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Standing wave solutions for a Fisher type equation with a nonlocal convection

Masayasu MIMURA and Kazutaka OHARA

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Abstract. This paper is concerned with stationary solitary wave solutions of a nonlinear diffusion equation described by

$$u_t = du_{xx} - [(K*u)u]_x + ku(1-u).$$

It is proved that there is no such solution for kernels $K(x) \in L_1(\mathbf{R})$, and that for some specific kernel $K(x) \notin L_1(\mathbf{R})$ there is a range of value of the total distribution $\int_{\mathbf{R}} u dx$ for fixed d and k over which such solutions exist.

1. Introduction

Recently there has been considerable interest in biological models governed by a class of reaction-diffusion equations. The simplest model describing the dynamics of one species that moves by diffusion, in one dimension, is expressed by

$$(1) \qquad u_t = du_{xx} + k(1-u/\alpha)u,$$

in which the Pearl-Verhulst law is employed for the population growth. Here d is the diffusion constant, k, α are respectively an intrinsic growth rate, a carrying capacity for the species. The equation, which is called the Fisher equation, is also encountered in population genetics. It was analyzed precisely by seeking traveling wave solutions which take the form $u(x, t) = u(x-ct)$ with the constant speed c. As boundary conditions, we have $u(-\infty) = \alpha$, $u(+\infty) = 0$. Traveling wave solutions exist for any fixed $c \geqq c^* = 2\sqrt{kd}$, and a solution of the initial value problem for (1) with a fairly wide class of initial functions forms asymptotically a traveling wave with the minimum speed $c = c^*$. Furthermore, if the initial function is of compact support, it spreads out with time where the fronts move in both directions with the asymptotic speed c^* (see Uchiyama [4]).

On the other hand, we often meet diffusion-convection equations in the absence of growth and/or death terms

$$u_t = du_{xx} - (Vu)_x,$$

where V is the convection velocity of the species. Recently Nagai and Mimura [3] have studied a nonlinear diffusion equation with a nonlocal convection of the form

(2) $$u_t = d(u^m)_{xx} - [(K*u)u]_x,$$

where $m \geqq 1$ and $K*u = \int_{\boldsymbol{R}} K(x-y)u(y)dy$. They assume specifically the kernel K to satisfy

(i) K is an odd function in $\boldsymbol{R}$;
(ii) K is monotone increasing in $-\infty < x < 0$ and in $0 < x < +\infty$.

A representative example of the kernel is

(3) $$K(x) = \begin{cases} a\, e^{\lambda x} & (-\infty < x < 0), \\ -a\, e^{-\lambda x} & (0 < x < +\infty), \end{cases}$$

where a, λ are positive constants. The convection term in (2) provides the mechanism that $u(x, t)$ moves to the right when

$$\int_{-\infty}^{x} e^{-\lambda(x-y)}\, u(y)dy < \int_{x}^{\infty} e^{\lambda(x-y)}\, u(y)dy$$

and to the left when the inequality is reversed. For this reason, the convection effect satisfying (i), (ii) may be phenomenologically understood to be a kind of nonlocal aggregation of individuals. Intuitively, one can expect that by an interplay of the balance between "diffusion" and "aggregation", (2) may exhibit a pulse-like solution. Indeed, that is verified when $\lambda = 0$ in (3) (see [3]). Suppose $u(x, 0)$ ($\geqq 0$) has compact support. Then a solution $u(x, t)$ of (2) with $m = 1$ evolves into a stationary solitary wave of the form

$$\frac{I^2 \exp(I(x-s)/d)}{d(1+\exp(I(x-s)/d))^2}$$

with $I = \int_{\boldsymbol{R}} u(x, 0)dx$, where s is determined by the initial function $u(x, 0)$. One finds that such spatially localized standing waves show a great contrast to spreading waves occurring in (1).

In this paper, we consider a linear diffusion model (2) with $m = 1$ incorporated with the population growth in (1),

(4) $$u_t = du_{xx} - [(K*u)u]_x + ku(1-u),$$

which is regarded as a Fisher type equation with a nonlocal convection.

Murray [2] and his coworker Gibbs considered the effect of (local) convection on the Fisher equation from the viewpoint of traveling waves when the convection in (4) takes the form $[H(u)]_x$. Here we study the existence of stationary solitary waves of (4) in the case when $K(x)$ possesses the effect of *aggregation*. It is emphasized that (4) contains three qualitatively different effects of "diffusion", "aggregation" and "growth". The results on (1) and (2) motivate us to address the question whether an initial distribution of compact support evolves into a localized standing wave or a spreading wave whose fronts move in both directions. When $K(x)$ takes the specific form (3), it is numerically observed that for $0<\lambda\leqq\infty$ an initial distribution of compact support spreads out in both directions with $\|u(\cdot, t)\|_{L^1}\to\infty$ as $t\to\infty$ and, on the other hand, for $\lambda=0$ it tends to a stationary solitary wave and $\|u(\cdot, t)\|_{L^1}$ approaches some finite value as $t\to\infty$. (Fig. 1.)

In Section 2, we show the non-existence of stationary solitary wave solutions of (4) for any kernels $k\in L_1(\mathbf{R})$. We also consider a specific form (3) with $\lambda=0$ as an example of $k\notin L_1(\mathbf{R})$, although it is very simple and then show the existence of stationary wave solutions of (4). In Section 3, we give the proofs of the results shown in Section 2. Finally, in Section 4, we mention some remarks on the results.

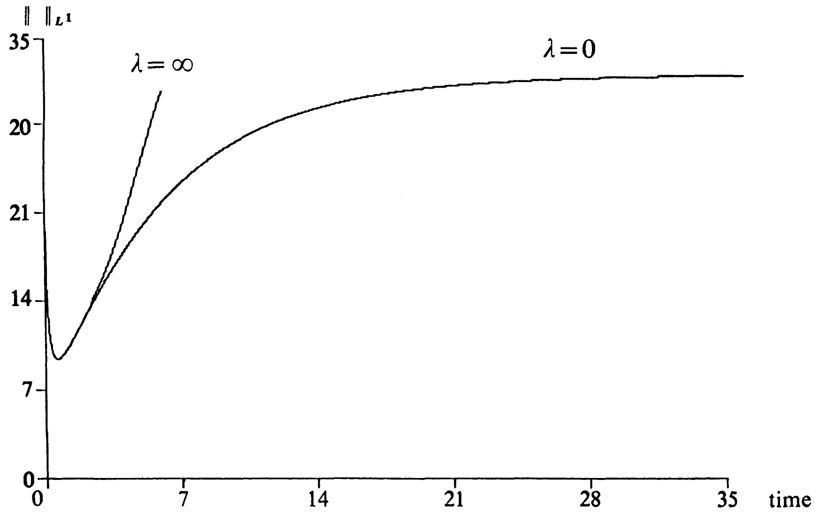

Figure 1. Time development of $\|u(\cdot, t)\|_{L^1}$ of (3), (4) with $\lambda=0$ and $\lambda=\infty$. $d=5$, $a=1$, $k=40$ and

$$u(x, 0)=\begin{cases} 3 & -1/2<x<1/2 \\ 0 & \text{otherwise} \end{cases}$$

2. Results

We consider the stationary equation of (4),

$$(5) \qquad 0 = du'' - [(K*u)u]' + ku(1-u), \quad x\in\mathbf{R},$$

where $'$ denotes d/dx. We first define a solution of (5) by a nonnegative function

u such that

(i) $u \in C^2(\boldsymbol{R}) \cap L_1(\boldsymbol{R}) \cap L_\infty(\boldsymbol{R})$;
(ii) u satisfies (5).

LEMMA 1. *Let $u(x)$ be a solution of (5) with $K \in L_1(\boldsymbol{R}) \cup L_\infty(\boldsymbol{R})$. Then $u(\pm\infty)=u'(\pm\infty)=0$. Moreover $\max_{x\in\boldsymbol{R}} u(x)>1$.*

THEOREM A. *Suppose $K \in L_1(\boldsymbol{R})$. Then (5) has no solutions except $u \equiv 0$.*

Theorem A suggests that the behavior of solutions of (4) with $K \in L_1(\boldsymbol{R})$ seems to be analogous to that of (1). We will not argue this problem in this paper.

We next consider the case when K is not of class $L_1(\boldsymbol{R})$. As an example of such kernels, we specify $K(x)$ with $\lambda=0$ in (3),

$$K(x) = \begin{cases} a & (-\infty<x<0) \\ -a & (0<x<+\infty), \end{cases}$$

where a is a positive constant. The resulting equation is

$$0 = du'' + \left[\left(\int_{-\infty}^{x} u(y)dy - \int_{x}^{+\infty} u(y)dy\right)u\right]' + ku(1-u), \tag{6}$$

where d, k are taken to be d/a, k/a, respectively.

LEMMA 2. *Let $u(x)$ be a solution of (6). Then there is only one paoint $x_0 \in \boldsymbol{R}$ such that $u'(x_0)=0$, and $u(x)$ is symmetric with respect to $x=x_0$, that is, $u(x)=u(2x_0-x)$ for any $x \in \boldsymbol{R}$.*

THEOREM B. *Let $d>0$ be arbitrarily fixed. For each $k>0$, there exists $I_d(k)>0$ such that*

(i) *there is no solution of (6) for $0<I<I_d(k)$;*
(ii) *there is a solution of (6) for $I_d(k) \leqq I$;*
(iii) *for fixed k, $\lim_{d\downarrow 0} I_d(k)=0$, and for fixed d, $\liminf_{k\downarrow 0} I_d(k)>0$,*

where $I=\int_{-\infty}^{\infty} u(y)dy$ and $I_d(k)$ is estimated as follows:

$$I_d(k) = 2\sqrt{dk} \qquad (k\geqq 4),$$
$$\sqrt{4dk(5-k)} \geqq I_d(k) \geqq 2\sqrt{dk} \qquad (4>k\geqq 2),$$
$$2\sqrt{(4+k)d} \geqq I_d(k) \geqq \max\{2\sqrt{dk},\ 2^{1/4}\sqrt{d}\} \qquad (2>k>0).$$

Moreover, for each (I, d, k), where $I \geqq I_d(k)$, a solution of (6) uniqeuly exists.

3. Proofs

PROOF of LEMMA 1.

Define $F(x)$ by

$$F(x) = du' - (K*u)u + k\int_{-\infty}^{x} u(y)(1-u(y))dy, \tag{7}$$

where u is a solution of (5). Then $F'(x)\equiv 0$, that is $F(x)=\text{constant}=\xi$. Since $u(x)$ is bounded and integrable, $u'(x)$ is also bounded. Then it is easy to see that $u(x)\to 0$ as $x\to\pm\infty$. From (7), $\xi=F(-\infty)=du'(-\infty)$, and hence $u'(-\infty)=0$. On the other hand, we have

$$0 = F(+\infty) = du'(+\infty) + k\int_{R} u(y)(1-u(y))dy,$$

which implies $u'(+\infty)=\int_{R} u(y)(1-u(y))dy=0$. From this we also see that $\max_{x\in R} u(x)>1$. ■

PROOF of THEOREM A.

First note that

$$\left|\int_a^b (K*u)dx\right| \leqq \|K\|_{L_1}\|u\|_{L_1}.$$

for any a, b ($b>a$). Lemma 1 leads to $F(x)=0$, that is

$$u'(x) = \frac{(K*u)(x)}{d}u(x) - \frac{1}{d}g(x),$$

where $g(x)=k\int_{-\infty}^{x} u(y)(1-u(y))dy$. Therefore

$$u(x) = \exp\left(\frac{1}{d}\int_a^x (K*u)(s)ds\right)\times \left[u(a) + \int_a^x \exp\left(-\frac{1}{d}\int_a^y (K*u)(s)ds\right)\left(-\frac{g(y)}{d}\right)dy\right] \quad (x>a). \tag{8}$$

We note $g(+\infty)=0$ and $g'(x)>0$ for sufficiently large x. That is, if a is chosen to be sufficiently large such that $0<u(a)<1$, then $g(x)<0$ for $x>a$. Thus from (8), we obtain

$$\begin{aligned} u(x) &\geqq u(a)\exp\left(\frac{1}{d}\int_a^x (K*u)(s)ds\right) \\ &\geqq u(a)\exp\left(-\frac{1}{d}\|K\|_{L_1}\cdot\|u\|_{L_1}\right) \quad (x>a), \end{aligned}$$

which contradicts $u \in L_1(\boldsymbol{R})$. ■

Next we will prove Lemma 2 and Theorem B. To find a solution of (6), it is convenient to introduce a new variable $v(x)$ defined by

$$v(x) = \int_{-\infty}^{x} u(y)dy.$$

Then (6) becomes

$$du'' + \{(2v-I)u\}' + ku(1-u) = 0, \tag{9}$$

where $I=\int_{-\infty}^{\infty} u(y)dy$ which is a total population number to be determined. Substituting $u=v'$ into (9) gives

$$dv''' + \{(2v-I)v'\}' + kv'(1-v') = 0. \tag{10}$$

We investigate (10) in the three-dimensional phase space so that (10) is written as

$$\begin{aligned} v' &= p, \\ p' &= q, \qquad\qquad x \in \boldsymbol{R} \\ q' &= (1/d)\cdot\{(I-2v)q - 2p^2 - kp(1-p)\}. \end{aligned} \tag{11}$$

The boundary conditions of (11) at $x=\pm\infty$ are

$$(v, p, q)|_{x=-\infty} = (0, 0, 0), (v, p, q)|_{x=+\infty} = (I, 0, 0). \tag{12}$$

We are concerned with the eixstence of solutions $(v, p, q; I)$ of (11), (12) instead of (6).

PROOF of LEMMA 2.

First we show that there is only one point x_0 such that $u'(x_0)=0$. Suppose the conclusion is not true. Then the following three cases can happen. 1) $u(x_0)<\max u=u(x_1)$, or 2) $u(x_0)=\max u(x_1)$ and there exists x_2 between x_0 and x_1 such that $u(x_2)<\max u$, or 3) there exists an interval $[x_0, x_1]$ such that $u(x)\equiv\max u=u(x_0)=u(x_1)$ for $x\in[x_0, x_1]$. Consider the first case. We may assume $x_0<x_1$. Let us show $u''(x_0)<0$. Suppose to the contrary that $u''(x_0)\geqq 0$. Then it follows from (9) that at $x=x_0$,

$$0 \geqq - du'' - (2v-I)u' = 2u^2 + ku(1-u), \tag{13}$$

which leads to $u(x_0)=0$ for $0<k\leqq 2$. Considering the initial value problem (10) subject to the initial conditions

$$(14)\qquad \begin{aligned} v(x_0) &= \int_{-\infty}^{x_0} u(y)dy, \\ v'(x_0) &= 0, \\ v''(x_0) &= 0, \end{aligned}$$

we have $v(x)=\text{constant}$, which is a contradiction. Consider the case $k>2$. From the fact that $u'(x_1)=0$, $u''(x_1)\leqq 0$, it follows that $2u^2(x_1)+ku(x_1)(1-u(x_1))\geqq 0$, that is, $u(x_1)\leqq k/(k-2)$. Thus, $2u^2(x_0)+ku(x_0)(1-u(x_0))\geqq 0$. By (13), we have $u(x_0)=0$, which is again a contradiction. Therefore $u''(x_0)<0$. Hence there exists x^* between x_0 and x_1 such that $u'(x^*)=0$, $u''(x^*)\geqq 0$ and $u(x^*)<\max u$. A similar argument leads to a contradiction. The second and third cases may be treated analogously, so we omit the proofs.

Next we show the symmetry of a solution $u(x)$. Suppose $u(x)$ is not symmetric. We note that if $u(x)$ is a solution of (6), $u(-x)$ is also a solution of (6). Let us write two solutions of (11) as $(v(x), p(x), q(x))$ $(\bar{v}(x), \bar{p}(x), \bar{q}(x))$ corresponding to $u(x)$, $u(-x)$, respectively. Regarding p, q as functions of v, we rewrite (11), (12) as

$$(15)\qquad \begin{aligned} \frac{dp}{dv} &= \frac{q}{p}, \\ \frac{dq}{dv} &= \frac{1}{d}\left\{(I-2v)\frac{q}{p} - k - (2-k)p\right\}, \end{aligned}$$

$$(16)\qquad p(0) = p(I) = q(0) = q(I) = 0.$$

$\bar{p}$, $\bar{q}$ also satisfy (15), (16). Since $u(x)$ is assumed not to be symmetric, we may assume without loss of generality that $\bar{q}(I/2)<0<q(I/2)$. We note $p(I/2)=\bar{p}(I/2)$ and $p'(I/2)>\bar{p}'(I/2)$. Substituting the first equation of (15) into the second, we obtain

$$(17)\qquad \frac{dq}{dv} = \frac{1}{d}\left\{(I-2v)\frac{dp}{dv} - k - (2-k)p\right\}.$$

Integrating (17) with respect to v in $[v_1, v_2]$, we have

$$(18)\qquad \begin{aligned} q(v_2) - q(v_1) = \frac{1}{d}\{&(I-2v_2)p(v_2) \\ &- (I-2v_1)p(v_1) - k(v_2-v_1) + k\int_{v_1}^{v_2} p(v)dv\}. \end{aligned}$$

Put $p^*=p-\bar{p}$, $q^*=q-\bar{q}$. Then (18) yields

$$(19)\qquad q^*(v_2) - q^*(v_1) = \frac{1}{d}\{(I-2v_2)p^*(v_2) - (I-2v_1)p^*(v_1) + k\int_{v_1}^{v_2} p^*(v)dv\}.$$

Setting $v_1=0$, $v_2=I/2$, in (19), we have

$$0 < q^*(I/2) = \frac{k}{d}\int_0^{I/2} p^*(v)dv. \tag{20}$$

If $\bar{p}(v)>p(v)$ for $0<v<I/2$, then $p^*(v)<0$ for $0<v<I/2$ which contradicts (20). Therefore there exists v_0 $(0<v_0<I/2)$ such that $p(v_0)=\bar{p}(v_0)$. Let v^* be the largest one for which $p(v^*)=\bar{p}(v^*)$. Setting $v_1=0$, $v_2=v^*$ in (19), we have $q^*(v^*)=\frac{k}{d}\int_0^{v^*} p^*(v)dv>0$, that is, $q(v^*)>\bar{q}(v^*)$, which implies $p'(v^*)=q(v^*)/p(v^*)>\bar{q}(v^*)/\bar{p}(v^*)=\bar{p}'(v^*)$. This contradicts the choice of v^*. ■

From now on, we call a solution (p, q, v) of (11), (12) symmetric if only $p(x)$ is symmetric, and a solution (p, q) of (15), (16) symmetric if $p(v)$ is symmetric.

PROOF of THEOREM B.

We first show the following results as a consequence of Lemma 2.

COROLLARY 1. *For each (I, d, k), a symmetric solution of (11), (12) is unique*

PROOF. Suppose that there are two symmetric solutions, say $(p(v), q(v))$, $(\bar{p}(v), \bar{q}(v))$. We may assume $p(I/2)>\bar{p}(I/2)$. Putting $p^*=p-\bar{p}$, $q^*=q-\bar{q}$, and then setting $v_1=0$, $v_2=I/2$ in (19), we have

$$0 = \frac{k}{d}\int_0^{I/2} p^*(v)dv.$$

Then, there is v_0 $(0<v_0<I/2)$ such that $p^*(v_0)=0$. Let v^* be the largest one for which $p^*(v^*)=0$. Setting $v_1=0$, $v_2=v^*$ in (19), we find

$$q^*(v^*) = \frac{k}{d}\int_0^{v^*} p^*(v)dv < 0,$$

which means $q(v^*)<\bar{q}(v^*)$ or $p'(v^*)<\bar{p}'(v^*)$. This is a contradiction. ■

Theorem B is proved by employing the "shooting method" (for instance Dunbar [1]).

If a trajectory $(v(x), p(x), q(x))$ connecting $(v(-\infty), p(-\infty), q(-\infty))=(0, 0, 0)=O$ with $(v(x^*), p(x^*), q(x^*))=(I/2, p_0, 0)=P$ is found for some I, x^* and p_0, then a function $(\tilde{v}, \tilde{p}, \tilde{q})$ of the form

$$(\tilde{v}, \tilde{p}, \tilde{q}) = \begin{cases} (v(x),\ p(x),\ q(x)) & (-\infty<x\leqq x^*) \\ (I-v(2x^*-x),\quad p(2x^*-x),\quad -q(2x^*-x)) & (x^*\leqq x<+\infty) \end{cases}$$

becomes a symmetric solution of (11), (12). Thus, we may look for a solution of (11) connecting O and P for some I, x^* and p_0. We consider a trajectory originating from O. To do so, we linearize (11) so that the Jacobian matrix of the linearized system is

$$A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -k/d & I/d \end{bmatrix}.$$

The eigenvalues of A are

$$0 \quad \text{and} \quad (I\pm\sqrt{I^2-4dk})/2d=\mu\pm \qquad (I^2-4dk\geqq 0)$$

or

$$0 \quad \text{and} \quad (I\pm i\sqrt{4dk-I^2})/2d \qquad (I^2-4dk<0).$$

For $I^2-4dk<0$, O is a node and there is a one-dimensional unstable manifold U_1 corresponding to the largest eigenvalue μ_+. In terms of a parameter m, points on U_1 are parametrically represented as

$$U_1(m)=m\begin{bmatrix} 1 \\ \mu_+ \\ \mu_+^2 \end{bmatrix}+o(|m|)$$

in a small neighborhood of O. Also with two parameters there is a two-dimensional unstable manifold U_2 of trajectories originating from the origin. It takes the representation of

$$U_2(m,n)=m\begin{bmatrix} 1 \\ \mu_+ \\ \mu_+^2 \end{bmatrix}+n\begin{bmatrix} 1 \\ \mu_- \\ \mu_-^2 \end{bmatrix}+o(|m|+|n|)$$

in a small neighborhood of O. We note $U_1\subseteq U_2$. A trajectory on the manifold U_1 approaches tangentially to the eigenvector corresponding to μ_+ as $x\to-\infty$. All trajectories on the manifold U_2 except a trajectory on U_1 approach O tangentially to the eigenvector corresponding to μ_- as $x\to-\infty$. For $I^2-4dk=0$, O is a degenerate node, where there exists a two-dimensional unstable manifold U in a neighborhood of O. In this case all trajectories on the manifold U approach O tangentially to the eigenvector corresponding to $\mu_+=\mu_-=I/2d$ as $x\to-\infty$. For $I^2-4dk<0$, O is a spiral point. Therefore a trajectory originating from the origin must enter the region $p<0$. So we must require $I^2-4dk\geqq 0$, and in fact we require $I^2-4dk>0$ until the end of Lemma 11.

We show that a trajectory on the portion of the one-dimensional manifold U_1 in the octant $\{(v,p,q)|v>0, p>0, q>0\}$ attains the quarter plane $D=$

$\{(v, p, q)|v=I/2, p>0, q>0\}$ under suitable conditions. We first consider the case $k \geqq 4$.

LEMMA 3. *Let* $k \geqq 4$. *Suppose* $p(0)>(I/2d)\cdot v(0)$, $q(0)>(I/2d)\cdot p(0)$. *Then for any* $x>0$ *so long as* $p(x)>0$, $p(x)>(I/2d)\cdot v(x)$, $q(x)>(I/2d)\cdot p(x)$.

PROOF. Suppose that there exists the smallest $x_0>0$ such that $p(x_0)=(I/2d)\cdot v(x_0)$, $q(x_0)>(I/2d)\cdot p(x_0)$. Then it is obvious to see $p'(x_0)-(I/2d)\cdot v'(x_0) \leqq 0$. On the other hand, we have $p'(x_0)-(I/2d)\cdot v'(x_0)=q(x_0)-(I/2d)$ $p(x_0)>0$ which is a contradiction. Next suppose the smallest $x_0>0$ exists such that $p(x_0) \geqq (I/2d)\cdot v(x_0)$, $q(x_0)=(I/2d)\cdot p(x_0)$. Then, it turns out that

$$\begin{aligned} q'(x_0) - \frac{I}{2d}p'(x^0) &= \frac{1}{d}\{(I-2v(x_0))q(x_0) \\ &\quad - kp(x_0) - (2-k)p^2(x_0)\} - \frac{I}{2d}q(x_0) \\ &= \frac{I^2}{4d^2}p(x_0) - \frac{I}{d^2}v(x_0)p(x_0) - \frac{k}{d}p(x_0) - \frac{2-k}{d}p^2(x_0) \\ &\geqq \frac{1}{d}p(x_0)\left\{\frac{I^2}{4d} - k + (k-4)p(x_0)\right\} > 0, \end{aligned}$$

which contradicts the inequality $q'(x_0)-(I/2d)\cdot p'(x_0) \leqq 0$. ■

LEMMA 4. *Let* $k \geqq 4$. *Suppose* $p(0)<sv(0)$, $q(0)<sp(0)$ *with* $s=(k-2)I/2d$. *Then for any* $x>0$ *so long as* $p(x)>0$, $p(x)<sv(x)$, $q(x)<sp(x)$.

The proof is almost similar to that of Lemma 3. So we omit it.

LEMMA 5. *For* $k \geqq 4$, *let* V_1 *be a cone defined by*

$$V_1 = \{(v, p, q)|sv \geqq p \geqq (I/2d)\cdot v,\ sp \geqq q \geqq (I/2d)\cdot p,\ 0 \leqq v \leqq I/2\},$$

where s *is the constant given in Lemma* 4. *Then for any* $x \in \boldsymbol{R}$ *so long as* $0< v(x) \leqq I/2$, *a trajectory on the manifold* U_1 *is trapped in* V_1.

PROOF. We note that in a neighborhood of O, U_1 lies in V_1. Hence, Lemmas 3 and 4 immediately imply the result. ■

Thus, we find that for $k \geqq 4$ a trajectory on U_1 attains the plane $D=\{(v, p, q)|\cdot v=I/2, p>0, q>0\}$. We next consider the case $2 \leqq k<4$.

LEMMA 6. *Let* $2 \leqq k<4$. *Suppose* $p(0)>(I/2d)\cdot v(0)$, $q(0)>(I/2d)\cdot p(0)$. *Then for any* $x>0$ *so long as* $0<p(x)<(I^2-4dk)/(16-4k)d$, $p(x)>(I/2d)\cdot v(x)$, $q(x)>(I/2d)\cdot p(x)$.

LEMMA 7. *Let* $0<k<4$. *Suppose* $p(0)<(I/d)\cdot v(0)$, $q(0)<(I/d)\cdot p(0)$. *Then for any* $x>0$ *so long as* $p(x)>0$, $p(x)<(I/d)\cdot v(x)$, $q(x)<(I/d)\cdot p(x)$.

LEMMA 8. *Let* $2 \leqq k < 4$. *Suppose* $q(0) > (k/d) \cdot \{I/2 - v(0)\}$. *Then for any* $x > 0$ *as long as* $0 < v(x) < I/2$, $q(x) > (k/d) \cdot \{I/2 - v(x)\}$.

As the proofs of Lemmas 6, 7 and 8 can be carried out in a similar way to Lemma 3, we omit them.

We define a cone V_2 by

$$V_2 = \{(v, p, q) | (I/2d) \cdot v \leqq p \leqq (I/d) \cdot v,$$
$$(I/2d) \cdot p \leqq q \leqq (I/d) \cdot p,\ 0 \leqq p \leqq (I^2 - 4dk)/(16 - 4k)d\}.$$

We note that U_1 lies in V_2 in a neighborhood of the origin. Then it follows from Lemmas 7 and 8 that for $2 \leqq k < 4$, a trajectory on U_1 is trapped in V_2 as long as $0 < p \leqq (I^2 - 4dk)/(16 - 4k)d$. Hence it enters $S_2 = \{(v, p, q) | p = (I^2 - 4dk)/(16 - 4k)d,\ (d/I) \cdot p \leqq v \leqq (2d/I) \cdot p,\ (I/2d) \cdot p \leqq q \leqq (I/d) \cdot p\}$, which is the cross section of V_2. Fix I such that $I^2 \geqq 4dk(5-k)$. Then we know that S_2 is contained in $\{(v, q) | q \geqq k(I/2 - v)/d\}$ and therefore from Lemma 8, a trajectory on U_1 must enter the plane

$$\bar{D} = \{(v, p, q) | v = I/2,\ p > 0,\ q \geqq 0\}.$$

We will show that the trajectory never reaches the line

$$l = \{(v, p, q) | v = I/2,\ p > 0,\ q = 0\}.$$

Suppose this is not ture, that is, the trajectory attanins l for the first time at some x, say $x = x_0$. Define $f(x)$ by

$$f(x) = q(x) - (k/d) \cdot (I/2 - v(x)).$$

Then one can easily see that $f(x_0) = 0$ and

$$f'(x_0) = (k-2)p^2(x_0)/d > 0 \qquad (k > 2),$$
$$f'(x_0) = f''(x_0) = 0,\ f'''(x_0) = 8p^2(x_0)/d^2 > 0 \qquad (k = 2),$$

which is a contradiction. Thus, we find that for $2 \leqq k < 4$ and $I^2 \geqq 4dk(5-k)$, a trajectory of (11) on U_1 enters D.

Finally we consider the case $0 < k < 2$. We note Lemma 7 also holds in this case.

LEMMA 9. *Let I be fixed such that $I^2 > (16 + 4k)d$, and let $\varepsilon > 0$ be a sufficiently small number (depending on I, k, and d). Suppose*

$$q(0) > \{\mu_+ - \varepsilon k - (2/d) \cdot v(0)\} p(0), \quad p(0) > (I/2d) \cdot v(0).$$

Then for any $x > 0$ *as long as* $p(x) > 0$, *and* $0 < v(x) < 4d/I$,

$$q(x) > \{\mu_+ - \varepsilon k - (2/d)\cdot v(x)\}p(x), \quad p(x) > (I/2d)\cdot v(x).$$

PROOF. Suppose that there exists the smallest $x_0>0$ such that $p(x_0)=(I/2d)\cdot v(x_0)$ and $q(x_0)>\{\mu_+-\varepsilon k-(2/d)\cdot v(x_0)\}p(x_0)$. Then $p'(x_0)-(I/2d)\cdot v'(x_0)\leqq 0$. On the other hand, it follows that

$$p'(x_0) - \frac{I}{2d}v'(x_0) = q(x_0) - \frac{I}{2d}p(x_0) > \left(\frac{\sqrt{I^2-4dk}}{2d} - \frac{8}{I} - \varepsilon k\right)p(x_0) > 0,$$

which is a contradiction. Next suppose that there exists the smallest x_0 such that $p(x_0)\geqq(I/2d)\cdot v(x_0)$ and $q(x_0)=\{\mu_+-\varepsilon k-(2/d)\cdot v(x_0)\}p(x_0)$. Then it follows that

$$\begin{aligned} q'(x_0) &- \left\{\mu_+ - \varepsilon k - \frac{2}{d}v(x_0)\right\}p'(x_0) + \frac{2}{d}v'(x_0)p(x_0) \\ &= \left\{\left(\frac{I}{d} - \mu_+ + \varepsilon k\right)\left(\mu_+ - \varepsilon k - \frac{2}{d}v(x_0)\right)p(x_0)\right. \\ &\quad \left. - \frac{k}{d}p(x_0) + \frac{k}{d}p^2(x_0)\right\} \\ &\geqq kp(x_0)\left\{\varepsilon\left(\mu_+ - \varepsilon k - \frac{k}{d\mu_+}\right) + \frac{1}{d}v(x_0)\left(\frac{I}{2d} - \frac{2}{d\mu_+} - 2\varepsilon\right)\right\} > 0 \end{aligned}$$

This contradicts the choise of $x_0>0$. ■

LEMMA 10. *Suppose* $q(0)>(1/d)\cdot\{k+(2-k)p(0)\}(I/2-v(0))$. *Then for any* $x>0$ *as long as* $0<v(x)<I/2$, $q(x)>(1/d)\cdot\{k+(2-k)p(x)\}(I/2-v(x))$.

The proof is carried out in a similar way to that of Lemma 3.

We now define a cone V_3 by

$$V_3 = \{(v, p, q)|(\mu_+-\varepsilon k-2v/d)p\leqq q\leqq(I/d)\cdot p,\ (I/2d)\cdot v\leqq p\leqq(I/d)\cdot v,\ 0\leqq p\leqq 2\}.$$

Then Lemmas 7 and 9 imply that a trajectory on U_1 is trapped in V_3 while p satisfies $0<p(x)\leqq 2$. Hence it enters the cross section of V_3 at $p=2$, say S_3,

$$S_3 = \{(v, p, q)|2d/I\leqq v\leqq 4d/I,\ p=2,\ 2(\mu_+-\varepsilon k-2v/d)\leqq q\leqq 2I/d\}.$$

It follows that for $(v, p, q)\in S_3$,

$$\begin{aligned} q - \frac{1}{d}\{k + (2-k)p\}\left(\frac{I}{2} - v\right) &> 2\left(\mu_+ - \varepsilon k - \frac{2v}{d}\right) - \frac{4-k}{2d}I + \frac{1}{d}(4-k)v \\ &= 2\left(\mu_+ - \varepsilon k - \frac{4-k}{4d}I\right) - \frac{k}{d}v \\ &\geqq k\left(\frac{I}{4d} - 2\varepsilon - \frac{4}{I}\right). \end{aligned}$$

Here we used $\mu_+ - (4-k)I/4d \geqq kI/8d$ if $I^2 \geqq 64d/(8-k)$. Thus if I is chosen so that $I^2 > 16d$, and if $\varepsilon > 0$ is sufficiently small, then S_3 is on the upside of the surface $q = \{k + (2-k)p\}(I/2 - v)/d$. Therefore Lemmas 7, 9 and 10 imply that a trajectory on U_1 reaches D if $I^2 > (16 + 4k)d$.

LEMMA 11. *Let I, d, k be fixed such that $I^2 - 4dk > 0$. If a trajectory on U_1 reaches D it is a symmetric solution of* (11), (12).

PROOF. Let C_2 be a curve in which U_2 intersects the plane $q = 0$. Then one finds that C_2 has a portion which lines in the quarter plane $\{(v, p, q) | v > 0, p > 0, q = 0\}$, and a trajectory leaving any point on this portion immediately enters the octant of $\{v > 0, p > 0, q > 0\}$ in a neighborhood of O. Let γ be a curve on U_2 in a small neighborhood of the origin, connecting a point on U_1 and a point on C_2. We introduce a parameter s into γ in such a way that $\gamma = \gamma(s)$ $(0 \leqq s \leqq 1)$, $\gamma(0) \in C_2$ and $\gamma(1) \in U_1$. Furthermore, let A and B be

$$A = \{s_0 | \text{a trajectory leaving } \gamma(s_0) \text{ attains}$$
$$D^* = \{(v, p, q) | 0 < v < I/2,\ p > 0,\ q = 0\}\} \quad \text{and}$$
$$B = \{s_0 | \text{a trajectory leaving } \gamma(s_0) \text{ attains } D\},$$

respectively. The previous discussion shows that $0 \in A$, $1 \in B$. Define s^* by $\sup A = s^*$. Then $0 < s^* < 1$. By the continuity of solutions on initial values, it turns out that if $s^* \in A$, then $s \in A$, where $s > s^*$ and $s - s^*$ is sufficiently small, which is a contradiction. Using a similar argument to the above, we also find $s^* \notin B$. By Lemma 4 (or Lemma 7), any trajectory originating from the curve γ must reach D^* or $\bar{D}$. Thus, we conclude that a trajectory leaving $\gamma(s^*)$ attains the line l, which gives a symmetric solution of (11), (12). ■

PROPOSITION 1. *Fix $d > 0$. Then for each $k > 0$, the set*

$$E_k = \{I | I > 0, \text{ a solution of (11) leaving } U_2 \text{ attains } l \text{ for } I\}$$

is connected.

PROOF. We first show two lemmas.

LEMMA 12. *Consider* (15). *Fix (I, d, k) under $I^2 > 4dk$. Let $(p_1(v), q_1(v))$ (resp. $(p_2(v), q_2(v))$) be a trajectory which lies on U_1 (resp. U_2). If there is $s_1 \in (0, I/2)$ such that $q_1(s_1) = 0$, then there is also $s_2 \in (0, I/2)$ such that $q_2(s_2) = 0$.*

PROOF. Let $(p_2(v), q_2(v))$ be a trajectory lying on U_2 but not on U_1. Since $\mu_+ > \mu_- > 0$, $q_1(v) > q_2(v)$ for sufficiently small $v > 0$. If $q_2(v) > 0$ for any $v \in (0, I/2)$, there is the smallest $\bar{v}(0 < \bar{v} < s_1)$ such that $q_1(\bar{v}) = q_2(\bar{v})$. Putting $q^* =$

q_1-q_2, $p^*=p_1-p_2$ and then setting $v_1=0$, $v_2=\bar{v}$ in (19), we have

$$0=\frac{1}{d}(I-2\bar{v})p^*(\bar{v})+\frac{k}{d}\int_0^{\bar{v}}p^*(v)dv,$$

from which, it does not hold that $p^*(v)>0$ for all $v\in(0, \bar{v})$. Then, noting $p^*(v)>0$ for sufficiently small v, we find that there is the smallest $v^*(0<v^*<\bar{v})$ for which $p^*(v^*)=0$. Again setting $v_1=0$, $v_2=v^*$ in (19), we have

$$q^*(v^*)=\frac{k}{d}\int_0^{v^*}p^*(v)dv>0$$

and then $q_1(v^*)>q_2(v^*)$, that is, $p_1'(v^*)>p_2'(v^*)$. This is a contradiction. ■

From this lemma, we found that for $I^2>4dk$, a necessary and sufficient condition for the existence of solutions of (11), (12) is that a trajectory $(p(v), q(v))$ lying on the unstable manifold U_1 satisfies $q(v)>0$ for any $v\in(0, I/2)$.

LEMMA 13. *Fix $d>0$, $k>0$. Let $(\bar{p}(v), \bar{q}(v))$ (resp. $(p(v), q(v))$) be a trajectory leaving U_1 for $\bar{I}$ (resp. I) with $I>\bar{I}>2\sqrt{dk}$. If $\bar{q}(v)>0$ for any $v\in(0, \bar{I}/2)$, then $q(v)>0$ for any $v\in(0, I/2)$ and $q(I/2)>\bar{q}(\bar{I}/2)$.*

PROOF. First show that $p(v)>\bar{p}(v)$ holds for any $v\in(0, \bar{I}/2]$. We see $p(v)>\bar{p}(v)$ for sufficiently small v. Suppose that v_0 $(0<v_0\leqq\bar{I}/2)$ be the smallest for which $p(v_0)=\bar{p}(v_0)$. Then, when we put $v_1=0$, $v_2=v_0$ in (18)

$$\begin{aligned}q(v_0)&=\frac{1}{d}\{(I-2v_0)p(v_0)-kv_0+k\int_0^{v_0}p(v)dv\}\\&>\frac{1}{d}\{(\bar{I}-2v_0)\bar{p}(v_0)-kv_0+k\int_0^{v_0}\bar{p}(v)dv\}=\bar{q}(v_0),\end{aligned}$$

which leads to $p'(v_0)>\bar{p}'(v_0)$. This is a contradiction. So $p(v)>\bar{p}(v)$ for $0<v\leqq\bar{I}/2$. Using this property we can show $q(v)>\bar{q}(v)$ for $0<v\leqq\bar{I}/2$, because

$$\begin{aligned}q(v)&=\frac{1}{d}\{(I-2v)p(v)-kv+k\int_0^{v}p(v)dv\}\\&>\frac{1}{d}\{(\bar{I}-2v)\bar{p}(v)-kv+k\int_0^{v}\bar{p}(v)dv\}\\&=\bar{q}(v).\end{aligned}$$

Finally we show $q(I/2)>\bar{q}(\bar{I}/2)$. Suppose that this is not true, that is there is $v^*\in(\bar{I}/2, I/2]$ such that $q(v^*)=\bar{q}(\bar{I}/2)$. Let v^* be the smallest one for which $q(v^*)=\bar{q}(\bar{I}/2)$. Then (18) leads to

$$
\begin{aligned}
0 &= q(v^*) - \bar{q}(\bar{I}/2) \\
&= \frac{1}{d}\left\{(I-2v^*)p(v^*) - kv^* + k\int_0^{v^*} p(v)dv + \frac{k\bar{I}}{2} - k\int_0^{I/2} \bar{p}(v)dv\right\} \\
&= \frac{1}{d}\left\{(I-2v^*)p(v^*) + k(\bar{I}/2-v^*) + k\int_0^{I/2} (p(v) - \bar{p}(v))dv\right. \\
&\quad \left. + k\int_{I/2}^{v^*} p(v)dv\right\} > \frac{k}{d}\left\{(\bar{I}/2-v^*) + \int_{I/2}^{v^*} p(v)dv\right\}.
\end{aligned}
$$

If $p(v)>1$ holds for $\bar{I}/2 \leqq v \leqq v^*$, then the right hand side is positive. This is a contradiction. Thus it suffices to show $p(v)>1$. If $(\bar{p}, \bar{q})$ is a symmetric solution with parameters $(\bar{I}, d, k)$, then $\bar{p}(\bar{I}/2)>1$ by Lemma 1. If it is not a symmetric solution, Lemma 11 yields a symmetric solution (p_1, q_1) $(\neq(\bar{p}, \bar{q}))$ with parameters $(\bar{I}, d, k)$. Then it is easy to see that $\bar{p}(v)>p_1(v)$ for $0<v\leqq \bar{I}/2$. Since $p_1(\bar{I}/2)>1$, it follows that $\bar{p}(\bar{I}/2)>1$. Since $p(v)$ is monotone increasing for $\bar{I}/2\leqq v\leqq v^*$, $p(v)>p(\bar{I}/2)$ for $\bar{I}/2<v\leqq v^*$. Thus, by noting $p(\bar{I}/2)>\bar{p}(\bar{I}/2)>1$, $p(v)>1$ holds for $\bar{I}/2\leqq v\leqq v^*$. ∎

We remark that Lemma 13 also holds for $I=2\sqrt{dk}$ if $(\bar{p}, \bar{q})$ is taken to be a symmetric solution of (11) with parameters $(\bar{I}, d, k)$.

Thus Lemmas 12 and 13 immediately lead to Proposition 1. ∎

We define $I_d(k)$ by $I_d(k)=\inf E_k$ (in Proposition 1). Then, we find that for each fixed d, k there is a solution of (11), (12) for $I>I_d(k)$ and there is no solution for $0<I<I_d(k)$.

LEMMA 14. *The problem* (11), (12) *has a solution for* $I=I_d(k)$.

PROOF. Take a decreasing sequence $\{I_n\}$ such that $I_n \downarrow I_0=I_d(k)$, $I_n<I_0+1$. Let u_n be a solution of (11), (12) for $I=I_n$ $(>2\sqrt{dk})$ with $\max u_n=u_n(0)$. It follows from Lemmas 4 and 7 that u_n, u_n' are uniformly bounded. Therefore, from (6) u_n'' is also uniformly bounded. Furthermore, we find that u_n''' is also uniformly bounded. This information shows that $\{u_n\}$, $\{u_n'\}$, $\{u_n''\}$ are equicontinuous. Thus, we find that there exist suitable subsequences $\{u_n\}$, $\{u_n'\}$, $\{u_n''\}$ and a function u such that $u_n\to u$, $u_n'\to u'$, $u_n''\to u''$ uniformly on compact subsets of $\boldsymbol{R}$. Using the property $u_n(x)=u_n(-x)$, we find

$$\int_{-\infty}^{x} u_n(y)dy - \int_{x}^{+\infty} u_n(y)dy = 2\int_0^x u_n(y)dy.$$

Then u_n satisfies

$$du_n'' + \left(2\int_0^x u_n(y)dy\right)u_n' + 2u_n^2 + ku_n(1-u_n) = 0,$$

and then as $n\to\infty$,

$$du'' + \left(2\int_0^x u(y)dy\right)u' + 2u^2 + ku(1-u) = 0.$$

On the other hand, noting

$$\int_{-\infty}^x u(y)dy$$

$$= \int_{-\infty}^x \lim_{n\to\infty} u_n(y)dy \leqq \liminf_{n\to\infty} \int_{-\infty}^x u_n(y)dy \leqq \lim_{n\to\infty} I_n = I_0,$$

we find that $u \in L_1(\boldsymbol{R})$, $u(x)=u(-x)$ and $u(0)\geqq 1$. Thus $u(x)$ becomes a symmetric solution. Finally putting $I=\int_{-\infty}^{\infty} u(y)dy$, we have $I\leqq I_0=I_d(k)$, so that by the definition of $I_d(k)$, $I=I_0$. ■

Finally we show a non-existence result.

PROPOSITION 2. *Let I, d, k be fixed such that $0<k<2$ and $I^2\geqq 4dk$. If $I^4\leqq 2d^2$, then there exist no solutions of* (11) *and* (12).

PROOF. By Lemma 7, a symmetric solution must satisfy $q\leqq(I/d)\cdot p$, $p\leqq (I/d)\cdot v$, as long as $0<v\leqq I/2$. (Lemma 7 also holds when $I^2=4dk$). This means that $u'(x)\leqq I^3/2d^2$ if $u'(x)>0$. From Lemma 1, there exists $x_0 \in \boldsymbol{R}$ such that $u(x_0)=1$ and $u'(x_0)>0$. Then we have

$$u(x) \geqq \max\{0,\ 1-(I^3/2d^2)\cdot(x_0-x)\}$$

for $x<x_0$. One can easily see that

$$\frac{I}{2} > \int_{-\infty}^{x_0} u(y)\,dy > \frac{d^2}{I^3},$$

that is, $I^4>2d^2$, which completes the proof. ■

Thus, we arrive at Theorem B.

4. Discussion

The paper is mainly devoted to the proof of the existence of stationary wave solutions when the kernel K takes the specific form. Unfortunately we are unable to discuss stability of the solutions. It is the main difficiculty that many infinite number of solutions $u(x)$ exist for $\int_{-\infty}^{\infty} u(x)dx \geq I_d(k)$. Here, we give less rigorous and heuristic argument on stability by using singular perturbation techniques. Let us consider the following initial value problem of (6) with $d=\varepsilon$, $k=1+1/\varepsilon$:

$$u_t = \varepsilon u_{xx} + [(2v-I(t))u]_x + (1+1/\varepsilon)u(1-u),$$

(21)
$$u(x, 0) = u_0(x) = \begin{cases} 1 & (|x|<I_0/2), \\ 0 & (|x|>I_0/2) \end{cases}$$

where $I(t)=\int_{-\infty}^{\infty} u(y, t)dy$, $v(x, t)=\int_{-\infty}^{x} u(y, t)dy$. Let us consider in which direction the fronts will move when $\varepsilon(>0)$ is sufficiently small. We observe the movement of the front at $x=I_0/2$ only. Introducing the stretched variables $\tau=t/\varepsilon$, $\xi=(x-I_0/2)/\varepsilon$ in (21), we obtain

(22)
$$u_\tau = u_{\xi\xi} + [(2v-I(\varepsilon\tau))u]_\xi + (1+\varepsilon)u(1-u).$$

When ε formally tends to zero, (22) may be approximately reduced to

(23)
$$u_\tau = u_{\xi\xi} + I_0 u_\xi + u(1-u) \quad \xi \in \boldsymbol{R}.$$

The initial condition is

(24)
$$u(\xi, 0) = \begin{cases} 1 & (\xi<0), \\ 0 & (\xi>0). \end{cases}$$

It is well known that there exists a traveling wave solution $U(\xi+(I_0-2)\tau)$ and a function $\theta(\tau)$ with $\theta(\tau)=O(\log \tau)$ such that the solution $u(\tau, \xi)$ of (23), (24) satisfies

$$u(\xi, \tau) - U(\xi+(I_0-2)\tau+\theta(\tau)) \longrightarrow 0 \text{ uniformly in } \xi, \text{ as } \tau \longrightarrow \infty.$$

This result implies that the front at $x=I_0/2$ moves to the left if $I_0>2$, and if $I_0<2$ it moves to the right. On the other hand, Theorem B states that stationary solutions of (21) with the boundary conditions $u(-\infty)=u(+\infty)=0$ exist for each $I=I(\infty)\geqq 2$, when ε tends to 0. Thus, we conjecture that a realizable stationary solution is the one with $I=2$ (the minimum total population number), if initial functions are of compact support. This is numerically confirmed, though the above argument is admittedly far from being rigorous.

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Department of Mathemaitcs,
Faculty of Science,
Hiroshima University,