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Generalizations of Witt algebras over a field of characteristic zero

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Introduction

In this paper we investigate the structure of generalizations of Witt algebras over a field $\mathfrak{k}$ of characteristic zero, and consider a class of infinite-dimensional simple Lie algebras over $\mathfrak{k}$. Let I be a non-empty index set and G be an additive subgroup of $\prod_{i\in I}\mathfrak{k}_i^+$, where $\mathfrak{k}_i^+$ $(i\in I)$ are copies of the additive group $\mathfrak{k}$. Let $W(G, I)$ be the Lie algebra over $\mathfrak{k}$ with basis $\{w(a, i)\,|\,a\in G,\ i\in I\}$ and the multiplication

$$[w(a, i), w(b, j)] = a_j w(a+b, i) - b_i w(a+b, j),$$

where $i, j\in I$ and $a=(a_i)_{i\in I}$, $b=(b_i)_{i\in I}\in G$. The Lie algebra $W(G, I)$ is infinite-dimensional if $G\neq 0$.

We note that if the field $\mathfrak{k}$ is of characteristic $p>0$, then $W(G, I)$ is isomorphic to the generalized Witt algebra defined by Kaplansky [3]. It is known that the generalized Witt algebra is simple if G is "total" and $\mathfrak{k}$ is of characteristic $p>2$ [3] (see also Ree [5], Seligman [6], and Wilson [7]). It is also known that $W(G, I)$ is simple if $|I|=1$, $G\neq 0$, and $\mathfrak{k}$ is of characteristic $\neq 2$ [2, p. 206].

The main results of this paper are as follows: If $G\neq 0$, then $W(G, I)$ is a direct sum of the unique maximal ideal R of $W(G, I)$ and a simple subalgebra S of $W(G, I)$, where S is isomorphic to $W(H, J)$ for some H and J (Theorem 3.1). If $G\neq 0$, then the following statements are equivalent: (i) $W(G, I)$ is simple; (ii) $R=0$; (iii) the center of $W(G, I)$ is 0; (iv) G is "total" (Corollary 3.2). $W(G, I)$ is a finitely generated Lie algebra if and only if I is a finite set and G is a finitely generated group (Theorem 4.1). If $I=\{1,\dots, n\}$ and $G=\oplus_{i=1}^n\mathbf{Z}_i$, then $W(G, I)$ is isomorphic to the derivation algebra of $\mathfrak{k}[x_1, x_1^{-1},\dots, x_n, x_n^{-1}]$ (Proposition 4.2).

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1. Notation and preliminary results

Throughout this paper the ground field $\mathfrak{k}$ is of characteristic zero and Lie algebras over $\mathfrak{k}$ are not necessarily finite-dimensional. Let L be a Lie algebra over $\mathfrak{k}$. If L has no ideals except 0 and L, and if $L^2 \neq 0$, we call L *simple*. L is *perfect* if $L^2 = L$. If H is a subalgebra of L we write $H \leq L$, and if H is an ideal we write $H \lhd L$. Let $H \leq L$. Then $I_L(H)$ and $C_L(H)$ denote the idealizer and the centralizer of H in L, respectively. We write $\zeta(L)$ for the center of L. If S is a subset of L we let $\langle S \rangle$ denote the subalgebra of L generated by S. For n-fold products we use the notation: $[a, {}_0 b] = a$, $[a, {}_{n+1} b] = [[a, {}_n b], b]$ for all $n \geq 0$, where $a, b \in L$. For a set A we denote by $|A|$ the cardinality of A. Notation and terminology not mentioned above may be found in [2].

We simply write W instead of $W(G, I)$ if there would be no confusion. Since I is supposed to be non-empty, W has basis elements $w(0, i)$ $(i \in I)$, and hence $\dim W \geq 1$. For each $a \in G$ let W_a be the subspace of W spanned by $\{w(a, i) \mid i \in I\}$. Then it is clear that $W = \oplus_{a \in G} W_a$ and $[W_a, W_b] \subseteq W_{a+b}$ $(a, b \in G)$. Hence W is a G-graded Lie algebra. Let H_a denote $H \cap W_a$ for a subalgebra H of W and $a \in G$.

Let W_0^* be the dual space of W_0. Then we can identify W_0^* and $\prod_{i \in I} \mathfrak{k}_i^+$ by the group isomorphism $\phi: \prod_{i \in I} \mathfrak{k}_i^+ \to W_0^*$ defined by $\phi(a)(w(0, i)) = a_i$ $(i \in I)$ for each $a = (a_i)_{i \in I} \in \prod_{i \in I} \mathfrak{k}_i^+$. Hence G is a subgroup of W_0^*. Let $a \in G$, $x \in W_0$ and $x = \sum_{i \in I} \alpha_i w(0, i)$, where $\alpha_i \in \mathfrak{k}$ and $\alpha_i = 0$ for all but a finite number of indices i. Then $a(x) = \sum_i \alpha_i a(w(0, i)) = \sum_i a_i \alpha_i$. If $a \neq 0$, then $a_i \neq 0$ for some $i \in I$, and hence $a(x) = a_i \neq 0$ for $x = w(0, i) \in W_0$.

For each $a \in G$ let t_a: $W \to W$ be the linear automorphism of W defined by

$$t_a(w(b, i)) = w(a+b, i) \quad (b \in G,\ i \in I).$$

Then clearly $W_a = t_a(W_0)$ and $W = \oplus_{a \in G} t_a(W_0)$. We begin with the following technical lemma.

LEMMA 1.1. *Let $a, b \in G$ and $x, y \in W_0$. Then*
(i) $[t_a(x), t_b(y)] = a(y) t_{a+b}(x) - b(x) t_{a+b}(y)$.
(ii) $[t_a(x), y] = a(y) t_a(x)$.
(iii) *If $a \neq 0$ then $t_a(x) = a(z)^{-1}[t_a(x), z]$ for some $z \in W_0$.*

PROOF. (i) Let $t_a(x) = \sum_{i \in I} \alpha_i w(a, i)$, $t_b(y) = \sum_{j \in I} \beta_j w(b, j)$, where α_i and β_j are zero for all but finite sets of $i \in I$ and $j \in I$. Then $[t_a(x), t_b(y)] = \sum_{i,j} \alpha_i \beta_j \cdot [w(a, i), \ w(b, j)] = (\sum_j a_j \beta_j) \ (\sum_i \alpha_i w(a+b, i)) - (\sum_i b_i \alpha_i) \ (\sum_j \beta_j w(a+b, j)) = a(y) t_{a+b}(x) - b(x) t_{a+b}(y)$.

(ii) follows from (i) by letting $b = 0$, and (iii) follows from (ii) since $a(z) \neq 0$ for some $z \in W_0$.

It is clear that W_0 is an abelian subalgebra of W. Furthermore, we have the following

LEMMA 1.2. (i) $\zeta(W) \subseteq W_0 = I_W(W_0)$.

(ii) $\zeta(W) = \{x \in W_0 \mid a(x) = 0 \quad \textit{for any} \quad a \in G\}$.

PROOF. (i) Let $x \in I_W(W_0)$. Then $x = \sum_{a \in G} x_a$, where $x_a \in W_a$. By Lemma 1.1 (ii) we have

$$\textstyle\sum_{a \in G} a(y)x_a = [x, y] \in W_0 \quad (y \in W_0).$$

Hence $a(y)x_a = 0$ if $a \neq 0$. However, if $a \neq 0$ then $a(y) \neq 0$ for some $y \in W_0$, whence $x_a = 0$ for any $a \neq 0$. Thus $x = x_0 \in W_0$ and $I_W(W_0) = W_0$. Clearly $\zeta(W) \subseteq C_W(W_0) \subseteq I_W(W_0) = W_0$.

(ii) Since $\zeta(W) \subseteq W_0$ by (i), it follows immediately from Lemma 1.1 (ii) that $x \in \zeta(W)$ if and only if $x \in W_0$ and $a(x) = 0$ for any $a \in G$.

Note that W_0 is a Cartan subalgebra of W by Lemma 1.2 (i). Let $x \in W_0$. Then $[t_a(y), x] = a(x)t_a(y)$ for any $a \in G$ and $y \in W_0$. Hence $G \backslash \{0\}$ is the set of roots of W relative to W_0, and $W = W_0 \oplus (\oplus_{a \in G \backslash \{0\}} W_a)$ is a root space decomposition.

LEMMA 1.3. *Let H be an ideal of W. Then $H = W$ if $W_a \subseteq H$ for some $a \in G$.*

PROOF. If $G = 0$, then clearly $W = W_0 = H$. So we assume that $G \neq 0$. If $W_0 \subseteq H$ and b is a non-zero element of G, then there exists $x \in W_0$ such that

$$t_b(y) = b(x)^{-1}[t_b(y), x] \in H \quad (y \in W_0)$$

by Lemma 1.1 (iii). Hence $W_b \subseteq H$ $(0 \neq b \in G)$ and $H = W$. If $W_a \subseteq H$ for some $a \neq 0$, then $a(x) \neq 0$ for some $x \in W_0$, and hence for any $y \in W_0$

$$y = a(x)^{-1}[t_a(x), t_{-a}(y) - \frac{1}{2}a(y)a(x)^{-1}t_{-a}(x)] \in H$$

since $t_a(x) \in W_a$. Thus $W_0 \subseteq H$ and by the argument above we have $W = H$.

Now we have the following

PROPOSITION 1.4. (i) *W is abelian if and only if $G = 0$.*
(ii) *W is non-abelian and perfect if and only if $G \neq 0$.*

PROOF. If $G = 0$, then clearly $W^2 = 0$. So let $G \neq 0$ and a be a non-zero element of G. Then by Lemma 1.1 (iii) there exists $x \in W_0$ such that

$$t_a(y) = a(x)^{-1}[t_a(y), x] \in W^2 \quad (y \in W_0).$$

Hence $W_a \subseteq W^2$, and so $W = W^2$ by Lemma 1.3. Thus W is non-abelian and perfect. The 'only if' parts are obvious.

COROLLARY 1.5. *$\zeta(W) \subsetneq W_0$ if and only if $G \neq 0$.*

PROOF. If $G=0$ then $W_0 \subseteq \zeta(W) \subseteq W_0$ by the proposition and Lemma 1.2 (i). Thus $\zeta(W)=W_0$. Conversely if $\zeta(W) \supseteq W_0$, then $\zeta(W)=W$ by Lemma 1.3, i.e. W is abelian. Hence $G=0$ by the proposition.

REMARK 1.6. If $|I|=1$ and $G=\mathbf{Z}$, then W is simple and satisfies the maximal condition for subalgebras (see [1], [2], and [4]). But if $|I|>1$, then in general W has non-trivial ideals, and dose not satisfy the maximal condition for subalgebras. For example let $I=\{1, 2\}$ and $G=\langle a\rangle \leq \mathfrak{k}^+ \oplus \mathfrak{k}^+$, where $a=(1, 0)$. Then $G \simeq \mathbf{Z}$ and W has a basis $\{w(na, i) \mid n \in \mathbf{Z}, i=1, 2\}$ over $\mathfrak{k}$. Let H be the subspace of W spanned by $\{w(na, 2) \mid n \in \mathbf{Z}\}$. Then it is easy to see that H is an infinite-dimensional abelian ideal of W.

2. Ideals of $W(G, I)$

In this section we show that W has a radical. We begin with the following

LEMMA 2.1. *Every ideal of W is G-homogeneous.*

PROOF. Let H be a non-zero ideal of W. Let x be a non-zero element of H and $x=\sum_{a\in G} x_a$, where $x_a \in W_a$. Set $A(x)=\{a \in G \mid x_a \neq 0\}$. Clearly $A(x)$ is a finite set. We show by induction on $|A(x)|$ that $x_a \in H$ for any $a \in A(x)$, and we conclude that $H=\bigoplus_{a\in G} H_a$. If $|A(x)|=1$ the result is obvious. Suppose that $|A(x)|>1$. Let $a, b \in A(x)$ and $a \neq b$. Then there is y in W_0 such that $a(y) \neq b(y)$. Let $n=|A(x)|$ and $\{c_1,\dots, c_n\}=\{a(y) \mid a \in A(x)\}$, where $c_r \neq c_s$ whenever $r \neq s$. For each $r \in \{1,\dots, n\}$, set

$$A_r(x) = \{a \in A(x) \mid a(y)=c_r\}, \quad x_r = \textstyle\sum_{a\in A_r(x)} x_a .$$

Then $x=\sum_{r=1}^n x_r \in H$. Since $H \lhd W$, we have $\sum_{r=1}^n c_r x_r=[x, y] \in H$. Hence it follows by the second induction on m that

$$(*) \qquad \textstyle\sum_{r=1}^n c_r^m x_r = [x, {}_m y] \in H \qquad (m = 0, 1,\dots, n-1).$$

Now the coefficients c_r^m make an $n \times n$ matrix (c_r^m), and $\det(c_r^m)$ is a Vandermonde determinant, which is non-zero. Consequently from $(*)$ we have

$$x_r = \textstyle\sum_{a\in A_r(x)} x_a \in H \qquad (r = 1,\dots, n).$$

Since $|A(x_r)|=|A_r(x)|<n$, we have $x_a \in H$ for $a \in A_r(x)$ by the inductive hypothesis. Therefore $x_a \in H$ for any $a \in A(x)$.

We give a criterion for an ideal of W to be proper.

LEMMA 2.2. *Let H be an ideal of W, and let $G \neq 0$. Then*
(i) *$H \neq 0$ if and only if $H_0 \neq 0$.*

(ii) *$H \neq W$ if and only if $H_0 \subseteq \zeta(W)$.*

PROOF. (i) Let $H \neq 0$. Then $H_a \neq 0$ for some $a \in G$ by Lemma 2.1. We may assume that $a \neq 0$ since if $a=0$ it is trivial that $H_0 \neq 0$. Let $x \in H_a$ and $x \neq 0$. Then $x = t_a(y)$ for some $0 \neq y \in W_0$. If $a(y) \neq 0$, then by Lemma 1.1 (i) we have $y = \frac{1}{2} a(y)^{-1}[x, t_{-a}(y)] \in [H_a, L_{-a}] \subseteq H_0$. If $a(y)=0$, since $a(z) \neq 0$ for some $z \in W_0$, $y = a(z)^{-1}[x, t_{-a}(z)] \in H_0$. In both cases we have $H_0 \neq 0$. The converse is trivial.

(ii) Let $H_0 \subseteq \zeta(W)$. Since $G \neq 0$, $\zeta(W) \subsetneq W_0$ by Corollary 1.5. Hence $H_0 \neq W_0$, and so $H \neq W$. Conversely, assume that $H_0 \not\subseteq \zeta(W)$. Let $x \in H_0 \backslash \zeta(W)$. Then $[t_a(y), x] = a(x) t_a(y) \neq 0$ for some $a \in G$, $t_a(y) \in W_a$, where $0 \neq y \in W_0$. Hence $a(x) \neq 0$, and so $t_a(z) = a(x)^{-1}[t_a(z), x] \in H$ for any $z \in W_0$, i.e. $W_a \subseteq H$. Therefore $H = W$ by Lemma 1.3.

Now we have the main theorem of this section.

THEOREM 2.3. *Let $G \neq 0$. Then there exists a proper ideal R of W which satisfies the following properties*:

(i) *$R_0 = \zeta(W)$ and $R_a = t_a(\zeta(W))$ for any $a \in G$.*

(ii) *R is abelian.*

(iii) *R contains every proper ideal of W.*

PROOF. We set $R_a = t_a(\zeta(W))$ for each $a \in G$, and $R = \oplus_{a \in G} R_a$. By Corollary 1.5 we have $\zeta(W) \subsetneq W_0$, whence R is a proper subspace of W. Let $u \in R_a$ and $v \in W_b$. Then $u = t_a(x)$, $v = t_b(y)$ for some $x \in \zeta(W)$, $y \in W_0$. Since $b(x)=0$ by Lemma 1.2 (ii), we have $[u, v] = a(y) t_{a+b}(x)$ by Lemma 1.1 (i). Hence $[u, v] \in R_{a+b}$ by definition of R, and so $R \lhd W$. Further, if $v \in R_b$, then $y \in \zeta(W)$ and $a(y)=0$. Hence $[u, v]=0$, i.e. R is abelian. Thus R is a proper ideal of W satisfying (i) and (ii).

Let $H = \oplus_{a \in G} H_a$ be a proper ideal of W. By Lemma 2.2 (ii), $H_0 \subseteq \zeta(W) = R_0$. If $H_a = 0$ for any $0 \neq a \in G$, then $H = H_0 \subseteq R_0 \subseteq R$. So let $H_a \neq 0$ for some $0 \neq a \in G$. If u is a non-zero element of H_a, then $u = t_a(x)$ for some $0 \neq x \in W_0$. Since $a \neq 0$, $a(y) \neq 0$ for some $0 \neq y \in W_0$. Now we have

$$(*) \qquad a(y)x + a(x)y = [t_a(x), t_{-a}(y)] \in [H_a, W_{-a}] \subseteq H_0 \subseteq \zeta(W).$$

Hence $2a(x)a(y)t_a(x) = [t_a(x), a(y)x + a(x)y] = 0$, and so $a(x)=0$. Therefore $a(y)x = [t_a(x), t_{-a}(y)] \in \zeta(W)$ from $(*)$, and hence $x \in \zeta(W)$, i.e. $u \in R_a$. Thus $H_a \subseteq R_a$ and it follows that $H \subseteq R$. This completes the proof.

COROLLARY 2.4. *Let $G \neq 0$. Then every proper ideal of W is abelian.*

We call R of Theorem 2.3 the *radical* of W.

3. The structure of $W(G, I)$

In this section we give a structure theorem for $W(G, I)$, which is one of the main results of this paper.

THEOREM 3.1. *Let $G \neq 0$, and let R be the radical of W. Then there exists a subalgebra S of W which satisfies the following conditions*:

(i) $W = R \oplus S$.

(ii) *S is simple.*

(iii) *S is isomorphic to $W(H, J)$ for some H, J.*

PROOF. Let ϕ: $W_0 \to W_0/R_0$ be the natural map. Then $\{\phi(w(0, i)) \mid i \in I\}$ spans W_0/R_0, which is non-zero by Corollary 1.5. Hence there exists a non-empty subset J of I such that $\{\phi(w(0, j)) \mid j \in J\}$ is a basis of W_0/R_0. Let S_0 be the subspace of W_0 spanned by $\{w(0, j) \mid j \in J\}$. Then clearly $W_0 = R_0 \oplus S_0$. Let $S_a = t_a(S_0)$ for each $a \in G$, and let $S = \oplus_{a \in G} S_a$. Then $W_a = t_a(R_0) \oplus t_a(S_0) = R_a \oplus S_a$, whence

$$W = \oplus_{a \in G} W_a = R \oplus S.$$

We claim that S is a simple subalgebra of W. Let $u \in S_a$, $v \in S_b$. Then $u = t_a(x)$, $v = t_b(y)$ for some x, $y \in S_0$, and $[u, v] = a(y)t_{a+b}(x) - b(x)t_{a+b}(y) \in S_{a+b}$ by Lemma 1.1 (i). Hence $S \leq W$. Clearly $S \simeq W/R$, and so S has no proper ideals by Theorem 2.3. Furthermore, since $\dim S_a \geq 1$ for each $a \in G$, S is not abelian. Thus S is simple, as claimed.

Now we show (iii). Let ψ: $S_0 \to W_0$ be the inclusion map, and ψ^*: $W_0^* \to S_0^*$ be the dual map of ψ. We fix bases $\{w(0, i) \mid i \in I\}$ of W_0 and $\{w(0, j) \mid j \in J\}$ of S_0. Then we can identify W_0^*, S_0^* with $\prod_{i \in I} \mathfrak{k}_i^+$, $\prod_{j \in J} \mathfrak{k}_j^+$, respectively. For any $a = (a_i)_{i \in I} \in G$ we have

$$\psi^*(a)(w(0, j)) = a(w(0, j)) = a_j \qquad (j \in J),$$

and so $\psi^*(a) = (a_j)_{j \in J}$. We claim that $\psi^*|_G$ is injective. Let $\psi^*(a) = 0$, where $a \in G$. Then $a(x) = 0$ for any $x \in S_0$. On the other hand $a(y) = 0$ for any $y \in \zeta(W) = R_0$ by Lemma 1.2 (ii). Thus $a(z) = 0$ for any $z \in R_0 \oplus S_0 = W_0$, i.e. $a = 0$ as claimed. Therefore $\psi^*|_G$: $G \to \psi^*(G)$ is a group isomorphism, and $\psi^*(G) \leq \prod_{j \in J} \mathfrak{k}_j^+$. Let $H = \psi^*(G)$. It is easy to see that the linear map ρ: $S \to W(H, J)$ defined by $\rho(w(a, j)) = w(\psi^*(a), j)$ $(a \in G, j \in I)$ is an isomorphism.

Let the field $\mathfrak{k}$ be of characteristic $p > 0$. Then an additive subgroup G of $\prod_{i \in I} \mathfrak{k}_i^+$ is called total by Kaplansky [3] if the only element $\alpha = (\alpha_i)_{i \in I}$, where $\alpha_i = 0$ for all but a finite set of i, such that $\sum_{i \in I} a_i \alpha_i = 0$ for any $a = (a_i)_{i \in I} \in G$ is the zero-element. It is known that if characteristic $p > 2$ and G is total then $W(G, I)$ is simple as remarked in the introduction.

We use the same terminology for a field of characteristic zero. Then we have the following

COROLLARY 3.2. *Let $G \neq 0$. Then the following conditions are equivalent*:
(i) *W is simple.*
(ii) *The radical R of W is zero.*
(iii) *The center $\zeta(W)$ of W is zero.*
(iv) *G is total.*

PROOF. Clearly (i)⇒(iii), (iii)⇒(ii) by Theorem 2.3 (i), and (ii)⇒(i) by the above theorem.

Let $\alpha=(\alpha_i)_{i\in I}$, where $\alpha_i \in \mathfrak{k}$ and $\alpha_i=0$ for all but a finite set of i, and let $x=\sum_{i\in I}\alpha_i w(0, i)$ in W_0. We consider that $G \leq W_0^*$ as before. Then $a(x)=\sum_{i\in I} a_i\alpha_i$ for any $a=(a_i)_{i\in I}\in G$. Hence G is total if and only if $\{x \in W_0 \mid a(x)=0$ for any $a\in G\}=0$, which is equivalent to $\zeta(W)=0$ by Lemma 1.2 (ii).

We give a sufficient condition for W to be simple.

COROLLARY 3.3. *If the subspace of $\prod_{i\in I}\mathfrak{k}_i^{\dagger}$ spanned by G contains the direct sum $\bigoplus_{i\in I}\mathfrak{k}_i^{\dagger}$, then W is simple.*

PROOF. For each $j\in I$ let $e^{(j)}=(\delta_{ji})_{i\in I}$, where δ_{ji} is the Kronecker delta. Then clearly $e^{(j)}\in \bigoplus_{i\in I}\mathfrak{k}_i^{\dagger}$, and hence $e^{(j)}=\sum_r \alpha_r a_r$ for some finite sets $\{\alpha_r\}\subseteq \mathfrak{k}$, $\{a_r\}\subseteq G$. Let $x=\sum_{i\in I}\beta_i w(0, i)\in\zeta(W)$, where $\beta_i=0$ for all but a finite set of i. Then we have $e^{(j)}(x)=\sum_{i\in I}\beta_i e^{(j)}(w(0, i))=\beta_j$. But $e^{(j)}(x)=\sum_r \alpha_r a_r(x)=0$ by Lemma 1.2 (ii). Thus $\beta_j=0$ for any $j\in I$, i.e. $x=0$. Hence $\zeta(W)=0$, and therefore W is simple by Corollary 3.2.

4. Finitely generated Lie algebras

In this section we consider finitely generated Lie algebras.

THEOREM 4.1. *W is finitely generated if and only if I is finite and G is finitely generated.*

PROOF. Let $W=\langle x_1,\dots, x_n\rangle$, where n is a positive integer. Then there exists a finite set of basis elements $\{w(a_r, i_r)\mid r=1,\dots, m\}$ such that $x_1,\dots, x_n$ are spanned by $\{w(a_r, i_r)\mid r=1,\dots, m\}$. Hence $L=\langle w(a_r, i_r)\mid r=1,\dots, m\rangle$, so that for any $a\in G$, $i\in I$,

$$w(a, i) = \sum_{1\leq r_1,\dots,r_h\leq m}\alpha_{r_1,\dots,r_h}[w(a_{r_1}, i_{r_1}),\dots, w(a_{r_h}, i_{r_h})],$$

where $\alpha_{r_1,\dots,r_h}\in\mathfrak{k}$. It is easy to see that

$$[w(a_{r_1}, i_{r_1}),\dots, w(a_{r_h}, i_{r_h})] = \sum_{s=1}^{m} \beta_s w(a_{r_1}+\cdots+a_{r_h}, i_s),$$

for some $\beta_s \in \mathfrak{k}$. Thus $a=a_{r_1}+\cdots+a_{r_h}$ for some $r_1,\dots, r_h \in \{1,\dots, m\}$ and $i=i_s$ for some $s\in\{1,\dots, m\}$. Therefore I is finite and $G=\langle a_1,\dots, a_m\rangle$.

Conversely, suppose that $|I|=n$ and G is finitely generated. If $G=0$ then W is finite-dimensional since $\{w(0, i) \mid i\in I\}$ is a basis of W. So assume that $G\neq 0$. Since G is torsion-free, G is a free abelian group of finite rank. Let $G=\oplus_{h=1}^{m}\langle a^{(h)}\rangle$, where m is the rank of G, and let

$$F = \langle w(-2a^{(h)}, i), w(3a^{(h)}, i) \mid 1 \leq h \leq m, i\in I\rangle.$$

Clearly F is finitely generated. We show by induction on m that $W=F$.

Let $m=1$. Then $G=\langle a^{(1)}\rangle$. Since $a^{(1)}=(a_i^{(1)})_{i\in I}\neq 0$, there is $j\in I$ such that $a_j^{(1)}\neq 0$. Since $[w(ra^{(1)}, j), w(sa^{(1)}, j)]=(r-s)a_j^{(1)}w((r+s)a^{(1)}, j)$, it is not hard to see that $\{w(ra^{(1)}, j) \mid r\in \mathbf{Z}\}\subseteq\langle w(-2a^{(1)}, j), w(3a^{(1)}, j)\rangle\subseteq F$. Hence for $r\in\mathbf{Z}$ and $i\neq j$ we have

$$\begin{aligned} w(ra^{(1)}, i) = (2a_j^{(1)})^{-1}([w((r+2)a^{(1)}, j), w(-2a^{(1)}, i)] \\ - (r+2)a_i^{(1)}w(ra^{(1)}, j))\in F, \end{aligned}$$

i.e. $\{w(ra^{(1)}, i) \mid r\in\mathbf{Z}\}\subseteq F$, where $i\in I$ and $i\neq j$. Thus $W(G, I)=F$.

Let $m>1$, and let

$$H = \oplus_{h=1}^{m-1}\langle a^{(h)}\rangle, \quad K = \langle a^{(m)}\rangle.$$

Then $G=H\oplus K$. Inductively we may assume that $W(H, I)\subseteq F$, $W(K, I)\subseteq F$. Let x be a non-zero element of G. Then $x=y+z$ for some $y=(y_i)_{i\in I}\in H$, $z=(z_i)_{i\in I}\in K$. It is clear that $y\neq z$, whence $y_j\neq z_j$ for some $j\in I$. Hence

$$w(x, j) = (y_j-z_j)^{-1}[w(y, j), w(z, j)]\in[w(H, I), W(K, I)] \subseteq F.$$

Now either $y_j\neq 0$ or $z_j\neq 0$. If $y_j\neq 0$, then

$$w(x, i) = y_j^{-1}([w(y, i), w(z, j)] + z_i w(x, j))\in F$$

for any $i\neq j$. If $z_j\neq 0$, then similarly $w(x, i)\in F$ for any $i\neq j$. Thus $\{w(x, i) \mid i\in I\}\subseteq F$ for $0\neq x\in G$. It is clear that $\{w(0, i) \mid i\in I\}\subseteq W(H, I)\subseteq F$. Therefore $\{w(x, i) \mid x\in G, i\in I\}\subseteq F$, i.e. $W(G, I)=F$.

Finally we have the following

PROPOSITION 4.2. *Let $I=\{1,\dots, n\}$, n a positive integer, and let $G=\oplus_{i=1}^{n}\mathbf{Z}_i$ with copies $\mathbf{Z}_i$ of $\mathbf{Z}$. Then $W(G, I)$ is isomorphic to the derivation algebra of $\mathfrak{k}[x_1, x_1^{-1},\dots, x_n, x_n^{-1}]$ in indeterminates $x_1,\dots, x_n$.*

PROOF. Let $R=\mathfrak{k}[x_1, x_1^{-1},\dots, x_n, x_n^{-1}]$. For $x=\prod_{j=1}^{n} x_j\in R$ and $a\in G$ we write $x^a=\prod_{j=1}^n x_j^{a_j}$, where $a=(a_j)_{j\in I}$. For $a\in G$, $i\in I$ we define a linear endomorphism $\delta(a, i)\colon R\to R$ by

$$x^r\delta(a, i) = r_i x^{r+a} \qquad (r = (r_j)_{j\in I}\in G).$$

Let $D=\{\delta(a, i)\mid a\in G,\ i\in I\}$. It is easy to see that $\delta(a, i)$ is a derivation of R. Straightforward calculation shows that

$$\delta(a, i)\delta(b, j) - \delta(b, j)\delta(a, i) = a_j\delta(a+b, i) - b_i\delta(a+b, j)$$

for any $\delta(a, i)$, $\delta(b, j)\in D$, i.e.

$$[\delta(a, i), \delta(b, j)] = a_j\delta(a+b, i) - b_i\delta(a+b, j).$$

We claim that D spans $\operatorname{Der} R$. Let δ be a derivation of R. Then for each $i\in I$ we have $x_i\delta=\sum_{a\in G}\alpha(a, i)x^a$, where $\alpha(a, i)\in\mathfrak{k}$ and $\alpha(a, i)=0$ for all but a finite set of a. Let $e^{(i)}=(\delta_{ij})_{j\in I}$ with the Kronecker delta δ_{ij}, and let

$$\delta' = \sum_{i=1}^n \sum_{a\in G}\alpha(a, i)\delta(a-e^{(i)}, i).$$

Then $\delta'\in D$. Since $x_i\delta(a-e^{(j)}, j)=0$ whenever $i\neq j$, we have

$$x_i\delta' = \sum_{a\in G}\alpha(a, i)x_i\delta(a-e^{(i)}, i) = \sum_{a\in G}\alpha(a, i)x^a = x_i\delta \quad (i\in I).$$

Clearly the value $x_i^{-1}\delta'$ is determined by $x_i\delta'$. Therefore $\delta'=\delta$, and hence D spans $\operatorname{Der} R$, as claimed.

Furthermore, we show that D is linearly independent. Suppose that $\sum_{i=1}^n\sum_{a\in G}\alpha(a, i)\delta(a, i)=0$, where $\alpha(a, i)\in\mathfrak{k}$ and $\alpha(a, i)=0$ for all but a finite set of a. Then we have

$$x_j\sum_{i=1}^n\sum_{a\in G}\alpha(a, i)\delta(a, i) = \sum_{a\in G}\alpha(a, j)x^{a+e^{(j)}} = 0 \qquad (j\in I).$$

Hence $\alpha(a, j)=0$ for any $a\in G$, $j\in I$.

Since $\operatorname{Der} R$ has a basis D, it is clear that $\operatorname{Der} R$ is isomorphic to $W(G, I)$, where $G=\bigoplus_{i=1}^n \mathbf{Z}_i$ and $I=\{1,\dots, n\}$, by the map $\delta(a, i)\mapsto w(a, i)$.

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