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Ideal boundary limit of discrete Dirichlet functions

Dedicated to Professor Yukio Kusunoki on his 60th birthday

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§1. Introduction

In the previous paper [5], we proved that every Dirichlet potential $u(x)$ of order $p>1$ on an infinite network $N=\{X, Y, K, r\}$ has limit 0 as x tends to the ideal boundary of N along p-almost every infinite path. Our aim of this paper is to prove the converse of this fact. In case $p=2$, our result has a continuous counterpart in [3], i.e., on a Riemannian manifold Ω, every Dirichlet function (=Tonelli function with finite Dirichlet integral) $u(x)$ has limit 0 as x tends to the ideal boundary of Ω along 2-almost every curve joining a fixed parametric ball to the ideal boundary of Ω if and only if u is a Dirichlet potential (i.e., the values of u on the harmonic boundary of Ω are 0). Since the proof in [3] is based on some results concerning continuous harmonic flows and the Royden compactification of Ω, it seems to be difficult to follow the reasoning in our case.

We shall prove in §2 that every Dirichlet function of order p on X can be decomposed uniquely into the sum of Dirichlet potential of order p and a p-harmonic function on X. We shall discuss in §3 the ideal boundary limit of a non-constant p-harmonic function with finite Dirichlet integral of order p. As an application, we shall prove that a Dirichlet function of order p is a Dirichlet potential of order p if and only if it has limit 0 as x tends to the ideal boundary of N along p-almost every infinite path.

We shall freely use the notation in [5] except for the reference numbers; references are rearranged in the present paper.

§2. Decomposition of $D^{(p)}(N)$

Let p and q be positive numbers such that $1/p+1/q=1$ and $1<p<\infty$ and let $\phi_p(t)$ be the real function on the real line R defined by $\phi_p(t)=|t|^{p-1}\,\mathrm{sign}\,(t)$. For each $w\in L(Y)$, let us define $\phi_p(w)\in L(Y)$ by $\phi_p(w)(y)=\phi_p(w(y))$ for $y\in Y$.

For each $u\in L(X)$, the p-Laplacian $\Delta_p u\in L(X)$ of u is defined by

$$\Delta_p u(x) = \sum_{y\in Y} K(x, y)\phi_p(du(y)),$$

where $du(y)=-r(y)^{-1}\sum_{x\in X} K(x, y)u(x)$ (a discrete derivative of u). We say that u is p-harmonic on a subset A of X if $\Delta_p u(x)=0$ on A. Denote by $\boldsymbol{HD}^{(p)}(N)$

the set of all $u \in \boldsymbol{D}^{(p)}(N)$ which is p-harmonic on X. Some properties of p-harmonic functions were discussed in [6] in a more general setting. It should be noted that $\boldsymbol{HD}^{(p)}(N)$ is not a linear space in general if $p \neq 2$.

For $w_1, w_2 \in L(Y)$, we consider the inner product

$$\langle\!\langle w_1, w_2 \rangle\!\rangle = \sum_{y \in Y} r(y) w_1(y) w_2(y)$$

of w_1 and w_2 if the sum is well-defined. It is easily seen that $\langle\!\langle w_1, w_2 \rangle\!\rangle$ is well-defined if the support of w_1 or w_2 is a finite set or if $H_p(w_1)$ (resp. $H_q(w_1)$) and $H_q(w_2)$ (resp. $H_p(w_2)$) are finite. For each $u \in \boldsymbol{D}^{(p)}(N)$, we have

$$D_p(u) = \langle\!\langle \phi_p(du), du \rangle\!\rangle = H_q(\phi_p(du)).$$

We begin with some lemmas.

LEMMA 2.1. *$\langle\!\langle \phi_p(w_1) - \phi_p(w_2), w_1 - w_2 \rangle\!\rangle \geq 0$ for all $w_1, w_2 \in L(Y)$ with finite energy of order p. The equality holds only if $w_1 = w_2$.*

PROOF. Since $f(t) = H_p(w_1 + t(w_2 - w_1))$ is a strictly convex function of $t \in R$ in case $w_1 \neq w_2$ and the derivative of $f(t)$ at $t=0$ is equal to $p\langle\!\langle \phi_p(w_1), w_2 - w_1 \rangle\!\rangle$, our assertion follows from [2; p. 25, Proposition 5.4].

LEMMA 2.2. (Clarkson's inequality) *For $u, v \in \boldsymbol{D}^{(p)}(N)$, the following inequalities hold:*

$$D_p(u+v) + D_p(u-v) \leq 2^{p-1}[D_p(u) + D_p(v)] \text{ in case } p \geq 2; \tag{2.1}$$

$$[D_p(u+v)]^{1/(p-1)} + [D_p(u-v)]^{1/(p-1)} \leq 2[D_p(u) + D_p(v)]^{1/(p-1)} \text{ in case } 1 < p \leq 2. \tag{2.2}$$

PROOF. Let $t \in R$, $0 \leq t \leq 1$. By [1], [4] or [7], we have

$$(1+t)^p + (1-t)^p \leq 2^{p-1}(1+t^p) \text{ in case } p \geq 2, \tag{2.3}$$

$$(1+t)^p + (1-t)^p \geq (1+t^q)^{p-1} \text{ in case } 1 < p \leq 2. \tag{2.4}$$

Let us put $s = (1-t)/(1+t)$. Then (2.4) is equivalent to

$$[(1+s)^q + (1-s)^q]^{p-1} \leq 2^{p-1}(1+s^p). \tag{2.4$'$}$$

We see easily that (2.1) follows from (2.3) and that (2.2) follows from (2.4)$'$ and the reverse Minkowski's inequality.

LEMMA 2.3. *$\langle\!\langle \phi_p(dh), dv \rangle\!\rangle = 0$ for every $v \in \boldsymbol{D}_0^{(p)}(N)$ and $h \in \boldsymbol{HD}^{(p)}(N)$.*

PROOF. Let $v \in \boldsymbol{D}_0^{(p)}(N)$ and $h \in \boldsymbol{HD}^{(p)}(N)$. Then there exists a sequence $\{f_n\}$ in $L_0(X)$ such that $\|v - f_n\|_p \to 0$ as $n \to \infty$. We have

$$\langle\!\langle \phi_p(dh), df_n \rangle\!\rangle = \sum_{y \in Y} r(y) [\phi_p(dh(y))] [df_n(y)]$$
$$= - \sum_{x \in X} f_n(x) [\Delta_p h(x)] = 0,$$

$$|\langle\!\langle \phi_p(dh), d(v-f_n)\rangle\!\rangle| \le [H_q(\phi_p(dh))]^{1/q}[H_p(d(v-f_n))]^{1/p}$$
$$= [D_p(h)]^{1/q}[D_p(v-f_n)]^{1/p} \longrightarrow 0$$

as $n\to\infty$, so that $\langle\!\langle \phi_p(dh), dv\rangle\!\rangle=0$.

We shall prove the following decomposition theorem:

THEOREM 2.1. *Assume that N is of hyperbolic type of order p. Then every $u\in \boldsymbol{D}^{(p)}(N)$ can be decomposed uniquely in the form: $u=v+h$, where $v\in \boldsymbol{D}_0^{(p)}(N)$ and $h\in \boldsymbol{HD}^{(p)}(N)$.*

PROOF. Let $u\in \boldsymbol{D}^{(p)}(N)$ and consider the following extremum problem:

(2.5) Find $\alpha=\inf\{D_p(u-f); f\in \boldsymbol{D}_0^{(p)}(N)\}$.

Clearly α is finite. Let $\{f_n\}$ be a sequence in $\boldsymbol{D}_0^{(p)}(N)$ such that $D_p(u-f_n)\to 0$ as $n\to\infty$. We show that $D_p(f_n-f_m)\to 0$ as n, $m\to\infty$. In case $p\ge 2$, we have by (2.1)

$$\begin{aligned}\alpha &\le D_p(u-(f_n+f_m)/2)\\ &\le D_p(u-(f_n+f_m)/2)+D_p((f_m-f_n)/2)\\ &\le 2^{p-1}[D_p((u-f_n)/2)+D_p((u-f_m)/2)]\\ &= 2^{-1}[D_p(u-f_n)+D_p(u-f_m)] \longrightarrow \alpha\end{aligned}$$

as n, $m\to\infty$. In case $1<p\le 2$, we have by (2.2)

$$\begin{aligned}\alpha^{1/(p-1)} &\le [D_p(u-(f_n+f_m)/2)]^{1/(p-1)}\\ &\le [D_p(u-(f_n+f_m)/2)]^{1/(p-1)}+[D_p((f_m-f_n)/2)]^{1/(p-1)}\\ &\le 2[D_p((u-f_n)/2)+D_p((u-f_m)/2)]^{1/(p-1)} \longrightarrow \alpha^{1/(p-1)}\end{aligned}$$

as n, $m\to\infty$. Thus we have $D_p(f_m-f_n)\to 0$ as n, $m\to\infty$. Since $[D_p(v)]^{1/p}$ is a pseudonorm, we see easily that $\{D_p(f_n)\}$ is bounded.

Next we show that $\{|f_n(b)|\}$ is bounded, where $b\in X$ is a fixed element such that $\|u\|_p\doteq[D_p(u)+|u(b)|^p]^{1/p}$ (cf. [5]). Supposing the contrary, we may assume that $|f_n(b)|\to\infty$ as $n\to\infty$ by choosing a subsequence if necessary. Put $f'_n(x)=f_n(x)/f_n(b)$. Then $f'_n(b)=1$ and $f'_n\in \boldsymbol{D}_0^{(p)}(N)$. Since $\{D_p(f_n)\}$ is bounded, we have $D_p(f'_n)=D_p(f_n)/|f_n(b)|^p\to 0$ as $n\to\infty$, so that $\|f'_n-1\|_p=[D_p(f'_n)]^{1/p}\to 0$ as $n\to\infty$. Namely $1\in \boldsymbol{D}_0^{(p)}(N)$. This contradicts the assumption that N is of hyperbolic type of order p (cf. [10]). Therefore $\{f_n(b)\}$ is bounded. By choosing a subsequence if necessary, we may assume that $\{f_n(b)\}$ converges. Then $\{f_n\}$ is a Cauchy sequence in the reflexive Banach space $\boldsymbol{D}^{(p)}(N)$. There exists $v\in \boldsymbol{D}^{(p)}(N)$ such that $\|f_n-v\|_p\to 0$ as $n\to\infty$. Since $\boldsymbol{D}_0^{(p)}(N)$ is closed, $v\in \boldsymbol{D}_0^{(p)}(N)$. Let us put $h=u-v$ and show that $h\in \boldsymbol{HD}^{(p)}(N)$. For any $f\in L_0(X)$ and $t\in R$, we have $v+tf\in \boldsymbol{D}_0^{(p)}(N)$ and $D_p(h)=\alpha\leq D_p(h-tf)$, so that the derivative of $D_p(h-tf)$ with respect to t is zero at $t=0$. If follows that

$$(2.6)\qquad 0 = \sum_{y\in Y} r(y)\,[\phi_p(dh(y))]\,[df(y)] = \langle\!\langle \phi_p(dh),\, df\rangle\!\rangle\,.$$

Denote by ε_z the characteristic function of the set $\{z\}\subset X$. By taking $f=\varepsilon_z$ in (2.6), we have $\Delta_p h(z)=0$ for every $z\in X$. Since $h\in \boldsymbol{D}^{(p)}(N)$, we conclude that $h\in \boldsymbol{HD}^{(p)}(N)$, which shows a decomposition of u.

To prove the uniqueness of the decomposition, let us assume that $u=v_1+h_1=v_2+h_2$ with $v_i\in \boldsymbol{D}_0^{(p)}(N)$ and $h_i\in \boldsymbol{HD}^{(p)}(N)$ ($i=1, 2$). Since $v_2-v_1\in \boldsymbol{D}_0^{(p)}(N)$, we have by Lemma 2.3

$$\begin{aligned}&\langle\!\langle \phi_p(dh_1)-\phi_p(dh_2),\, dh_1-dh_2\rangle\!\rangle = \langle\!\langle \phi_p(dh_1)-\phi_p(dh_2),\, d(v_2-v_1)\rangle\!\rangle\\ &\quad = \langle\!\langle \phi_p(dh_1),\, d(v_2-v_1)\rangle\!\rangle - \langle\!\langle \phi_p(dh_2),\, d(v_2-v_1)\rangle\!\rangle = 0.\end{aligned}$$

Thus $h_1=h_2$ by Lemma 2.1, so that $v_1=v_2$. This completes the proof.

REMARK 2.1. In case $p=2$, Theorem 2.1 is a discrete analogue of Royden's decomposition of a Dirichlet function (cf. [11]).

LEMMA 2.4. *Let $u\in \boldsymbol{D}_0^{(p)}(N)$ and $w\in L(Y)$. If $u\in L^+(X)$ and $\sum_{y\in Y} K(x, y)\cdot w(y)\geq 0$ for all $x\in X$, then*

$$\sum_{x\in X} u(x) \sum_{y\in Y} K(x, y)w(y) \leq [D_p(u)]^{1/p}[H_q(w)]^{1/q}.$$

PROOF. It suffices to prove our inequality in case $H_q(w)$ is finite. There exists a sequence $\{f_n\}$ in $L_0(X)$ such that $\|u-f_n\|_p\to 0$ as $n\to\infty$. Put $u_n(x)=\max[f_n(x), 0]$. Then $u_n\in L_0^+(X)$. Since $Ts=\max(s, 0)$ is a normal contraction of R, i.e., $|Ts_1-Ts_2|\leq|s_1-s_2|$ for any s_1, $s_2\in R$, we have $D_p(u_n)\leq D_p(f_n)$. By our assumption that $u\in L^+(X)$, we have

$$|u_n(x)-u(x)| = |Tf_n(x)-Tu(x)| \leq |f_n(x)-u(x)|\,.$$

Since $\{f_n\}$ converges pointwise to u and $D_p(f_n)\to D_p(u)$ as $n\to\infty$, $u_n(x)\to u(x)$ as $n\to\infty$ for each $x\in X$ and $\limsup_{n\to\infty} D_p(u_n)\leq D_p(u)$. We have

$$\begin{aligned}\sum_{x\in X} u_n(x) \sum_{y\in Y} K(x, y)w(y) &= \sum_{y\in Y} w(y) \sum_{x\in X} K(x, y)u_n(x)\\ &\leq [H_q(w)]^{1/q}[D_p(u_n)]^{1/p},\end{aligned}$$

so that

$$\begin{aligned}\sum_{x\in X} u(x) \sum_{y\in Y} K(x, y)w(y) &\leq \liminf_{n\to\infty} \sum_{x\in X} u_n(x) \sum_{y\in Y} K(x, y)w(y)\\ &\leq \limsup_{n\to\infty} [H_q(w)]^{1/q}[D_p(u_n)]^{1/p}\\ &\leq [H_q(w)]^{1/q}[D_p(u)]^{1/p}.\end{aligned}$$

§3. Main results

Denote by $P_{a,\infty}(N)$ the set of all paths from $a \in X$ to the ideal boundary ∞ of N and by $P_\infty(N)$ the union of $P_{a,\infty}(N)$ for all $a \in X$. We call an element of $P_\infty(N)$ an infinite path.

For every $u \in \boldsymbol{D}^{(p)}(N)$, $u(x)$ has a limit as x tends to the ideal boundary ∞ of N along p-almost every $P \in P_\infty(N)$ (cf. [5; Theorem 3.1]). We denote this limit simply by $u(P)$.

We shall prove

THEOREM 3.1. *Let $h \in \boldsymbol{HD}^{(p)}(N)$ be nonconstant. Then there is no constant c such that $h(P)=c$ for p-almost every infinite path P.*

PROOF. First we show that N is of hyperbolic type of order p. Supposing the contrary, we have $\boldsymbol{D}_0^{(p)}(N)=\boldsymbol{D}^{(p)}(N)$ by [10; Theorem 3.2], so that $D_p(h)=\langle\langle \phi_p(dh), dh\rangle\rangle=0$ by Lemma 2.2, which contradicts the assumption that h is nonconstant.

Let us put $w_h(y)=\phi_p(dh(y))$, $Y(x)=\{y \in Y;\, K(x, y)\neq 0\}$ and $Y^+(x, h)=\{y \in Y(x);\, K(x, y)w_h(y)>0\}$. If $y \in Y^+(x, h)$ and $e(y)=\{x, x'\}$, then we have by definition

$$K(x, y)\,\mathrm{sign}\,[-K(x, y)(h(x)-h(x'))]>0,$$

so that $h(x)<h(x')$.

Since h is nonconstant, there exists $x_0 \in X$ such that $w_h(y)$ is not identically zero on $Y(x_0)$. By the relation $\Delta_p h(x_0)=\sum_{y\in Y} K(x_0, y)w_h(y)=0$, we see that $Y^+(x_0, h)\neq\emptyset$. Let us define subsets X_n^+ and Y_n^+ for $n\geq 1$ as follows:

$$Y_n^+ = \cup\{Y^+(x, h);\, x \in X_{n-1}^+\},$$

$$X_n^+ = \cup\{e(y)-X_{n-1}^+;\, y \in Y_n^+\},$$

where $X_0^+=\{x_0\}$. We put $X^+=\cup_{n=0}^\infty X_n^+$ and $Y^+=\cup_{n=1}^\infty Y_n^+$. Then $N^+=\langle X^+, Y^+\rangle$ is an infinite subnetwork of N. To see this, it suffices to show that $X_n^+\neq\emptyset$ for each n. We prove this by induction. By the above observation, $Y_1^+=Y^+(x_0, h)\neq\emptyset$, so that $X_1^+\neq\emptyset$. Suppose that $X_{n-1}^+\neq\emptyset$. Since X_{n-1}^+ is a finite set, there exists $a \in X_{n-1}^+$ such that $h(a)=\max\{h(x);\, x \in X_{n-1}^+\}$. By definition, we can find $y_1 \in Y_{n-1}^+$ such that $e(y_1)=\{a, x_1\}$ for some $x_1 \in X_{n-2}^+$ and $y_1 \in Y^+(x_1, h)$. We have $K(a, y_1)w_h(y_1)=-K(x_1, y_1)w_h(y_1)<0$ and $\Delta_p h(a)=\sum_{y\in Y} K(a, y)w_h(y)=0$, so that $Y^+(a, h)\neq\emptyset$. Let $y_2 \in Y^+(a, h)$ and $e(y_2)=\{a, x_2\}$. Then $h(a)<h(x_2)$ by the above observation, so that $x_2 \notin X_{n-1}^+$. Thus $x_2 \in X_n^+$, i.e., $X_n^+\neq\emptyset$.

Let us put $q^+(x)=\sum_{y\in Y^+} K(x, y)w_h(y)$. Then $q^+(x_0)>0$ and $q^+(x)\geq 0$ for all $x \in X^+$, since $Y^+(x, h) \subset Y^+$ for $x \in X^+$. Note that $\inf\{h(x); x \in X^+ - \{x_0\}\} > h(x_0)$. Let Γ^+ be the set of all paths $P \in P_{x_0,\infty}(N)$ contained in N^+, i.e., $C_X(P) \subset X^+$ and $C_Y(P) \subset Y^+$. Let us recall the extremal distance $EL_p(\{x_0\}, \infty; N^+)$ of order p of N^+ relative to $\{x_0\}$ and ∞:

$$EL_p(\{x_0\}, \infty; N^+)^{-1} = \inf\{H_p(W; N^+); W \in E(P_{x_0,\infty}(N^+))\},$$

where $H_p(w; N^+) = \sum_{y \in Y^+} r(y)|w(y)|^p$ and $E(P_{x_0,\infty}(N_+))$ is the set of all $W \in L^+(Y^+)$ such that $\sum_P r(y)W(y) \geq 1$ for all $P \in P_{x_0,\infty}(N^+)$. Then we see easily that $\lambda_p(\Gamma^+) = EL_p(\{x_0\}, \infty; N^+)$. Now we show that $\lambda_p(\Gamma^+) < \infty$. Supposing the contrary, we have $EL_p(\{x_0\}, \infty; N^+) = \infty$. Therefore N^+ is of parabolic type of order p by [10; Theorem 4.1], and hence $\boldsymbol{D}_0^{(p)}(N^+) = \boldsymbol{D}^{(p)}(N^+)$. Let $W \in E(P_{x_0,\infty}(N^+))$ and $H_p(W; N^+) < \infty$. Define $u \in L(X^+)$ by $u(x_0) = 0$ and

$$u(x) = \inf\{\textstyle\sum_P r(y)W(y); P \in P_{x_0,x}(N^+)\} \quad \text{for} \quad x \neq x_0,$$

where $P_{x_0,x}(N^+)$ is the set of all paths from x_0 to $x \in X_+$ in N^+. Then u is non-constant and $|\sum_{x \in X^+} K(x, y)u(x)| \leq r(y)W(y)$ on Y^+ by [9; Theorem 3]. Put $v(x) = \max[1 - u(x), 0]$. Then $v(x_0) = 1$, $v \in L^+(X^+)$ and

$$D_p(v; N^+) = \textstyle\sum_{y \in Y^+} r(y)|dv(y)|^p \leq D_p(u; N^+) \leq H_p(W; N^+) < \infty.$$

Since $v \in \boldsymbol{D}_0^{(p)}(N^+) \cap L^+(X^+)$ and $q^+(x) \geq 0$ on X^+, we have by Lemma 2.4

$$\begin{aligned} q^+(x_0) &= v(x_0) \textstyle\sum_{y \in Y^+} K(x_0, y)w_h(y) \\ &\leq \textstyle\sum_{x \in X^+} v(x) \sum_{y \in Y^+} K(x, y)w_h(y) \\ &\leq [D_p(v; N^+)]^{1/p}[H_q(w_h; N^+)]^{1/q} \\ &\leq [H_p(W; N^+)]^{1/p}[H_q(w_h)]^{1/q} = [H_p(W; N^+)]^{1/p}[D_p(h)]^{1/q}. \end{aligned}$$

It follows that $H_p(W; N^+) \geq [q^+(x_0)]^p[D_p(h)]^{-p/q} > 0$, so that $EL_p(\{x_0\}, \infty; N^+) < \infty$. This is a contradiction. Thus $\lambda_p(\Gamma^+) < \infty$ and $h(P) > h(x_0)$ for p-almost every $P \in \Gamma^+$.

Similarly we define an infinite subnetwork $N^- = \langle X^-, Y^- \rangle$ by $X^- = \cup_{n=0}^{\infty} X_n^-$ and $Y^- = \cup_{n=1}^{\infty} Y_n^-$, where $X_0^- = \{x_0\}$ and for $n \geq 1$

$$\begin{aligned} Y_n^- &= \cup\{Y^-(x, h); x \in X_{n-1}^-\}, \\ X_n^- &= \cup\{e(y) - X_{n-1}^-; y \in Y_n^-\}, \\ Y^-(x, h) &= \{y \in Y(x); K(x, y)w_h(y) < 0\}. \end{aligned}$$

Let us put $q^-(x) = \sum_{y \in Y^-} K(x, y)w_h(y)$. Then $q^-(x) \leq 0$ for all $x \in X^-$ and $q^-(x_0) < 0$. Let Γ^- be the set of all paths $P \in P_{x_0,\infty}(N)$ contained in N^-. Then we can prove similarly that $\lambda_p(\Gamma^-) < \infty$. Furthermore $h(P) < h(x_0)$ for p-almost every $P \in \Gamma^-$. This completes the proof.

COROLLARY. *$HD^{(p)}(N)$ consists of only constant functions if and only if for each $u \in D^{(p)}(N)$ there is a constant c_u such that $u(P)=c_u$ for p-almost every infinite path P.*

We shall prove

THEOREM 3.2. *Let $u \in D^{(p)}(N)$. Then $u \in D_0^{(p)}(N)$ if and only if $u(P)=0$ for p-almost every infinite path P.*

PROOF. In case N is of parabolic type of order p, our assertion is clear. We consider the case where N is of hyperbolic type of order p. By [5; Theorem 3.3], it suffices to show the "if" part. By Theorem 2.1, u can be decomposed in the form: $u=v+h$, where $v \in D_0^{(p)}(N)$ and $h \in HD^{(p)}(N)$. Assume that $u(P)=0$ for p-almost every infinite path P. Since $v(P)=0$ for p-almost every infinite path P by [5; Theorem 3.3], we have $h(P)=0$ for p-almost every infinite path P. It follows from Theorem 3.1 that $h=0$, and hence $u \in D_0^{(p)}(N)$.

We say as in [8] that $u \in L(X)$ vanishes at the ideal boundary of N if, for every $\varepsilon>0$, there exists a finite subset X' of X such that $|u(x)|<\varepsilon$ on $X-X'$.

As an application of Theorem 3.2, we have

COROLLARY. *Let $u \in D^{(p)}(N)$. If u vanishes at the ideal boundary of N, then $u \in D_0^{(p)}(N)$.*

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