the solution $(u_1(t, x; \varepsilon), u_2(t, x; \varepsilon))$ converges to an equilibrium solution near $(\tilde{n}_1\phi_0^1(x), \tilde{n}_2\phi_0^2(x))$ as $t \to +\infty$; see the last paragragh in Example 2.) This, together with the above observation (i), implies that both species can coexist if $c<c_1$. It would be of particular interest to consider the case where $3/5<c<c_1$. In this case, as just mentioned above, both species can coexist; on the other hand, if we replace the present values of $e_i(x)$ $(i=1, 2)$ by 0, the previous observations show that the only stable equilibrium point of (3.5) is $(0, r_2)$, which implies that, in the equations (3.4), only the second component will survive. An ecological interpretation of the above observations is that the coexistence of the two species is possible if the environmental potentials $E_i(x)$ $(i=1, 2)$ (defined by $e_i(x)=\frac{d}{dx}E_i(x)$) are spatially inhomogeneous, while it is not if $E_i(x)$ $(i=1, 2)$ are homogeneous (Figure 1). Note that, as in Example 1, the quantities $n_1(\varepsilon t)$ and $n_2(\varepsilon t)$ approximate the total volumes of $u_1(t, x; \varepsilon)$, $u_2(t, x; \varepsilon)$ respectively by order ε.

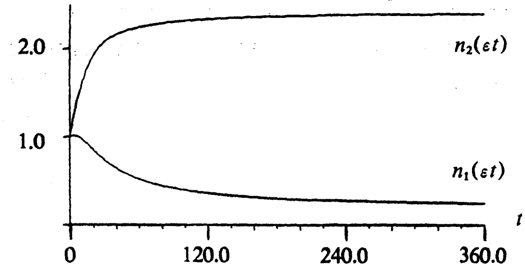

Figure 1–a: Evolution behavior of the solution of (3.6) with $\varepsilon=0.1$, $c=0.7$ $(3/5<c<c_1)$.

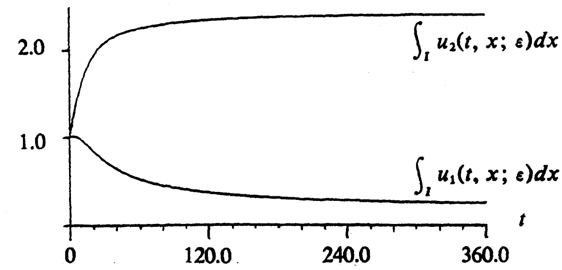

Figure 1–b: Evolution behavior of the total volume $\int_I u_i(t, x; \varepsilon)dx$ $(i=1, 2)$ of (3.4) with $\varepsilon=0.1$, $c=0.7$ $(3/5<c<c_1)$.

We continue the analysis of (3.4) for other values of c. Fix c arbitrarily in the interval $(c_1<c<c_2)$; then

$$\lim_{\tau\to\infty} Y(\tau) = \left(0, \frac{3e^2-1}{e^2+1}\phi_0^2\right) \equiv \bar{Y}.$$

By simple calculations, we see that $QF(\bar{Y})=0$ and $\det(QF'(\bar{Y})|_{B_1})\neq 0$. Thus, Theorem 3 implies the existence of a unique equilibrium solution $W(\varepsilon)=(w_1(\varepsilon), w_2(\varepsilon))$ of (3.4) with $W(0)=\bar{Y}$. We claim that $w_1(\varepsilon)=0$. To see this, let us consider (3.4) with $u_1\equiv 0$; namely,

$$\frac{\partial}{\partial t}u_2 + \frac{\partial}{\partial x}J_2 = \varepsilon\{r_2(x)-u_2\}u_2. \tag{3.7}$$

This type of equation was already discussed in Example 1. It is not difficult to see that, for sufficiently small ε, there exists an equilibrium solution $\tilde{w}_2(\varepsilon)$ of (3.7) with $(0, \tilde{w}_2(0))=\bar{Y}$. Thus, $(0, \tilde{w}_2(\varepsilon))$ is also an equilibrium solution of (3.4). From the uniqueness of equilibrium solutions of (3.4) in a neighborhood of $\bar{Y}$, it follows that the solution $W(\varepsilon)$ coincides with $(0, \tilde{w}_2(\varepsilon))$, proving our claim. Theorem 1 asserts that if the solution of (3.6) approaches the equilibrium point $(0, \bar{n}_2)$ as $\tau\to+\infty$ then the original solution $(u_1(t, x; \varepsilon), u_2(t, x; \varepsilon))$ eventually enters an ε-neighborhood of $\bar{Y}$. As a matter of fact, as mentioned before, we can also show that $(u_1(t, x; \varepsilon), u_2(t, x; \varepsilon))$ actually converges to the equilibrium solution $W(\varepsilon)$ as $t \to +\infty$. This can be shown as follows: The system (3.4) is of competition type, hence it is strongly order-preserving in the sense of Matano [5]. In such a system, an isolated equilibrium solution has to be either asymptotically stable or unstable (see [5; Theorem 7]); and an unstable equilibrium solution always has a non-empty unstable manifold that connects the equilibrium to another equilibrium (or, possibly, ∞) (see [5; Theorem 5 and Lemma 5.10]). As regards our present system (3.4), $W(\varepsilon)$ is contained in a positively invariant ε-neighborhood of $\bar{Y}$, denoted by V_ε, and is the unique equilibrium solution in this neighborhood. Combining the observations above, we easily find that $W(\varepsilon)$ is asymptotically stable. Moreover, carefully reading the proof of Theorem 7 of [5] (or Hirsch's "almost quasi-convergence theorem" [3] as well) shows that any solution of (3.4) that enters the interior of the neighborhood V_ε converges to the equilibrium solution $W(\varepsilon)$ as $t \to +\infty$. This proves our claim. In terms of ecology, this means that u_1 becomes extinct while u_2 will survive (Figure 2).

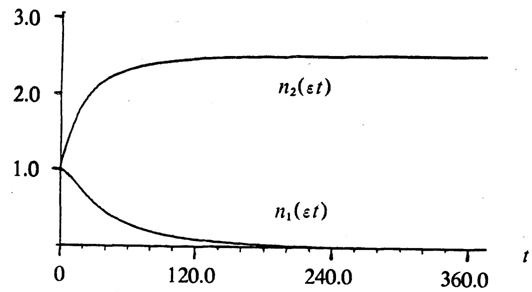

Figure 2–a: Evolution behavior of the solution of (3.6) with $\varepsilon=0.1$, $c=0.9$ $(c_1<c<c_2)$.

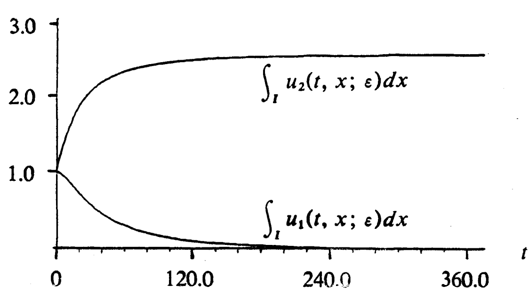

Figure 2–b: Evolution behavior of the total volume $\int_I u_i(t, x; \varepsilon)dx$ $(i=1, 2)$ of (3.4) with $\varepsilon=0.1$, $c=0.9$ $(c_1<c<c_2)$.

The other case is similarly analyzed, so we omit it.

Finally we give a brief consideration to models in an M-dimensional space, where $M \geqq 2$. The main results in Section 2 can directly applied to an M-dimensional version of (1.2), and the habitat I is replaced by a bounded region Ω in $\mathbf{R}^M$ with the smooth boundary $\partial\Omega$. For example, the equation of a single species model which we consider is the following:

$$
(3.8)\qquad \begin{cases} \dfrac{\partial}{\partial t}u + \operatorname{div} J(x, u) = \varepsilon f(x, u), \quad x \in \Omega, \\ u(0, x) = u_0(x), \\ J(x, u) = -\, d(x)\nabla u - u \cdot e(x), \\ \langle J(x, u), \nu \rangle = 0, \quad x \in \partial\Omega, \end{cases}
$$

where $d(x) > 0$ in $\bar{\Omega}$, $\langle\ ,\ \rangle$ is the Euclidean inner product and ν is an outward normal vector on $\partial\Omega$, $\nabla u = \operatorname{grad} u$ and $e(x) = {}^t(e_1(x), e_2(x), \dots, e_M(x))$. If each coefficient in (3.8) is sufficiently smooth and there exists a function $U(x)$ such that $e(x)/d(x) = \nabla U(x)$, then (3.8) can be treated similarly to (1.2) and all the calculations given at the beginning of this section are valid. We take $C(\Omega)$ with sup-norm as the space B and, as the domain of $A = \operatorname{div} J(x, \cdot)$, $D(A) = \{u \in W^{2,p}(\Omega) \mid u \in C(\Omega),\ Au \in C(\Omega),\ p > M,\ \langle J(x, u), \nu \rangle = 0 \text{ on } \partial\Omega\}$. To see that the conditions 1)–5) in Section 1 are satisfied, use ,for instance, Theorems 1 and 2 of Stewart [9] and the fact that

$$
\int_\Omega Au \cdot \bar{v} \cdot \exp(U(x))dx = \int_\Omega \langle \nabla u + u\nabla U(x), \overline{\nabla v + v\nabla U(x)} \rangle \cdot d(x) \cdot \exp(U(x))dx
$$

for u, $v \in D(A)$; we omit the details.

4. Proofs

PROOF OF THEOREM 1.

Throughout this section, we simply write $u(t; u_0, \varepsilon)$ as $u(t; \varepsilon)$ or u, $v(\tau; u_0)$ as $v(\tau)$ or v, and $\|\cdot\|_B$ as $\|\cdot\|$. Also, M, M_i, C, C_i and β, β_i $(i = 1, 2, \dots)$ mean positive constants independent of ε. Here M_1, M_2 are numbers given in conditions 2) and 4) in Section 1, respectively.

Transforming (1.1) by $w(t, \varepsilon) = u(t, \varepsilon) - e^{-tA}v(\varepsilon t)$, we have

$$
(4.1)\qquad \begin{cases} \dfrac{dw}{dt} + Aw = \varepsilon\{F(w + e^{-tA}v(\varepsilon t)) - QF(Qv)\} \quad t > 0, \\ w(0) = 0, \end{cases}
$$

which is written as

$$\begin{cases} \dfrac{dw}{dt} + \{A - \varepsilon F'(e^{-tA}v(\varepsilon t))\}w = \varepsilon\{F(e^{-tA}v(\varepsilon t)) - QF(Qv) + N(t, w; \varepsilon)\} \\ w(0) = 0, \end{cases}$$

where

$$N(t, w, \varepsilon) = F(w + e^{-tA}v(\varepsilon t)) - F(e^{-tA}v(\varepsilon t)) - F'(e^{-tA}v(\varepsilon t))w. \tag{4.2}$$

Denote by $X(t, \tau; \varepsilon)$ the solution of the operator equation

$$\begin{cases} \dfrac{dX}{dt} + \{(A - \varepsilon F'(e^{-tA}v(\varepsilon t))\}X = 0, \quad t > \tau, \\ X(\tau, \tau; \varepsilon) = I, \end{cases} \tag{4.3}$$

where I is the identity on B. Then (4.1) is reduced to

$$w(t, \varepsilon) = \varepsilon \int_0^t X(t, \tau; \varepsilon)\{F(e^{-\tau A}v(\varepsilon\tau)) - QF(Qv(\varepsilon\tau)) + N(\tau, w(\tau, \varepsilon); \varepsilon)\}d\tau. \tag{4.4}$$

Let us show that (4.4) has a solution for small ε. To do so, we prepare some lemmas. First rewrite (4.3) as

$$\begin{cases} \dfrac{dX}{dt} + A_\varepsilon X = B_\varepsilon(t)X, \\ X(\tau, \tau; \varepsilon) = I, \end{cases} \tag{4.5}$$

where

$$\begin{cases} A_\varepsilon = A - \varepsilon F'(\xi), \\ B_\varepsilon(t) = \varepsilon\{F'(e^{-tA}v(\varepsilon t)) - F'(\xi)\}. \end{cases} \tag{4.6}$$

Lemma 4.1. *There exist M_4, β and ε_0 such that*

$$\|e^{-tA_\varepsilon}\| \leqq M_4 e^{-\beta\varepsilon t}, \quad t > 0 \qquad \text{for} \quad \varepsilon \in (0, \varepsilon_0].$$

Proof. For $\beta > 0$ and θ $(0 < \theta < \pi/2)$, define a sector S_ε by

$$S_\varepsilon = \{\lambda \in \mathbf{C} \mid |\arg(\lambda - \beta\varepsilon)| > \theta,\ \lambda \neq \beta\varepsilon\}. \tag{4.7}$$

As will be shown in Appendix (Lemma 5.1), there exist β, θ, ε_0, C such that for any $\varepsilon \in (0, \varepsilon_0]$

$$S_\varepsilon \subset \rho(A_\varepsilon)$$

and

$$\|(\lambda - A_\varepsilon)^{-1}\| \leqq \frac{C}{|\lambda - \beta\varepsilon|} \qquad \text{for all} \quad \lambda \in S_\varepsilon,$$

where $\rho(A_\varepsilon)$ is the resolvent set of A_ε. Then, taking $J_\varepsilon = A_\varepsilon - \beta\varepsilon$, we easily find that

$$S_0 \subset \rho(J_\varepsilon)$$

and

$$\|(\lambda - J_\varepsilon)^{-1}\| < \frac{C}{|\lambda|} \qquad \text{for all} \quad \lambda \in S_0. \tag{4.8}$$

Denoting by Γ a contour in S_0 with $\arg\lambda \to \pm\theta$ as $|\lambda| \to \infty$, we see that

$$e^{-tJ_\varepsilon} = \frac{1}{2\pi i} \int_\Gamma e^{-\lambda t} (\lambda - J_\varepsilon)^{-1} d\lambda,$$

so that, by (4.8),

$$\sup_{t \geqq 0} e^{\beta\varepsilon t} \|e^{-tA_\varepsilon}\| = \|e^{-tJ_\varepsilon}\| < +\infty,$$

as required. ■

LEMMA 4.2. *There exists M_5 such that*

$$\int_0^\infty \|B_\varepsilon(t)\| dt \leqq M_5$$

PROOF. It follows that

$$\|B_\varepsilon(t)\| \leqq \varepsilon \int_0^1 \|F''(\theta\, e^{-tA} v(\varepsilon t) + (1-\theta)\xi)\| d\theta \cdot \| e^{-tA} v(\varepsilon t) - \xi\| . \tag{4.9}$$

Since $\{e^{-tA} v(\varepsilon t)\}_{t \geqq 0, \varepsilon > 0}$ is a bounded set in B, (4.9) reduces to

$$\|B_\varepsilon(t)\| \leqq \varepsilon C_1 \| e^{-tA} v(\varepsilon t) - \xi\| \leqq \varepsilon C_2 \{\|Qv(\varepsilon t) - \xi\|_1 + \|e^{-tA} P u_0\|\} \tag{4.10}$$

for some C_1 and C_2, where $\|\cdot\|_1$ means the norm on B_1. By the assumptions of Theorem 1

$$\|Qv(\tau) - \xi\|_1 \leqq C_3 e^{-\beta\tau}, \tag{4.11}$$

where β is the number given in Lemma 4.1, holds for some C_3. From (4.10), (4.11) and condition 4) in Section 1, it follows that

$$\|B_\varepsilon(t)\| \leqq \varepsilon C_4 \{e^{-\beta\varepsilon t} + e^{-\lambda_1 t}\}, \tag{4.12}$$

for some C_4. The described estimate is obtained by integrating (4.12) over $[0, \infty)$. ■

Using Lemmas 4.1 and 4.2, we can show

LEMMA 4.3. *There exists M_6 such that*

$$\|X(t, \tau; \varepsilon)\| \leqq M_6 e^{-\beta\varepsilon(t-\tau)},$$

where β is the one in Lemma 4.1.

PROOF. Since (4.5) is written as

$$X(t, \tau; \varepsilon) = e^{-(t-\tau)A_\varepsilon} + \int_\tau^t e^{-(t-s)A_\varepsilon} B_\varepsilon(s) X(s, \tau; \varepsilon) ds,$$

it follows that

$$\|X(t, \tau; \varepsilon)\| \leqq C_5\left\{e^{-\beta\varepsilon(t-\tau)} + \int_\tau^t e^{-\beta\varepsilon(t-s)} \|B_\varepsilon(s)\| \cdot \|X(s, \tau; \varepsilon)\| ds\right\} \tag{4.13}$$

for some C_5. Applying Gronwall's inequality to (4.13) and then using Lemma 4.2, we obtain

$$e^{\beta\varepsilon(t-\tau)}\|X(t, \tau; \varepsilon)\| \leqq C_5 \exp\left(C_5 \int_\tau^t \|B_\varepsilon(s)\| ds\right) \leqq C_5 \exp\left(C_5 \int_0^\infty \|B_\varepsilon(s)\| ds\right) \leqq C_6,$$

as required. ∎

Rewrite (4.4) as

$$w(t, \varepsilon) = H_\varepsilon(w)(t), \tag{4.14}$$

where $H_\varepsilon(w)(t) = \varepsilon U(t, \varepsilon) + \varepsilon \int_0^t X(t, \tau; \varepsilon) N(\tau, w(\tau, \varepsilon); \varepsilon) d\tau$, and

$$U(t, s) = \int_0^t X(t, \tau; \varepsilon)\{F(e^{-\tau A} v(\varepsilon\tau)) - QF(Qv(\varepsilon\tau))\} d\tau. \tag{4.15}$$

It suffices to show that (4.14) has a unique solution $w(t, \varepsilon)$ such that

$$\|w(t, \varepsilon)\| \leqq O(\varepsilon) \text{ uniformly for } t \in [0, \infty).$$

LEMMA 4.4. *There exists M_7 such that*

$$\|U(t, \varepsilon)\| \leqq M_7.$$

Moreover, if ξ satisfies $F(\xi) = 0$, then there exist M_8 and β_1 such that

$$\|U(t, \varepsilon)\| \leqq M_8 e^{-\beta_1 \varepsilon t}.$$

PROOF. It follows from (4.14) that

$$\|U(t, \varepsilon\| \leqq \| \int_0^t X(t, \tau; \varepsilon)PF(e^{-\tau A} v(\varepsilon\tau))d\tau \|$$

$$+ \int_0^t \|X(t, \tau; \varepsilon)\| \cdot \|QF(e^{-\tau A} v(\varepsilon\tau)) - QF(Qv(\varepsilon\tau))\|_1 d\tau$$

$$\equiv K_1 + K_2.$$

First, note that

$$K_2 \leqq M_6 \int_0^t e^{-\beta\varepsilon(t-\tau)} \|F(e^{-\tau A} v(\varepsilon\tau)) - F(Qv(\varepsilon\tau))\| \cdot \|Q\| d\tau$$

$$\leqq M_6 \int_0^t e^{-\beta\varepsilon(t-\tau)} \int_0^1 \|F'(\theta\, e^{-\tau A} v(\varepsilon\tau) + (1-\theta)Qv(\varepsilon\tau))\| \cdot \|Q\| d\theta$$

$$\times \|e^{-\tau A} Pv(\varepsilon\tau)\| d\tau$$

$$\leqq C_7 \int_0^t e^{-\beta\varepsilon(t-\tau)} e^{-\lambda_1\tau} d\tau$$

for some C_7. Then we find

$$K_2 \leqq C_8\, e^{-\beta_2\varepsilon t}$$

for some C_8 and β_2. We next estimate K_1 as follows:

$$K_1 \leqq \int_0^t \|X(t, \tau; \varepsilon)P\{F(e^{-\tau A} v(\varepsilon\tau)) - F(\xi)\}\| d\tau$$

$$+ \left\| \int_0^t X(t, \tau; \varepsilon)PF(\xi)d\tau \right\|$$

$$\equiv K_{11} + K_{12}.$$

It follows that

$$K_{11} \leqq \int_0^t \|X(t, \tau; \varepsilon)P\| \cdot \int_0^1 \|F'(\theta e^{-\tau A} v(\varepsilon\tau) + (1-\theta)\xi)\| d\theta$$

$$\times \| e^{-\tau A} v(\varepsilon\tau) - \xi \| d\tau$$

$$\leqq C_9 \int_0^t e^{-\beta\varepsilon(t-\tau)} \{\| e^{-\tau A} Pv(\varepsilon\tau)\| + \|Qv(\varepsilon\tau) - \xi\|\} d\tau$$

$$\leqq C_{10} \int_0^t e^{-\beta\varepsilon(t-\tau)} (e^{-\lambda_1\tau} + e^{-\beta\varepsilon\tau}) d\tau$$

$$\leqq C_{11}\, e^{-\beta_3\varepsilon t}$$

for some C_9, C_{10}, C_{11} and β_3. Using the estimates on K_{11} and K_2, we have

$$\|U(t, \varepsilon)\| \leqq C_{12}\, e^{-\beta_4\varepsilon t} + \| \int_0^t X(t, \tau; \varepsilon)PF(\xi)d\tau \| \tag{4.16}$$

for some C_{12} and β_4. Thus, if ξ satisfies $PF(\xi)=0$, then we obtain the second assertion in Lemma 4.4. We next consider the case when $PF(\xi)\neq 0$. Define $z(t)$ by

$$z(t) = \int_0^t X(t, \tau; \varepsilon)PF(\xi)d\tau,$$

which is a solution of

$$\begin{cases} \dfrac{dz}{dt} + (A-\varepsilon F'(e^{-tA}v(\varepsilon t))\}z = PF(\xi), \\ z(0) = 0, \end{cases}$$

i.e.

$$\begin{cases} \dfrac{dz}{dt} + A_\varepsilon z = PF(\xi) + B_\varepsilon(t)z, \\ z(0) = 0. \end{cases} \tag{4.17}$$

(4.17) is equivalent to

$$z(t) = \int_0^t e^{-(t-s)A_\varepsilon}\{PF(\xi)+B_\varepsilon(s)z(s)\}ds. \tag{4.18}$$

Suppose that

$$\left\| \int_0^t e^{-(t-s)A_\varepsilon} PF(\xi)ds \right\| < +\infty,$$

which will be proved in Appendix. Then from (4.18) we can have

$$\|z(t)\| \leqq C_{13}(1+\int_0^t \|B_\varepsilon(s)\|\cdot\|z(s)\|ds) \tag{4.19}$$

for some C_{13}. Applying Gronwall's inequality to (4.19), we see

$$\|z(t)\| \leqq C_{13}\exp(C_{13}\int_0^\infty \|B_\varepsilon(s)\|ds) \leqq C_{14}. \tag{4.20}$$

Therefore, it follows from (4.16) that

$$\|U(t, \varepsilon)\| \leqq M_7$$

for some M_7. The proof is complete. ∎

We consider (4.4). Let $C([0, \infty); B)$ be the Banach space of all bounded continuous functions from $[0, \infty)$ into B with the norm $|||w||| = \sup_{t\geqq 0}\|w(t)\|$, and for any fixed r, let

$$V_r = \{w | w \in C([0, \infty); B),\ |||w||| \leqq r\}.$$

Suppose $w \in V_r$. Then it follows from (4.14) that

$$\begin{aligned} \|H_\varepsilon(w)(t)\| &\leqq \varepsilon M_9(1+\int_0^t e^{-\beta\varepsilon(t-\tau)} \|N(\tau, w(\tau); \varepsilon)\| d\tau) \\ &= \varepsilon M_9(1+\int_0^t e^{-\beta(t-\tau)} \| \int_0^1 F'(\theta w(\tau)+e^{-\tau A} v(\varepsilon\tau))d\theta w(\tau) \\ &\quad - F'(e^{-\tau A} v(\varepsilon\tau))w(\tau)\| d\tau) \\ &\leqq \varepsilon M_{10}(1+\int_0^t e^{-\beta(t-\tau)} \|w(\tau)\|^2 d\tau) \\ &\leqq \varepsilon M_{10}(1+\int_0^t e^{-\beta\varepsilon(t-\tau)} d\tau |||w|||^2) \end{aligned}$$

for some M_9 and M_{10}. Thus, we have

$$|||H_\varepsilon(w)||| \leqq M(\varepsilon + |||w|||^2)$$

for some M. Put $\varepsilon_0 = \min\{1/(4M^2), r/(2M)\}$. Then it turns out for any $\varepsilon \in (0, \varepsilon_0]$, $V_{2M\varepsilon} \subset V_r$ and H_ε maps $V_{2M\varepsilon}$ into $V_{2M\varepsilon}$, because it follows that

$$|||H_\varepsilon(w)||| \leqq M(\varepsilon + |||w|||^2) \leqq M(\varepsilon + 4M^2\varepsilon^2) \leqq 2M\varepsilon$$

for all $w \in V_{2M\varepsilon}$. Moreover, there exists M_{11} such that

$$|||H_\varepsilon(w_1) - H_\varepsilon(w_2)||| \leqq (M_{11}\varepsilon/\beta) \cdot |||w_1 - w_2|||$$

for any $w_1, w_2 \in V_{2M\varepsilon}$. Consequently H_ε is a contraction on $V_{2M\varepsilon}$ for any $0<\varepsilon<\min\{\varepsilon_0, \beta/M_{11}\}$. Thus, there exists a unique fixed point w in $V_{2M\varepsilon}$, and $|||w||| \leqq 2M\varepsilon$. The proof of Theorem 1 is complete. ■

PROOF OF COROLLARY TO THEOREM 1.

This can be shown in the same way as Theorem 1, if we replace $C([0, \infty); B)$ by the space of continuous functions w: $[0, \infty) \to B$ such that

$$|||w||| \equiv \sup_{t \geqq 0} \|e^{\beta_1 \varepsilon t} w(t)\| < \infty,$$

and use the second sasertion of Lemma 4.4. So we omit the details. ■

PROOF OF THEOREM 2.

Let $T>0$ be the number given in the assumption of Theorem 2. Consider the equation (4.14):

$$w(t) = H_\varepsilon(w)(t) = \varepsilon U(t, \varepsilon) + \varepsilon \int_0^t X(t, \tau; \varepsilon) N(\tau, w(\tau, \varepsilon); \varepsilon) d\tau$$

on the space $C([0, T/\varepsilon]; B)$ with the norm

$$|||w|||_{\varepsilon,T} = \sup_{0\leqq t\leqq T/\varepsilon} \|w(t)\|.$$

Let $V_r^T = \{w \in C([0, T/\varepsilon]; B) \,\big|\, |||w|||_{\varepsilon,T} \leqq r\}$. To prove Theorem 2, it suffices to show the existence of $\varepsilon_T > 0$ and $M_T > 0$ depending only on T such that H_ε $(0 < \varepsilon \leqq \varepsilon_T)$ is a contraction mapping on

$$V^T_{M_T\varepsilon} = \{w \,\big|\, w \in C([0, T/\varepsilon]; B),\ |||w|||_{\varepsilon,T} \leqq M_T\varepsilon\}.$$

Here we fix $T > 0$ arbitrarily and denote various constants depending only on T by C_i^T, M_i^T, ε_i^T $(i = 1, 2, \ldots)$.

Lemma 4.5. *Let $X(t, \tau; \varepsilon)$ be the solution of the equation* (4.3). *Then there exists $C_1^T > 0$ such that*

$$\|X(t, \tau; \varepsilon)\| \leqq C_1^T \qquad \textit{for} \quad t, \tau \in [0, T/\varepsilon].$$

Proof. From (4.3), we have

$$X(t, \tau; \varepsilon) = e^{-(t-\tau)A} + \varepsilon\int_0^t e^{-(t-s)A} F'(e^{-sA} v(\varepsilon s))X(s, \tau; \varepsilon)ds. \tag{4.21}$$

Since $\{e^{-tA} v(\varepsilon t)\}_{t\in[0,T/\varepsilon],\varepsilon>0}$ is a bounded set in B,

$$\|F'(e^{-sA} v(\varepsilon s))\| \leqq C_2^T \qquad \text{for} \quad s \in [0, T/\varepsilon].$$

So (4.21) gives

$$\|X(t; \tau; \varepsilon)\| \leqq M_1 + \varepsilon\int_\tau^t M_1 C_2^T \|X(s, \tau; \varepsilon)\| ds.$$

Applying Gronwall's inequality, we get the result. ■

Lemma 4.6. *There exists $C_3^T > 0$ such that*

$$\|X(t, \tau; \varepsilon)P\| \leqq M_2 e^{-\lambda_1(t-\tau)} + \varepsilon C_3^T \qquad \textit{for} \quad t, \tau \in [0, T/\varepsilon],$$

where λ_1 is the number given in 4) *of Section* 1.

Proof. From (4.21), it follows that

$$PX(t, \tau; \varepsilon)P = e^{-(t-\tau)A} P + \varepsilon\int_\tau^t P e^{-(t-s)A} F'(e^{-sA} v(\varepsilon s))X(s, \tau; \varepsilon)Pds,$$

so that by Lemma 4.5

$$\begin{aligned} \|PX(t, \tau; \varepsilon)P\| &\leqq M_2 e^{-\lambda_1(t-\tau)} + \varepsilon\int_\tau^t M_2 e^{-\lambda_1(t-s)} C_2^T \|X(s, \tau; \varepsilon)P\| ds \\ &\leqq M_2 e^{-\lambda_1(t-\tau)} + \varepsilon C_4^T. \end{aligned} \tag{4.22}$$

Since $Qe^{-tA}=Q$, (4.21) gives

$$QX(t,\tau;\varepsilon)P=\varepsilon\int_\tau^t QF'(e^{-sA}v(\varepsilon s))PX(s,\tau;\varepsilon)Pds$$
$$+\varepsilon\int_\tau^t QF'(e^{-sA}v(\varepsilon s))QX(s,\tau;\varepsilon)Pds.$$

Hence, it follows from (4.22) that

$$\|QX(t,\tau;\varepsilon)P\|\leqq\varepsilon C_5^T+\varepsilon C_5^T\int_\tau^t\|QX(s,\tau;\varepsilon)P\|ds,$$

so that, by Gronwall's inequality,

$$\|QX(t,\tau;\varepsilon)P\|\leqq\varepsilon C_5^T e^{\varepsilon C_5^T(t-\tau)}\leqq\varepsilon C_6^T.$$

Consequently

$$\|X(t,\tau;\varepsilon)P\|\leqq\|QX(t,\tau;\varepsilon)P\|+\|PX(t,\tau;\varepsilon)P\|\leqq M_2e^{-\lambda_1(t-\tau)}+\varepsilon C_3^T$$

for some $C_3^T>0$. ■

LEMMA 4.7. *There exists $C_7^T>0$ such that*

$$\|U(t,\varepsilon)\|\leqq C_7^T\qquad\text{for}\quad t\in[0,T/\varepsilon],$$

where $U(t,\varepsilon)$ is the function given in (4.15).

PROOF. By Lemma 4.5 and Lemma 4.6, we have

$$\begin{aligned}\|U(t,\varepsilon)\|&\leqq\left\|\int_0^t X(t,s;\varepsilon)PF(e^{-sA}v(\varepsilon s))ds\right\|\\&\quad+\left\|\int_0^t X(t,s;\varepsilon)Q\{F(e^{-sA}v(\varepsilon s))-F(Qv(\varepsilon s))\}ds\right\|\\&\leqq C_8^T\int_0^t\|X(t,s;\varepsilon)P\|ds\\&\quad+C_8^T\int_0^t\|X(t,s;\varepsilon)Q\|\cdot\|F(e^{-sA}v(\varepsilon s))-F(Qv(\varepsilon s))\|ds\\&\leqq C_9^T+C_9^T\int_0^t\int_0^1\|F'(\theta e^{-sA}v(\varepsilon s)+(1-\theta)Qv(\varepsilon s))\|d\theta\cdot\|Pe^{-sA}v(\varepsilon s)\|ds\\&\leqq C_9^T+C_{10}^T\int_0^t M_2e^{-\lambda_1 s}ds\leqq C_7^T\end{aligned}$$

for $t\in[0,T/\varepsilon]$. This shows the result. ■

We now consider the equation (4.4). For any fixed $r>0$ and for any $w\in V_r^T$, it follows from Lemma 4.7 that

$$\begin{aligned}\|H_\varepsilon(w)(t)\| &\leqq \varepsilon C_7^T + \varepsilon\int_0^t C_1^T\|N(\tau, w(\tau); \varepsilon)\|d\tau \\ &\leqq \varepsilon C_7^T + \varepsilon C_1^T\int_0^t\int_0^1 \|F'(e^{-\tau A}v(\varepsilon\tau)+\theta w(\tau))-F'(e^{-\tau A}v(\varepsilon\tau))\|d\theta\cdot\|w(\tau)\|d\tau \\ &\leqq \varepsilon C_{11}^T(1+\int_0^t \|w(\tau)\|^2 d\tau) \\ &\leqq C_{12}^T(\varepsilon+|||w|||_{\varepsilon,T}^2)\end{aligned}$$

for $t\in[0, T/\varepsilon]$. Hence as is seen in the proof of Theorem 1, we can find constants $\varepsilon_1^T>0$ and $M_1^T>0$ such that for all $\varepsilon\in(0, \varepsilon_1^T]$, H_ε maps $V_{\varepsilon M_1^T}^T$ into itself. Moreover, there exists $M_2^T>0$ such that

$$|||H_\varepsilon(w_1)-H_\varepsilon(w_2)|||_{\varepsilon,T} \leqq \varepsilon M_2^T|||w_1-w_2|||_{\varepsilon,T}$$

for any $w_1, w_2\in V_{\varepsilon M_1^T}^T$. Consequently H_ε is a contraction on $V_{\varepsilon M_1^T}^T$ for $0<\varepsilon<\min\{\varepsilon_1^T, 1/_{M_2^T}\}$. The proof is complete. ■

PROOF OF THEOREM 3.

Decompose $w\in B$ into $w=w_1+w_2$ with $w_1\in QB=B_1=\mathrm{Ker}\,A$ and $w_2\in PB\equiv B_2$. Then the equations $Aw-\varepsilon F(w)=0$, $w(0)=\xi$ reduce to

$$QF(w_1+w_2) = 0, \tag{4.23}$$

$$A_2w_2 - \varepsilon PF(w_1+w_2) = 0, \tag{4.24}$$

where $A_2=A|_{B_2}$. Put

$$G_2(w_2; \varepsilon, w_1) = A_2w_2 - \varepsilon PF(w_1+w_2);$$

then $G_2\in C^2(D\times\mathbf{R}^1\times B_1; B_2)$ with $D=D(A)\cap B_2$. Since $G_2(0; 0, w_1)=0$, $\frac{\partial}{\partial w}G_2(0; 0, w_1)=A_2$ and A_2 is invertible, the standard implicit function theorem implies that there uniquely exists $w_2(\varepsilon, v)\in C^2(U(w_1); B_2)$ such that $w_2(0, w_1)=0$ and $G_2(w_2(\varepsilon, v); \varepsilon, v)=0$. Here $U(w_1)=\{(\varepsilon, v)\,|\,|\varepsilon|<\varepsilon_0,\ v\in B(w_1, \delta)$ for some ε_0 and δ depending on $w_1\}$, and $B(w_1, \delta)$ is the open ball in B_1 with radius δ centered at w_1. Substituting $w_2=w_2(\varepsilon, w_1)$ into (4.23), we have

$$\begin{cases} QF(w_1+w_2(\varepsilon, w_1)) = 0 \\ w_1(0) = \xi, \end{cases} \tag{4.25}$$

where ξ is the value satisfying $QF(\xi)=0$ and $\det(QF'(\xi)|_{B_1})\neq 0$. Define $G_1\in C^2(U(\xi); B_1)$ by $G_1(w_1, \varepsilon)=QF(w_1+w_2(\varepsilon, w_1))$. Then we have

$$G_1(\xi, 0) = 0, \quad \frac{\partial}{\partial w_1}G_1(\xi, 0) = QF'(\xi)\Big(I_1 + \frac{\partial}{\partial w_1}w_2(0, \xi)\Big), \tag{4.26}$$

where I_1 is the identity on B_1. Here we define G by

$$G(w_1, \varepsilon) = G_2(w_2(\varepsilon, w_1); \varepsilon, w_1) = A_2 w_2(\varepsilon, w_1) - \varepsilon PF(w_1 + w_2(\varepsilon, w_1)).$$

Since $G(w_1, \varepsilon) = 0$ on $U(\xi)$ and

$$\begin{aligned} 0 &= \frac{\partial}{\partial w_1} G(w_1, \varepsilon) \\ &= \frac{\partial}{\partial w_2} G_2(w_2(\varepsilon, w_1); \varepsilon, w_1) \frac{\partial}{\partial w_1} w_2(\varepsilon, w_1) \\ &\quad + \frac{\partial}{\partial w_1} G_2(w_2(\varepsilon, w_1); \varepsilon, w_1) I_1 \\ &= (A_2 - \varepsilon PF'(w_2(\varepsilon, w_1) + w_1)) \frac{\partial}{\partial w_1} w_2(\varepsilon, w_1) - \varepsilon PF'(w_1 + w_2(\varepsilon, w_1)), \end{aligned}$$

we find that $\frac{\partial}{\partial w_1} G(\xi, 0) = A_2 \frac{\partial}{\partial w_1} w_2(0, \xi) = 0$, which implies $\frac{\partial}{\partial w_1} w_2(0, \xi) = 0$ because $\frac{\partial}{\partial w_1} w_2(0, \xi)$ maps B_1 into B_2. Therefore (4.26) becomes $\frac{\partial}{\partial w_1} G_1(\xi, 0) = QF'(\xi)|_{B_1}$. Hence, by the implicit function theorem it follows from the assumption $\det(QF'(\xi)|_{B_1}) \neq 0$, that there exists a constant $\varepsilon_0 > 0$ such that (4.25) has a unique solution $w_1(\varepsilon)$ satisfying

$$w_1(\varepsilon) \in C^2((-\varepsilon_0, \varepsilon_0); B_1) \text{ and } w_1(0) = \xi.$$

We finally show that $w(\varepsilon) = w_1(\varepsilon) + w_2(\varepsilon, w_1(\varepsilon))$ is a unique solution of (2.2). Assume that the equilibrium solution of (2.2) is parameterized by $s \in H_0 = (-s_0, s_0)$ for some $s_0 > 0$ as follows:

$$\text{(4.27)} \qquad \begin{cases} A\tilde{w}(s) = \varepsilon(s) F(\tilde{w}(s)), \\ \varepsilon(0) = 0, \quad \tilde{w}(0) = \xi, \end{cases}$$

which is equivalent to

$$\text{(4.28)} \qquad \begin{cases} A_2 \tilde{w}_2(s) = \varepsilon(s) PF(\tilde{w}_1(s) + \tilde{w}_2(s)), \\ 0 = QF(\tilde{w}_1(s) + \tilde{w}_2(s)), \\ \varepsilon(0) = 0, \quad \tilde{w}_1(0) = \xi, \quad \tilde{w}_2(0) = 0, \end{cases}$$

where $\tilde{w}_1(s) = Q\tilde{w}(s)$, $\tilde{w}_2(s) = P\tilde{w}(s)$. First defining

$$\tilde{G}_2(w_2, s) = A_2 w_2 - \varepsilon(s) PF(\tilde{w}_1(s) + w_2),$$

we find that $\tilde{G}_2(0, 0) = 0$ and $\tilde{G}_2 \in C^2(D \times H_0; B_2)$. Since

$$\frac{\partial}{\partial w_2} \tilde{G}_2(w_2, s) = A_2 - \varepsilon(s) PF'(\tilde{w}_1(s) + w_2),$$

we have $\frac{\partial}{\partial w_2}\tilde{G}_2(0, 0)=A_2$. By the implicit function theorem, there exists a unique solution $\bar{w}_2(s)$ on H_1 such that $\tilde{G}_2(\bar{w}_2(s), s)=0$ and $\bar{w}_2(0)=0$, where $H_1=(-s_1, s_1)$ is an open interval containing 0. Define $\tilde{\bar{w}}_2(s)=w_2(\varepsilon(s), \tilde{w}_1(s))$. Then $\tilde{\bar{w}}_2(s)$ satisfies $\tilde{G}_2(\tilde{\bar{w}}_2(s), s)=0$ and $\tilde{\bar{w}}_2(0)=w_2(0, \xi)=0$. By the uniqueness, $\bar{w}_2(s)=\tilde{\bar{w}}_2(s)=\tilde{w}_2(s)$ on $H_2=(-s_2, s_2)\subset(-\varepsilon_0, \varepsilon_0)\cap H_1\cap H_0$.

Second, define

$$\tilde{G}_1(w_1, s) = QF(w_1+w_2(\varepsilon(s), w_1))\,;$$

then $\tilde{G}_1$ satisfies

$$\tilde{G}_1(\xi, 0) = QF(\xi+w_2(0, \xi)) = QF(\xi) = 0$$

and

$$\frac{\partial}{\partial w_1}\tilde{G}_1(w_1, s) = QF'(w_1+w_2(\varepsilon(s), w_1))\Big(I_1+\frac{\partial}{\partial w_1}w_2(\varepsilon(s), w_1)\Big).$$

So $\frac{\partial}{\partial w_1}\tilde{G}_1(\xi, 0)=QF'(\xi)\Big(I_1+\frac{\partial}{\partial w_1}w_2(0, \xi)\Big)=QF'(\xi)|_{B_1}$. By the assumption of Theorem 3, it turns out that there exists a unique function $\bar{w}_1(s)$ defined on an interval $H_3=(-s_3, s_3)$ such that $\tilde{G}_1(\bar{w}_1(s), s)=0$ and $\bar{w}_1(0)=\xi$. If we define $\tilde{\bar{w}}_1(s)=w_1(\varepsilon(s))$, then $\tilde{G}_1(\tilde{\bar{w}}_1(s), s)=0$ and $\tilde{\bar{w}}_1(0)=\xi$. So by the uniqueness, $\bar{w}_1(s)=\tilde{\bar{w}}_1(s)=\tilde{w}_1(s)$ on $H_4=(-\varepsilon_0, \varepsilon_0)\cap H_3\cap H_0$. Hence

$$\tilde{w}(s) = \tilde{w}_1(s) + \tilde{w}_2(s) = w_1(\varepsilon(s)) + w_2(\varepsilon(s), w_1(\varepsilon(s))) = w(\varepsilon(s)).$$

Thus, the proof is complete. ■

PROOF OF THEOREM 4.

Since $D(A)$ is dense in $X^{1/2}$ from the general theory, it suffices to show the following:

i) two norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_{1/2}$ are equivalent on $D(A)$, where $\|\cdot\|_{H^1}$ and $\|\cdot\|_{1/2}$ denote the norm on $H^1(I)$ with $I=(0, L)$and the norm on $X^{1/2}$ respectively,

ii) $D(A)$ is dense in $H^1(I)$.

First, we will show i). Here we write the norm on X by $\|\cdot\|$. Then

$$\begin{aligned}
\|u\|_{1/2}^2 &= \|A^{1/2}u\|^2 + \|u\|^2 \\
&= (Au, u) + \|u\|^2 \\
&= \int_I \left|u_x+\frac{e(x)}{d(x)}u\right|^2 d(x)k(x)dx+\|u\|^2 \\
&= \left\|d(x)^{1/2}u_x + \frac{e(x)}{d(x)^{1/2}}u\right\|^2 + \|u\|^2 \\
&\geqq \left|\,\|d(x)^{1/2}u_x\| - \left\|\frac{e(x)}{d(x)^{1/2}}u\right\|\,\right|^2 + \|u\|^2
\end{aligned} \tag{4.29}$$

for all $u \in D(A)$. From $(Au, u) \geqq 0$, $\|u\|_{1/2} \geqq \|u\|$. So we have

$$\|u\|_{1/2}^2 - \|u\|^2 \geqq \left| \|d(x)^{1/2}u_x\| - \left\| \frac{e(x)}{d(x)^{1/2}} u \right\| \right|^2$$

$$\|d(x)^{1/2}u_x\| \leqq (\|u\|_{1/2}^2 - \|u\|^2)^{1/2} + \left\| \frac{e(x)}{d(x)^{1/2}} u \right\|.$$

Since $\|u_x\| \leqq C_{14}\|d(x)^{1/2}u_x\|$ and $|e(x)/d(x)^{1/2}|$ is bounded, it follows that $\|u_x\| \leqq C_{15}\|u\|_{1/2}$ for some constants C_{14}, C_{15}. Hence $\|u\|_{H^1} \leqq C_{16}\|u\|_{1/2}$ for some constant C_{16} because the norm $\|\cdot\|$ and L^2-norm are equivalent. Conversely, it follows from (4.29) that

$$\begin{aligned} \|u\|_{1/2}^2 &\leqq \left| \|d(x)^{1/2}u_x\| + \left\| \frac{e(x)}{d(x)^{1/2}} u \right\| \right|^2 + \|u\|^2 \\ &\leqq C_{17}(\|u_x\|^2 + \|u\|^2) \\ &\leqq C_{18}\|u\|_{H^1}^2 \end{aligned}$$

for some constants C_{17}, C_{18}.

We next prove ii). Put $(\psi u)(x) = k(x) \cdot u(x)$ for $u \in H^1(I)$. Then it is easy to see that ψ is a homeomorphism on $H^1(I)$ and $\psi \cdot D(A) = \{u \in H^2(I) | u_x = 0$ on $0, L\}$. Therefore it suffices to show that the set $D = \{u \in C^\infty(I) | u_x = 0$ on $0, L\}$ is dense in $H^1(I)$. Since all $u \in H^1(I)$ is represented as $u(x) = \int_0^x u_x dx + u(0)$ with $u_x \in L^2(I)$, we find that

$$|u(x) - v(x)| \leqq (L \int_I |u_x - v_x|^2 dx)^{1/2} + |u(0) - v(0)|)$$

for all u, $v \in H^1(I)$.

Thus, $\int_I |u(x) - v(x)|^2 dx \leqq 2(L^2 \int_I |u_x - v_x|^2 dx + |u(0) - v(0)|^2)$ and

$$\|u - v\|_{H^1}^2 \leqq C_{19}(\int_I |u_x - v_x|^2 dx + |u(0) - v(0)|^2) \tag{4.30}$$

for some constant C_{19}. Since $C_0^\infty(I) = \{u \in C^\infty(I) | u(0) = u(L) = 0\}$ is dense in $L^2(I)$, there exists for $\varepsilon > 0$ and $u \in H^1(I)$, $w \in C_0^\infty(I)$ such that $\int_I |u_x - w|^2 dx < \varepsilon$. Defining $v(x)$ by

$$v(x) = \int_0^x w(x) dx + u(0),$$

we see by (4.30) that $v \in D$ and $\|u - v\|_{H^1}^2 \leqq C_{19}\varepsilon$. The proof is complete. ■

5. Appendix

Lemma 5.1. *There exist positive constants* β, $\theta(0<\theta<\pi/2)$, ε_0, C *such that*

(i) *for* $\varepsilon\in(0, \varepsilon_0]$, *the sector* $S_\varepsilon=\{\lambda\in\mathbf{C}\,\big|\,|\arg(\lambda-\beta\varepsilon)|>\theta,\ \lambda\neq\beta\varepsilon\}$ *is contained in* $\rho(A_\varepsilon)$,

(ii) $$\|(\lambda-A_\varepsilon)^{-1}\| \leqq \frac{C}{|\lambda-\beta\varepsilon|} \quad \textit{for all} \quad \lambda\in S_\varepsilon.$$

Here A_ε *is the operator defined in* (4.6).

Proof. From the assumption of Theorem 1, there exists a $\gamma>0$ such that

$$\mathrm{Re}\,\sigma(QL|_{B_1}) \leqq -\gamma,$$

where $L=F'(\xi)$. Since $QL|_{B_1}$ is a linear mapping on the finite dimensional space B_1, we can take a sector $S\subset\rho(-QL|_{B_1})$ so that

$$S = \{\lambda\in\mathbf{C}\,\big|\,|\arg(\lambda-\beta)|>\theta,\ \lambda\neq\beta\}$$

for some $0<\beta<\gamma$, $0<\theta<\frac{\pi}{2}$, and a constant C_{20} such that

$$\text{(5.1)}\qquad \begin{aligned} &\|(\lambda+QL|_{B_1})^{-1}\|_1 \leqq C_{20},\\ &\|(\lambda+QL|_{B_1})^{-1}\|_1 \leqq \frac{C_{20}}{|\lambda-\beta|} \end{aligned}$$

for all $\lambda\in S$. We now define S_ε by

$$S_\varepsilon = \varepsilon S = \{\varepsilon\lambda|\lambda\in S\}$$

and show that there is a number $\varepsilon_0>0$ such that if $\varepsilon\in(0, \varepsilon_0]$, then

$$S_\varepsilon \subset \rho(A_\varepsilon),$$

$$\|(\lambda-A_\varepsilon)^{-1}\| \leqq \frac{C}{|\lambda-\beta\varepsilon|}$$

for all $\lambda\in S_\varepsilon$. To do so, it suffices to prove that the equation

$$\text{(5.2)}\qquad (\lambda-A_\varepsilon)u = v$$

is solvable for all $v\in B$ and $\lambda\in S_\varepsilon$. Put $u_1=Qu$, $u_2=Pu$, $v_1=Qv$, and $v_2=Pv$. Then (5.2) is rewritten as

$$\text{(5.3)}\qquad \begin{cases} \lambda u_1 + \varepsilon QL(u_1+u_2) = v_1,\\ \lambda u_2 - Au_2 + \varepsilon PL(u_1+u_2) = v_2, \end{cases}$$

where $A_\varepsilon = A - \varepsilon L$. The first equation of (5.3) is written as

$$(\lambda + \varepsilon QL|_{B_1})u_1 = v_1 - \varepsilon QLu_2,$$

Since $\lambda \in \rho(-\varepsilon QL|_{B_1})$, this implies that

$$u_1 = (\lambda + \varepsilon QL|_{B_1})^{-1}(v_1 - \varepsilon QLu_2). \tag{5.4}$$

Substituting (5.4) into the second equation of (5.3), we obtain

$$(\lambda - A)u_2 = -\varepsilon PL\{L_\lambda(v_1 - \varepsilon QLu_2) + u_2\} + v_2, \tag{5.5}$$

where $L_\lambda = (\lambda + \varepsilon QL|_{B_1})^{-1}$. Denote PB by B_2 and $A|_{B_2}$ by A_2. We note that if ε is small enough, there exsts a constant C_{21} so that

$$\|(\lambda - A_2)^{-1}\|_2 \leqq C_{21} \qquad \text{for} \quad \lambda \in S_\varepsilon,$$

where $\|\cdot\|_2$ is the norm or the operator norm on B_2. Hence (5.5) is written as

$$u_2 = (\lambda - A_2)^{-1}[-\varepsilon PL\{L_\lambda(v_1 - \varepsilon QLu_2) + u_2\} + v_2],$$

that is,

$$\begin{aligned} J_{\varepsilon,\lambda}u_2 &\equiv \{I_2 - \varepsilon(\lambda - A_2)^{-1}PL(\varepsilon L_\lambda QL + I_2)\}u_2 \\ &= (\lambda - A_2)^{-1}(-\varepsilon PLL_\lambda v_1 + v_2), \end{aligned} \tag{5.6}$$

where I_2 is the identity on B_2. Since

$$\|\varepsilon L_\lambda\|_1 = \|(\lambda/\varepsilon + QL|_{B_1})^{-1}\| \leqq C_{20} \tag{5.7}$$

for all $\dfrac{\lambda}{\varepsilon} \in S$, we find that the inverse $J_{\varepsilon,\lambda}^{-1}$ is well defined so that

$$\|J_{\varepsilon,\lambda}^{-1}\| \leqq \frac{1}{1 - \varepsilon C_{22}} \leqq C_{23} \quad \text{for small } \varepsilon$$

for some C_{22}, C_{23}. So (5.6) reduces to

$$u_2 = J_{\varepsilon,\lambda}^{-1}(\lambda - A_2)^{-1}(-\varepsilon PLL_\lambda v_1 + v_2)$$

and therefore

$$\|u_2\|_2 \leqq C_{24}\|(\lambda - A_2)^{-1}\|_2(\|\varepsilon L_\lambda\| \cdot \|v_1\|_1 + \|v_2\|_2) \tag{5.8}$$

for some C_{24}. Since A_2 is sectorial and $\operatorname{Re}\sigma(A_2) > \alpha$ for some $\alpha > 0$, there exists some C_{25} such that if ε is small enough

$$\|(\lambda - A_2)^{-1}\|_2 \leqq \frac{C_{25}}{|\lambda - \alpha|}$$

for all $\lambda\in S_\varepsilon$. From this and (5.8), we have

$$\|u_2\|_2 \leqq \frac{C_{26}}{|\lambda-\alpha|}(\|v_1\|_1+\|v_2\|_2) \leqq \frac{C_{27}}{|\lambda-\alpha|}\|v\| \tag{5.9}$$

for some C_{26}, C_{27}. (5.9) and (5.4) imply that

$$\|u_1\|_1 \leqq \|L_\lambda\|_1\cdot\|v_1\|_1 + \|\varepsilon L_\lambda\|_1 \frac{C_{28}}{|\lambda-\alpha|}\|v\| \tag{5.10}$$

for some C_{28}. Since

$$\|L_\lambda\|_1 = \frac{1}{\varepsilon}\frac{C_{20}}{\left|\dfrac{\lambda}{\varepsilon}-\beta\right|} = \frac{C_{20}}{|\lambda-\varepsilon\beta|},$$

by (5.1), it follows that

$$\|u_1\|_1 \leqq C_{29}\left(\frac{1}{|\lambda-\beta\varepsilon|}+\frac{1}{|\lambda-\alpha|}\right)\|v\| \tag{5.11}$$

for some C_{29}. Thus, (5.9) and (5.11) yield

$$\begin{aligned}\|u\| &\leqq C_{30}\left(\frac{1}{|\lambda-\beta\varepsilon|}+\frac{1}{|\lambda-\alpha|}\right)\|v\| \\ &\leqq \frac{C_{31}}{|\lambda-\beta\varepsilon|}\|v\|\end{aligned}$$

for small ε. This proves Lemma 5.1. ■

LEMMA 5.2. *There exists some* $\varepsilon_0>0$ *and* $M_{12}>0$ *such that*

$$\left\|\int_0^t e^{-(t-s)A_\varepsilon}PF(\xi)ds\right\| \leqq M_{12}$$

for any $\varepsilon\in(0,\varepsilon_0]$. *Here* ξ *is one in assumptions of Theorem* 1 *in Section* 2.

PROOF. The function $w(t)=\int_0^t e^{-(t-s)A_\varepsilon}PF(\xi)ds$ satisfies

$$\begin{cases}\dfrac{dw}{dt}+A_\varepsilon w = PF(\xi), \\ w(0)=0.\end{cases} \tag{5.12}$$

Put $Z(t)=w(t)-A_\varepsilon^{-1}PF(\xi)$, then we have

$$\begin{cases}\dfrac{dZ}{dt}+A_\varepsilon Z = 0, \\ Z(0)=Z_0 \equiv -A_\varepsilon^{-1}PF(\xi).\end{cases} \tag{5.13}$$

Therefore it follows from Lemma 4.1 that

$$\|Z(t)\| \leqq M_1 e^{-\beta\varepsilon t} \|Z_0\|$$

so that

$$\|w(t)\| \leqq M_1 e^{-\beta\varepsilon t} \|Z_0\| + \|A_\varepsilon^{-1} PF(\xi)\| \leqq C_{32} \|A_\varepsilon^{-1} PF(\xi)\| . \tag{5.14}$$

We now show that $\|A_\varepsilon^{-1} PF(\xi)\|$ is bounded uniformly for small ε. Put $w = A_\varepsilon^{-1} PF(\xi)$; then

$$A_\varepsilon w = PF(\xi),$$

that is,

$$(A - \varepsilon F'(\xi))w = PF(\xi). \tag{5.15}$$

Define $w_1 = Qw$ and $w_2 = Pw$. Then (5.15) is written as

$$\begin{cases} -\varepsilon QF'(\xi)(w_1 + w_2) = 0, \\ A_2 w_2 - \varepsilon PF'(\xi)(w_1 + w_2) = PF(\xi). \end{cases} \tag{5.16}$$

From the first equation of (5.16), we obtain

$$w_1 = -Jw_2 \equiv -(QF'(\xi)|_{B_1})^{-1}(QF'(\xi)|_{B_2})w_2. \tag{5.17}$$

Substituting (5.17) into the second equation of (5.16), we have

$$\{A_2 - \varepsilon PF'(\xi)(-J + I_2)\}w_2 = PF(\xi)$$

and so

$$\{I_2 - \varepsilon A_2^{-1} PF'(\xi)(I_2 - J)\}w_2 = A_2^{-1} PF(\xi).$$

For small ε, the inverse $\{I_2 - \varepsilon A_2^{-1} PF'(\xi)(I_2 - J)\}^{-1}$ exists and is expressed as a Neumann series, so we have

$$\|w_2\|_2 \leqq \frac{\|A_2^{-1} PF(\xi)\|_2}{1 - \varepsilon \|A_2^{-1} PF'(\xi)\,(I_2 - J)\|_2} \leqq C_{33} \tag{5.18}$$

for small ε. Therefore (5.17) implies

$$\|w_1\|_1 \leqq \|J\|_{2,1} \cdot \|w_2\|_2 < C_{34}, \tag{5.19}$$

where $\|J\|_{2,1}$ denotes the operator norm of $J \in \mathscr{L}(B_2, B_1)$. Using (5.18) and (5.19), we have

$$\|w\| \leqq \|w_1\|_1 + \|w_2\|_2 \leqq C_{35},$$

as required. ∎

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