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A certain series of unitarizable representations of Lie superalgebras $\mathfrak{gl}(p|q)$

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0. Introduction

In this note, we construct a series of irreducible and unitarizable representations of Lie superalgebras $\mathfrak{gl}(p|q)$ $(1 \leqq p, q \leqq \infty)$. Our representations have two faces — a generalization of discrete series representations for $\mathfrak{su}(n, 1)$ and a supersymmetric analogue of the basic representation of the infinite rank Lie algebra $\mathfrak{gl}(\infty)$.

It is well known that the vertex representation of $\mathfrak{gl}(\infty)$ has deep connections to some areas in non-linear differential equations and mathematical physics such as KP-hierarchies, soliton theory, dual resonance models and statistical models. In section 4, we shall give a Boson picture of our representations for $\mathfrak{gl}(\infty|q)$ $(1 \leqq q \leqq \infty)$, which is a natural extension of the vertex representation of $\mathfrak{gl}(\infty)$.

1. Lie superalgebra $\mathfrak{gl}(p|q)$

Let $\boldsymbol{Z}$ be the set of all integers, and $\boldsymbol{N}$ (resp. $\boldsymbol{N}_0$) the set of all positive (resp. non-negative) integers. We set $\bar{\boldsymbol{N}} = \boldsymbol{N} \cup \{+\infty, \infty\}$, which is a totally ordered set with the natural linear order in $\boldsymbol{N}$ and $n < +\infty < \infty$ for every $n \in \boldsymbol{N}$. For $n \in \bar{\boldsymbol{N}}$, define subsets $\mathscr{S}_n$, $\mathscr{S}_n^*$ and $\mathscr{S}_n^\pm$ of $\boldsymbol{Z}$ as follows:

$$\mathscr{S}_n = \begin{cases} \boldsymbol{Z} & \text{if} \quad n = \infty \\ \{j \in \boldsymbol{N}_0 ; j < n\} & \text{if} \quad n < \infty, \end{cases}$$

$$\mathscr{S}_n^* = \mathscr{S}_n - \{0\}, \quad \mathscr{S}_n^\pm = \mathscr{S}_n \cap (\pm \boldsymbol{N}).$$

Fix p and q in $\bar{\boldsymbol{N}}$, and define the complex vector spaces $\mathfrak{gl}(p, q)_0$ and $\mathfrak{gl}(p, q)_1$ as follows:

$$\mathfrak{gl}(p, q)_0 = \left\{ \sum_{i,j \in \mathscr{S}_p} a_{ij} E_{ij}^{(00)} + \sum_{k,l \in \mathscr{S}_q} b_{kl} E_{kl}^{(11)} \right\}$$

$$\mathfrak{gl}(p, q)_1 = \left\{ \sum_{(m,n) \in \mathscr{S}_p \times \mathscr{S}_q} c_{mn} E_{mn}^{(01)} + \sum_{(r,s) \in \mathscr{S}_q \times \mathscr{S}_p} d_{rs} E_{rs}^{(10)} \right\},$$

where a_{ij}, b_{kl}, c_{mn} and d_{rs} are complex numbers such that for any integers u and v the number of non-zero a_{ij}, b_{kl}, c_{mn}, d_{rs} with $i, k, m, r > u$ and $j, l, n, s < v$ is finite.

Then $\mathfrak{gl}(p, q)=\mathfrak{gl}(p, q)_0 \oplus \mathfrak{gl}(p, q)_1$ is a Lie superalgebra with (anti-) commutation relations

$$[E_{ij}^{(\alpha\beta)}, E_{kl}^{(\gamma\varepsilon)}]' = \delta_{\beta\gamma}\delta_{jk}E_{il}^{(\alpha\varepsilon)} - (-1)^{(\alpha+\beta)(\gamma+\varepsilon)}\delta_{\alpha\varepsilon}\delta_{il}E_{kj}^{(\gamma\beta)}.$$

Now define Heaviside functions $Y_{\pm}$ and σ on $\boldsymbol{Z}$ by

$$Y_+(j) = \begin{cases} 1 & \text{if } j \geqq 0 \\ 0 & \text{if } j < 0, \end{cases}$$

$$Y_-(j) = 1 - Y_+(j)$$

and

$$\sigma(j) = \begin{cases} 1 & \text{if } j > 0 \\ -1 & \text{if } j \leqq 0, \end{cases}$$

and introduce the 1-cochain $\Psi_{p,q}$ on $\mathfrak{gl}(p, q)$ as follows:

$$\Psi_{p,q}(E_{ij}^{(00)}) = \delta_{p,\infty}\delta_{i,j}Y_-(i)$$

$$\Psi_{p,q}(E_{kl}^{(11)}) = -\delta_{q,\infty}\delta_{k,l}Y_-(k)$$

$$\Psi_{p,q}(E_{il}^{(01)}) = \Psi_{p,q}(E_{kj}^{(10)}) = 0.$$

We put

$$\mathfrak{gl}(p|q) = \begin{cases} \mathfrak{gl}(p, q) & \text{if } p, q \leqq +\infty \\ \mathfrak{gl}(p, q) \oplus \boldsymbol{C}c & \text{otherwise,} \end{cases}$$

and define the Lie bracket in $\mathfrak{gl}(p|q)$ by

$$[X, Y] = [X, Y]' \qquad \text{if } p, q \leqq +\infty$$

and

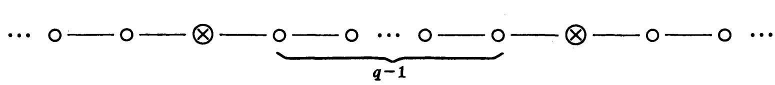
$$[X+\lambda c, Y+\mu c] = [X, Y]' + \Psi_{p,q}([X, Y]')c \qquad \text{otherwise.}$$

Then $\mathfrak{gl}(p|q)$ is a Lie superalgebra with the even part

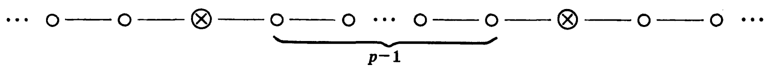
$$\mathfrak{gl}(p|q)_0 = \begin{cases} \mathfrak{gl}(p, q)_0 & \text{if } p, q < \infty \\ \mathfrak{gl}(p, q)_0 \oplus Cc & \text{otherwise} \end{cases}$$

and the odd part $\mathfrak{gl}(p|q)_1 = \mathfrak{gl}(p, q)_1$. Note that the Dynkin diagram of $\mathfrak{gl}(p|q)$ is given by the following:

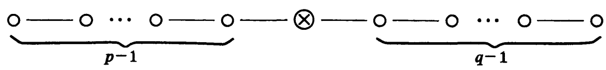
i) $\mathfrak{gl}(\infty|q)$ $(1 \leqq q < +\infty)$:



ii) $\mathfrak{gl}(p|\infty)$ $(1 \leqq p < +\infty)$:



iii) $\mathfrak{gl}(p|q)$ $(1 \leqq p,\ q < \infty)$:



The Lie superalgebra $\mathfrak{gl}(p|q)$ carries the involutive conjugate-linear automorphism ω defined by

$$\omega(E_{ij}^{(00)}) = -E_{ji}^{(00)}, \quad \omega(E_{ik}^{(01)}) = -\sigma(k)E_{ki}^{(10)}$$
$$\omega(E_{kl}^{(11)}) = -\sigma(k)\sigma(l)E_{lk}^{(11)}, \quad \omega(c) = -c,$$

which satisfies

$$\omega([X, Y]) = (-1)^{|X||Y|}[\omega(X), \omega(Y)]$$

for all homogeneous elements X and Y in $\mathfrak{gl}(p|q)$, where $|X|$ stands for the homogeneous degree of X, i.e., $|X|=j$ if $X \in \mathfrak{gl}(p|q)_j$.

2. Irreducible representations of $\mathfrak{gl}(p|q)$

Fix p and q in $\bar{N}$. Let $A(p)$ be the Clifford algebra over $\boldsymbol{C}$ generated by 1 and $\{\psi_j, \psi_j^*;\ j \in \mathscr{S}_p\}$ satisfying the anti-commutation relations $\psi_i\psi_j + \psi_j\psi_i = 0$, $\psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0$ and $\psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{i,j}$. Let $B(p)$ be the left ideal in $A(p)$ generated by $\{\psi_j;\ j \in \mathscr{S}_p^-\} \cup \{\psi_j^*;\ j \in \mathscr{S}_p \text{ and } j \geqq 0\}$. Let $\mathscr{K}_p$ denote the set of all finite sequences $I = (i'_r, \ldots, i'_1, i_1, \ldots, i_s)$ in $\mathscr{S}_p$ such that

$$i'_r < \cdots < i'_1 < 0 \leqq i_1 < \cdots < i_s.$$

For the above $I \in \mathscr{K}_p$, we put

$$|I| = \begin{cases} s - r & \text{if } I \text{ is not empty} \\ 0 & \text{if } I = \emptyset \text{ is the empty set.} \end{cases}$$

Then the complex vector space $U(p) = A(p)/B(p)$ is spanned by monomials

$$\{\xi_I = \psi_{i'_r}^* \cdots \psi_{i'_1}^* \psi_{i_1} \cdots \psi_{i_s};\ I \in \mathscr{K}_p\},$$

where $\xi_\phi = 1$.

Next we put

$$W(q) = \begin{cases} \boldsymbol{C}[y_j; j \in \mathscr{S}_q^*] & \text{if } q > 1 \\ \boldsymbol{C} & \text{if } q = 1 \end{cases}$$

and

$$a_j = Y_+(j)y_j - Y_-(j)\frac{\partial}{\partial y_j}, \quad a_j^* = Y_+(j)\frac{\partial}{\partial y_j} + Y_-(j)y_j$$

for $j \in \mathscr{S}_q^*$. Let $\mathscr{M}_q$ be the set of multi-indices $\alpha = (\alpha_j; j \in \mathscr{S}_q^*)$ with coefficients in $\boldsymbol{N}_0$ such that $\alpha_j = 0$ for all but finite j. For $\alpha \in \mathscr{M}_q$, we put

$$\alpha! = \prod_{j \in \mathscr{S}_q^*} \alpha_j!, \quad |\alpha| = \sum_{j \in \mathscr{S}_q^+} \alpha_j - \sum_{j \in \mathscr{S}_q^-} \alpha_j$$

and

$$F_\alpha = \prod_{j \in \mathscr{S}_q^*} y_j^{\alpha_j}.$$

Now we set $V(p, q) = U(p) \otimes W(q)$, and let

$$E = \sum_{i \in \mathscr{S}_p} :\psi_i \psi_i^*: + \sum_{k \in \mathscr{S}_q^*} :a_k a_k^*:$$

be the Euler operator on $V(p, q)$, where

$$:\psi_i \psi_j^*: = \psi_i \psi_j^* - Y_-(i)\delta_{i,j}$$

and

$$:a_k a_l^*: = a_k a_l^* + Y_-(k)\delta_{k,l}$$

are normal products. Note that $[E, \psi_i] = \psi_i$, $[E, \psi_i^*] = -\psi_i^*$, $[E, a_k] = a_k$ and $[E, a_k^*] = -a_k^*$. For each $m \in \boldsymbol{Z}$, the m-eigenspace $V(p, q)_m$ of E is spanned by monomials $\xi_I F_\alpha$ with $|I| + |\alpha| = m$, and is called the set of physical states with charge m.

Now, for any complex number ν, we define the linear operators on $V(p, q)$ as follows:

$$\pi_\nu(E_{ij}^{(00)}) = :\psi_i \psi_j^*:, \quad \pi_\nu(E_{i0}^{(01)}) = \psi_i, \quad \pi_\nu(E_{il}^{(01)}) = \psi_i a_l^*,$$

$$\pi_\nu(E_{0j}^{(10)}) = (\nu - E)\psi_j^*, \quad \pi_\nu(E_{kj}^{(10)}) = a_k \psi_j^*, \quad \pi_\nu(E_{kl}^{(11)}) = :a_k a_l^*:,$$

$$\pi_\nu(E_{k0}^{(11)}) = a_k, \quad \pi_\nu(E_{0l}^{(11)}) = (\nu - E)a_l^*, \quad \pi_\nu(E_{00}^{(11)}) = \nu - E,$$

$$\pi_\nu(c) = 1 \quad \text{if} \quad p \text{ or } q = \infty$$

for every $i, j \in \mathscr{S}_p$ and $k, l \in \mathscr{S}_q^*$. Then, by a simple computation, one obtains

THEOREM 2.1. *$(\pi_\nu, V(p, q))$ is a representation of* $\mathfrak{gl}(p|q)$.

In the case when $\nu \in \boldsymbol{R}$, we set

$$V^{(\nu)}(p, q) = \bigoplus_{m>\nu} V(p, q)_m .$$

Then it is also easy to see the following

THEOREM 2.2. 1) *In the case when* $p, q<\infty$:

i) *The representation* $(\pi_\nu, V(p, q))$ *is irreducible if and only if* $\nu \in \boldsymbol{C}-\boldsymbol{N}_0$.

ii) *If* $\nu \in \boldsymbol{N}_0$, $V^{(\nu)}(p, q)$ *is the unique proper submodule of* $(\pi_\nu, V(p, q))$.

2) *In the case when p or q is equal to* ∞:

i) *The representation* $(\pi_\nu, V(p, q))$ *is irreducible if and only if* $\nu \in \boldsymbol{C}-\boldsymbol{Z}$.

ii) *If* $\nu \in \boldsymbol{Z}$, $V^{(\nu)}(p, q)$ *is the unique proper submodule of* $(\pi_\nu, V(p, q))$.

3. Contravariant Hermitian form on $V^{(\nu)}(p, q)$.

Let π be a representation of the Lie superalgebra $\mathfrak{gl}(p|q)$ on a $\boldsymbol{Z}_2$-graded vector space $V=V_0+V_1$. A Hermitian form H on V is called *contravariant* if

$$H(\pi(X)u, v) + H(u, \pi(\omega(X))v) = 0$$

for all $X \in \mathfrak{gl}(p|q)$ and $u, v \in V$. And the representation π is called *unitarizable* if the contravariant form H is positive definite.

For p and q in $\bar{\boldsymbol{N}}$, we set

$$\Omega_{p,q} = \begin{cases} \boldsymbol{N}_0 \cup \{\nu \in \boldsymbol{R};\ \nu < 0\} & \text{if } \ p, q < \infty \\ \boldsymbol{Z} & \text{otherwise,} \end{cases}$$

and

$$\Delta_{p,q}^{(\nu)} = \begin{cases} \boldsymbol{N}_0 & \text{if } \ p, q < \infty \text{ and } \nu < 0 \\ \{m \in \boldsymbol{Z};\ m>\nu\} & \text{otherwise} \end{cases}$$

for $\nu \in \Omega_{p,q}$, and

$$\Xi_{p,q}^{(\nu)}(m) = \begin{cases} \Gamma(m-\nu)/\Gamma(-\nu) & \text{if } \ p, q < \infty \text{ and } \nu < 0 \\ (m-\nu-1)! & \text{otherwise} \end{cases}$$

for $\nu \in \Omega_{p,q}$ and $m \in \Delta_{p,q}^{(\nu)}$.

Fix $\nu \in \Omega_{p,q}$, and define the (positive-definite) Hermitian inner product $(\ ,\ )_\nu$ in $V^{(\nu)}(p, q)$ by requiring

$$(\xi_I F_\alpha, \xi_J F_\beta)_\nu = \Xi_{p,q}^{(\nu)}(m) \cdot \alpha!\ \delta_{I,J}\delta_{\alpha,\beta}$$

for all $\xi_I F_\alpha \in V(p, q)_m$ and $\xi_J F_\beta \in V(p, q)_n$ such that $m, n \in \Delta_{p,q}^{(\nu)}$. Then it is easy to see that

$$(\pi_\nu(X)u, v)_\nu + (u, \pi_\nu(\omega(X))v)_\nu = 0$$

for every $X \in \mathfrak{gl}(p|q)$ and $u, v \in V^{(\nu)}(p, q)$. Thus we have proved

THEOREM 3.1. *Let $p, q \in \bar{N}$ and $\nu \in \Omega_{p,q}$, then $(\pi_\nu, V^{(\nu)}(p, q))$ is an irreducible and unitarizable representation of* $\mathfrak{gl}(p|q)$.

4. Vertex representations of $\mathfrak{gl}(\infty|q)$ $(1 \leqq q \leqq \infty)$.

In this section, we consider the case when $p=\infty$, and rewrite the representation π_ν on $V(\infty, q)$ or $V^{(\nu)}(\infty, q)$ in terms of vertex operators. First we recall the so-called Boson-Fermion correspondence following [1].

Let

$$\hat{U} = \boldsymbol{C}[y_0, y_0^{-1}, x_1, x_2, \dots] = \bigoplus_{m \in \boldsymbol{Z}} \boldsymbol{C}[x] y_0^m$$

be the linear space of polynomial functions in y_0, y_0^{-1} and x_j's $(j \in \boldsymbol{N})$. For q in $\bar{\boldsymbol{N}}$, we set $\hat{V}(q) = \hat{U} \otimes W(q)$, which has a natural $\boldsymbol{Z}$-gradation

$$\hat{V}(q) = \bigoplus_{m \in \boldsymbol{Z}} (\hat{U}_m \otimes W(q))$$

and a $\boldsymbol{Z}_2$-gradation $\hat{V}(q) = \hat{V}(q)_{\bar{0}} \oplus \hat{V}(q)_{\bar{1}}$, where $\hat{U}_m = \boldsymbol{C}[x] y_0^m$ and

$$\hat{V}(q)_{\bar{i}} = \bigoplus_{m \equiv i (\mathrm{mod}\, 2)} (\hat{U}_m \otimes W(q)) \qquad (i=0, 1).$$

Note that each function f in $\hat{V}(q)$ is a finite sum

$$f = \sum_{m \in \boldsymbol{Z}} y_0^m f_m$$

where $f_m \in \boldsymbol{C}[x] \otimes W(q)$. Let

$$\hat{E} = y_0 \partial/\partial y_0 + \sum_{k \in \mathscr{S}_q^*} : a_k a_k^* : + \sum_{j \in \boldsymbol{N}} j x_j \partial/\partial x_j$$

be the Euler operator on $\hat{V}(q)$, and $\hat{V}(q)_m$ be the m-eigenspace of $\hat{E}$.

For each integer m, define a family of vertex operators with indeterminates u and v as follows:

$$\hat{X}^{(m)}(u, v) = \sum_{i,j \in \boldsymbol{Z}} \hat{X}_{ij}^{(m)} u^i v^{-j} = (u/v)^m [v/(u-v)]_m (\hat{X}(u, v) - (v/u)^m)$$

$$\hat{Y}^{(m)}(u) = \sum_{i \in \boldsymbol{Z}} \hat{Y}_i^{(m)} u^i = u^m \exp\left(\sum_{k=1}^{\infty} u^k x_k\right) \exp\left(-\sum_{k=1}^{\infty} u^{-k} D_k\right)$$

$$\hat{Z}^{(m)}(v) = \sum_{j \in \boldsymbol{Z}} \hat{Z}_j^{(m)} v^{-j} = v^{1-m} \exp\left(-\sum_{k=1}^{\infty} v^k x_k\right) \exp\left(\sum_{k=1}^{\infty} v^{-k} D_k\right),$$

where

$$D_k = k^{-1}\partial_k = k^{-1}\partial/\partial x_k,$$

$$\hat{X}(u, v) = \exp\left(\sum_{k=1}^{\infty} (u^k - v^k)x_k\right) \exp\left(-\sum_{k=1}^{\infty} (u^{-k} - v^{-k})D_k\right)$$

and

$$[v/(u-v)]_m = Y_-(m) \sum_{k=1}^{\infty} (v/u)^k - Y_+(m) \sum_{k=0}^{\infty} (u/v)^k.$$

We also introduce the following operators on $U(\infty)$:

$$Y(u) = \sum_{i\in Z} \psi_i u^i, \quad Z(v) = \sum_{j\in Z} \psi_j^* v^{-j}$$

and

$$H_n = \sum_{i\in Z} :\psi_i \psi_{i+n}^*: \qquad (n \in Z).$$

Let $\langle\ ,\ \rangle$ be the non-degenerate symmetric bilinear form on $U(\infty)$ such that $\langle \xi_I, \xi_J\rangle = \delta_{I,J}$, and T be the linear operator of $U(\infty)$ to $\hat{U}$ given by

$$T\xi = \sum_{m\in Z} y_0^m \langle \xi^{(m)}, \exp\left(\sum_{k=1}^{\infty} x_k H_k\right)\xi\rangle$$

for every $\xi \in U(\infty)$, where

$$\xi^{(m)} = \begin{cases} \psi_0\psi_1\cdots\psi_{m-1} & \text{if} \quad m > 0 \\ 1 & \text{if} \quad m = 0 \\ \psi_m^*\cdots\psi_{-1}^* & \text{if} \quad m < 0 \end{cases}$$

is called the normalized ground state vector with charge m in $U(\infty)$. The following lemma is due to [1]:

LEMMA 4.1. ([1]).

1) $[H_j, H_k] = j\,\delta_{j,-k}$,
2) $\partial_j \circ T = T \circ H_j$ *and* $jx_j \circ T = T \circ H_{-j}$ *for every* $j \in N$,
3) $T \circ :\psi_i \psi_j^*: \circ T^{-1}$ *on* $C[x]y_0^m = \hat{X}_{ij}^{(m)}$,
4) $T \circ Y(u) \circ T^{-1}$ *on* $C[x]y_0^m = y_0 \otimes (-1)^{mY_+(m)} \hat{Y}(u)$,
5) $T \circ Z(v) \circ T^{-1}$ *on* $C[x]y_0^m = y_0^{-1} \otimes (-1)^{(m-1)Y_+(m-1)} \hat{Z}(v)$

For $\nu \in C$, we define linear operators $\hat{\pi}_\nu(E_{ij}^{(\alpha\beta)})$ on $\hat{V}(q)$ by the following:

$$\hat{\pi}_\nu(E_{ij}^{(00)})f = \sum_{m\in Z} y_0^m \hat{X}_{ij}^{(m)} f_m$$

$$\hat{\pi}_\nu(E_{i0}^{(01)})f = \sum_{m\in Z} (-1)^{mY_+(m)} y_0^{m+1} \hat{Y}_i^{(m)} f_m$$

$$\hat{\pi}_\nu(E_{il}^{(01)})f = \sum_{m\in Z}(-1)^{mY+(m)}y_0^{m+1}\hat{Y}_i^{(m)}a_l^* f_m$$

$$\hat{\pi}_\nu(E_{0j}^{(10)})f = (\nu - \hat{E})\sum_{m\in Z}(-1)^{(m-1)Y+(m-1)}y_0^{m-1}\hat{Z}_j^{(m)}f_m$$

$$\hat{\pi}_\nu(E_{kj}^{(10)})f = \sum_{m\in Z}(-1)^{(m-1)Y+(m-1)}y_0^{m-1}\hat{Z}_j^{(m)}a_k f_m$$

$$\hat{\pi}_\nu(E_{kl}^{(11)})f = :a_k a_l^*: f, \quad \hat{\pi}_\nu(E_{k0}^{(11)})f = a_k f,$$

$$\hat{\pi}_\nu(E_{0l}^{(11)})f = (\nu-\hat{E})a_l^* f, \quad \hat{\pi}_\nu(E_{00}^{(11)})f = (\nu-\hat{E})f,$$

$$\hat{\pi}_\nu(c)f = f$$

for every $i, j \in Z$ and $k, l \in \mathscr{S}_q^*$ and $f = \sum_{m\in Z} y_0^m f_m \in \hat{V}(q)$.

From Theorems 2.1–3.1 and Lemma 4.1, one sees that $(\hat{\pi}_\nu, \hat{V}(q))$ is a representation of $\mathrm{gl}(\infty|q)$ equivalent to $(\pi_\nu, V(\infty, q))$, and that, in the case when $\nu \in Z$, the subspace

$$\hat{V}^{(\nu)}(q) = \bigoplus_{m>\nu} \hat{V}(q)_m$$

is invariant and unitarizable. Thus we have proved

THEOREM 4.1. *Let $q \in \bar{N}$ and $\nu \in C$.*

1) *If $\nu \notin Z$, $(\hat{\pi}_\nu, \hat{V}(q))$ is an irreducible* $\mathrm{gl}(\infty|q)$*-module.*
2) *If $\nu \in Z$, $(\hat{\pi}_\nu, \hat{V}^{(\nu)}(q))$ is an irreducible and unitarizable representation of* $\mathrm{gl}(\infty|q)$.

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