Hiroshima Math. J.
14 (1984), 547–550

On generic 1-parameter families of C^∞-maps of an n-manifold into a $(2n-1)$-manifold

Dedicated to Professor Tatsuo Homma on his 60th birthday

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(Received March 30, 1984)

Introduction

Let $n \geqq 2$. It is well known that a proper C^∞-map of an n-manifold M^n into a $(2n-1)$-manifold N^{2n-1} is stable if and only if it is regular except at the isolated singularities of Whitney umbrella and its sheets intersect in general position. In this note we clarify the normal forms of stable and generic C^∞-paths connecting two stable C^∞-maps of M^n into N^{2n-1}. We put

$$\mathscr{P} = \{f\colon M \times I \to N \times I \text{ proper } C^\infty\text{-maps with } p_N \circ f = p_M\,; f_0 \text{ and } f_1 \text{ are stable}\},$$

where p_X is the projection $X \times I \to I$ for $X = M$ or N and $f(x, \lambda) = (f_\lambda(x), \lambda)$.

Definition. Let $f, g \in \mathscr{P}$. We say that f is IA-equivalent to g or $f \underset{IA}{\sim} g$ if there exist diffeomorphisms k and K which satisfy the following commutative diagram (where the maps to I are the natural projections):

$$\begin{tikzcd}
M \times I \arrow[rr, "f"] \arrow[dr] \arrow[dd, "k"'] & & N \times I \arrow[dl] \arrow[dd, "K"] \\
& I & \\
M \times I \arrow[rr, "g"] \arrow[ur] & & N \times I \arrow[ul]
\end{tikzcd}$$

Accordingly we call $f \underset{IA}{\sim} g$ as germs at (x, λ) if there exist such k and K as germs.

Theorem. *Let $\mathscr{Q} = \{f \in \mathscr{P}\,; f$ satisfies one of the following conditions* (i)–(iv) *as germs at any point of $M \times I\}$. Then, $\mathscr{Q}$ is an open and dense subspace of $\mathscr{P}$ with respect to the fine C^∞-topology of $\mathscr{P}$ which is defined by the identification with a subspace of $C^\infty(M^n \times I, N^{2n-1})$.*

(i) $f \underset{IA}{\sim} (x_1, \dots, x_n, 0, \dots, 0, \lambda)$,
(ii) $f \underset{IA}{\sim} (x_1^2, x_2, \dots, x_n, x_1x_2, \dots, x_1x_n, \lambda)$ (*Whitney umbrella* $\times \mathbf{R}$),
(iii) $f \underset{IA}{\sim} (x_1^2, x_2, \dots, x_n, x_1(\pm\lambda - x_1^2 - x_2^2), x_1x_3, \dots, x_1x_n, \lambda)$ *and*
(iv) $f \underset{IA}{\sim} (x_1^2, x_2, \dots, x_n, x_1(\pm\lambda + x_1^2 - x_2^2), x_1x_3, \dots, x_1x_n, \lambda)$.

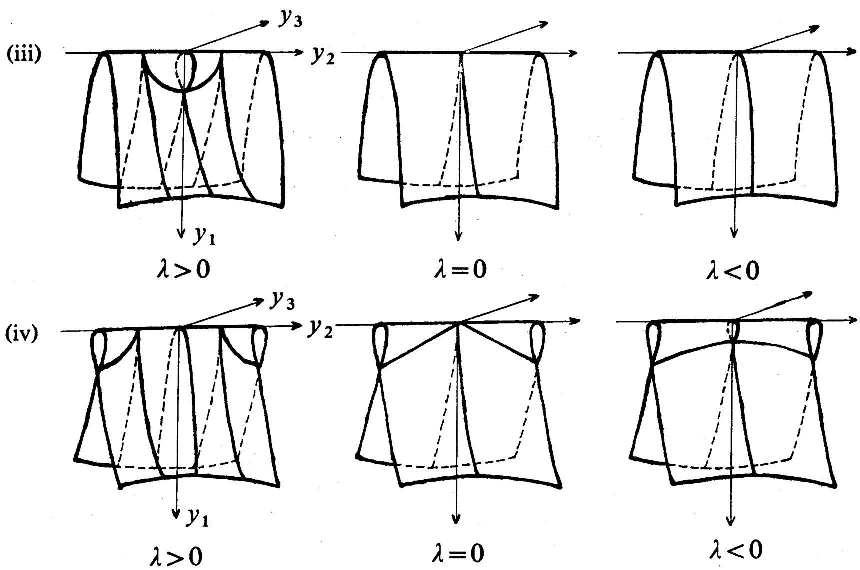

Figure 1.

We call $f\in\mathscr{P}$ is IA-stable if $\{g\in\mathscr{P};\ g\underset{IA}{\sim}f\}$ is an open subspace of $\mathscr{P}$. The proof of Theorem shows that the germs of (i)$\sim$(iv) are IA-stable as germs. If M is compact, we can define a subspace $\mathscr{Q}_0$ of $\mathscr{Q}$ with the condition that f_λ has at most one non-stable singular point for each λ and deformations of the sheets of f_λ are in general position. Then, it is not difficult to see that $\mathscr{Q}_0$ is an open and dense subspace of $\mathscr{P}$ and any $f\in\mathscr{Q}_0$ is IA-stable. This implies that any homotopy between two stable C^∞-maps of compact M^n into N^{2n-1} is approximated by a finite sequence of 'elementary' deformations up to isotopy. So, by noting that a nice PL-maps of a surface into a 3-manifold is isotopic to a stable C^∞-map, we get an alternative proof of the main theorem of Homma-Nagase [4] about the deformations of nice PL-maps of a closed surface into a 3-manifold.

We shall give a proof of Theorem based on the coordinate transformations, Malgrange's preparation theorem and Thom's transversality theorem. Since any miniversal deformation is parametrized stable (Cf. [2], [5]), we see that the singularities of codimension 1 have IA-stable versal deformations. Hence, our theorem implies that the normal forms of singularities of codimension 1 in the germs of C^∞-maps of $(\mathbf{R}^n, 0)$ into $(\mathbf{R}^{2n-1}, 0)$ are $(x_1^2, x_2,\dots, x_n, x_1x_2^2\pm x_1^3, x_1x_3,\dots, x_1x_n)$. The argument in this note can be easily generalized to the case of the generic paths of C^∞-maps of M^m into N^n when $(3m-1)/2<n\leqq 2m-1$ (Cf. [3]) by replacing (ii) with $(x_1^2, x_2,\dots, x_m, x_1x_{2m-n+1},\dots, x_1x_m)$ and (iii)–(iv) with $(x_1^2, x_2,\dots, x_m, x_1(\pm\lambda\pm x_1^2-x_2^2\pm x_3^2\pm\cdots\pm x_{3m-n+1}^2), x_1x_{2m-n+2},\dots, x_1x_m)$ in Theorem.

The authors thank Professor T. Fukuda and Dr. I. Nakai for the helpful discussions and Professor Y. Ando for a letter from which they have learned much.

Study of generic 1-parameter families and proof of Theorem

We write $x=(x_1,\dots, x_n)$, λ, $y=(y_1,\dots, y_{2n-1})$ as the local coordinates of M^n, I, N^{2n-1} respectively. We define submanifolds S_i^* of $J^1(M\times I, N)$ by

$$S_i^* = \{j^1f(x, \lambda)\in J^1(M\times I, N);\ \mathrm{rank}\,\partial(f_1,\dots,f_{2n-1})/\partial(x_1,\dots, x_n)=n-i \text{ at } (x, \lambda)\}\,.$$

This definition is compatible with the IA-equivalence. Put

$$\mathscr{Q}' = \{f\in \mathscr{P};\ j^1f \pitchfork S_i^*\ (i=1,\dots, n)\}\,.$$

Since $\cup_{i=1}^n S_i^*$ is closed in $J^1(M\times I, N)$, $\mathscr{Q}'$ is open and dense in $\mathscr{P}$ by Thom's transversality theorem. Assume $f\in \mathscr{Q}'$. We shall study the germs of f at (x_0, λ_0) with $f(x_0)=y_0$. By a parallel coordinate transformation, we may assume $(x_0, \lambda_0)=(0, 0)$ and $y_0=0$, where we consider $I=[-1, 1]$ instead of $I=[0, 1]$. Since codim $S_1^*=n$ and codim $S_i^*\geqq n+2$ $(i\geqq 2)$, it suffices to consider the following cases (1)–(3).

Case (1) $j^1f(0, 0)\notin S_1^*$: It is easy to see that $f \underset{IA}{\sim} (x, 0, \lambda)$ at $(0, 0)$.

Case (2) $j^1f_0 \pitchfork S_1$ *where* $S_1=\{\sigma\in J^1(M, N);\ \mathrm{rank}\,\sigma=n-1\}$: In an appropriate coordinate, we have $f_i(=y_i\circ f)=x_i$ $(2\leqq i\leqq n)$. Since j^1f_0 intersects with S_1 at $x=0$ transversally, $\partial_{x_1}f_1=\partial_{x_1}f_{n+1}=\cdots=\partial_{x_1}f_{2n-1}=0$ and $\Delta\neq 0$ at $(0, 0)$ where $\Delta=\det(\partial(\partial_{x_1}f_1, \partial_{x_1}f_{n+1},\dots, \partial_{x_1}f_{2n-1})/\partial(x_1,\dots, x_n))$. Hence, we may assume $\partial_{x_1x_1}f_1\neq 0$ and $\partial_{x_1x_1}f_{n+j}=0$ $(1\leqq j\leqq n-1)$ at $(0, 0)$. In particular $C^\infty_{(0,0)}(M\times I)/(f^*\mathfrak{m}_{(0,0)}(N\times I))C^\infty_{(0,0)}(M\times I)$ is generated by 1, x_1, where $C^\infty_{(0,0)}(M\times I)$ is the set of all germs of C^∞-functions defined on a neighborhood of $(x, \lambda)=(0, 0)$ in $M\times I$ and $\mathfrak{m}_{(0,0)}(N\times I)$ is the unique maximal ideal of $C^\infty_{(0,0)}(N\times I)$. By Malgrange's preparation theorem, we get $\alpha_1, \alpha_2\in C^\infty_{(0,0)}(N\times I)$ such that $x_1^2=(\alpha_1\circ f)(x, \lambda)+x_1(\alpha_2\circ f)(x, \lambda)$. We remark that $\alpha_2(0, 0)=0$ and $\partial_{y_1}\alpha_1(0, 0)\neq 0$.

Put $X_1=x_1-(1/2)(\alpha_2\circ f)(x, \lambda)$, $X_i=x_i$ $(2\leqq i\leqq n)$ and $Y_1=(\alpha_1+(1/4)\alpha_2^2)(y, \lambda)$, $Y_j=y_j$ $(2\leqq j\leqq 2n-1)$. Then, this is a coordinate change because of the above remark, and we get $Y_1=X_1^2$, $Y_i=X_i$ $(2\leqq i\leqq n)$, $Y_{n+j}=g_j(x, \lambda)$ $(1\leqq j\leqq n-1)$ for some $g_j\in C^\infty_{(0,0)}(M\times I)$. By using Malgrange's theorem again, we have $\beta_{j,1}, \beta_{j,2}\in C^\infty_{(0,0)}(\mathbf{R}^{n+1})$ $(1\leqq j\leqq n-1)$ such that $g_j(X_1, X', \lambda)=\beta_{j,1}(X_1^2, X', \lambda)+X_1\beta_{j,2}(X_1^2, X', \lambda)$ with an abbreviation $X'=(X_2,\dots, X_n)$. By putting $\bar{Y}_i=Y_i$ $(1\leqq i\leqq n)$, $\bar{Y}_{n+j}=Y_{n+j}-\beta_{j,1}(Y, \lambda)$ $(1\leqq j\leqq n-1)$, we get $\bar{Y}_1=X_1^2$, $\bar{Y}_i=X_i$ $(2\leqq i\leqq n)$, $\bar{Y}_{n+j}=X_1\beta_{j,2}(X_1^2, X', \lambda)$ $(1\leqq j\leqq n-1)$. Since j^1f_0 intersects with S_1 at $X=0$ transversally, we get $\beta_{j-1,2}(0, 0)=0$ and $\det(\partial_{X_i}\beta_{j-1,2}(0, 0))\neq 0$ $(2\leqq i, j\leqq n)$. This implies $f \underset{IA}{\sim} (x, x', x_1x', \lambda)$ at $(0, 0)$.

Case (3) j^1f_0 *intersects with* S_1 *non-transversally*: We may assume that

$f_1=x_1^2$, $f_i=x_i$ $(2\leqq i\leqq n)$ and $f_{n+j}=x_1h_j(x_1^2, x', \lambda)$ $(1\leqq j\leqq n-1)$. Since j^1f_0 intersects with S_1 non-transversally at $x=0$, we get $h_j(0, 0)=0$ $(1\leqq j\leqq n-1)$ and $\Delta'(0, 0)=0$ where $\Delta'=\det(\partial_{x_i}h_j)_{1\leqq i-1, j\leqq n-1}$. Since $j^1f \pitchfork S_1^*$, we have $\operatorname{rank}(\partial(\partial_{x_1}f_1, \partial_{x_1}f_{n+1},\dots, \partial_{x_1}f_{2n-1})/\partial(x_1,\dots, x_n, \lambda))=n$ at $(0, 0)$. So, we may assume $\partial_\lambda h_1(0, 0)\neq 0$ and $\det(\partial_{x_i}h_j(0, 0))_{2\leqq i-1, j\leqq n-1}\neq 0$. Moreover, by a coordinate change of $(x_3,\dots, x_n)$ and $(y_{n+2},\dots, y_{2n-1})$ we may assume $h_i=x_{i+1}$ $(2\leqq i\leqq n-1)$ from the last condition; other conditions reduce to $\partial_{x_2}h_1(0, 0)=0$ and $\partial_\lambda h_1(0, 0)\neq 0$. Hence, $h_1(x_1^2, x', \lambda)=a(x_1^2, x', \lambda)x_1^2+b(x_1^2, x', \lambda)x_2^2+c(x_1^2, x', \lambda)\lambda+\sum_{i=3}^n a_i(x_1^2, x', \lambda)x_i$ for some $a, b, c, a_i\in C^\infty_{(0,0)}(\mathbf{R}^{n+1})$ with $c(0, 0, 0)\neq 0$. This easily implies

$$f \underset{IA}{\sim} (x_1^2, x_2,\dots, x_n, x_1(\pm\lambda+\alpha(x_1^2, x', \lambda)x_1^2-\beta(x_1^2, x', \lambda)x_2^2), x_1x_3,\dots, x_1x_n, \lambda).$$

PROOF OF THEOREM. Note that $(x_1^2, x', x_1(\pm\lambda+\alpha(x_1^2, x', \lambda)x_1^2-\beta(x_1^2, x', \lambda)x_2^2), x_1x'', \lambda)\underset{IA}{\sim}(x_1^2, x', x_1(\pm\lambda+\operatorname{sign}\alpha(0, 0, 0)\beta(0, 0, 0)x_1^2-x_2^2), x_1x'', \lambda)$ if $\alpha(0, 0, 0)\neq 0$ and $\beta(0, 0, 0)\neq 0$, where $x'=(x_2,\dots, x_n)$ and $x''=(x_3,\dots, x_n)$. By the above argument we see that (i)–(iv) are IA-stable germs and so $\mathscr{Q}$ is open. We remember that $\mathscr{Q}'$ is dense in $\mathscr{P}$. For any $f'\in\mathscr{Q}'$ there is an $f\in\mathscr{Q}$ in any neighborhood of f'. In fact, if $f'\underset{IA}{\sim}(x_1^2, x', x_1(\pm\lambda+\alpha x_1^2-\beta x_2^2), x_1x'', \lambda)$ at (x, λ), we modify f' near (x, λ) to f so that $f\underset{IA}{\sim}(x_1^2, x', x_1(\pm\lambda+(\alpha+\gamma)x_1^2-(\beta+\gamma)x_2^2, x_1x'', \lambda)\underset{IA}{\sim}(x_1^2, x', x_1(\pm\lambda+x_1^2-x_2^2), x_1x'', \lambda)$ by using a germ of C^∞-function $\gamma(\lambda)\in C_0^\infty(\mathbf{R})$ close to 0 with $\gamma(0)>0$ and $\gamma=0$ outside a small neighborhood of 0. q.e.d.

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