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Oscillation criteria for functional differential inequalities with strongly bounded forcing term

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1. Introduction

In the last years there has been an increasing interest in studying the oscillatory behaviour of the solutions of differential equations and inequalities which involve forcing terms of the kind introduced by Kartsatos [9, 10]. As examples, we refer the reader to the papers of Chen and Yeh [2–4], Foster [5], Grace and Lalli [6–8], Kartsatos [11], Kusano et al. [12–14], and True [17]. The purpose of this paper is to establish some new oscillation criteria for higher order functional differential inequalities involving more general forcing functions. More precisely, we consider the class of perturbations which represent the so called strongly bounded functions (see [15]).

The functional differential inequalities under consideration are of the form

$$x(t)\{L_n x(t) + f(t, x(g_1(t)),\dots, x(g_m(t))) - h(t)\} \leqq 0, \ n \text{ even}, \tag{1}$$

and

$$x(t)\{L_n x(t) - f(t, x(g_1(t)),\dots, x(g_m(t))) - h(t)\} \geqq 0, \ n \text{ odd}, \tag{2}$$

where $n \geqq 2$ and L_n is the general disconjugate differential operator defined recursively by $L_0 x(t) = a_0(t)x(t)$ and

$$L_k x(t) = a_k(t)(L_{k-1}x(t))', \quad k = 1, 2,\dots, n.$$

We shall assume that $a_i(t)$, $i=0, 1,\dots, n$, are positive and continuous functions on $[t_0, \infty)$ and the operator L_n is in the first canonical form in the sense that

$$\int_{t_0}^{\infty} a_i^{-1}(t)dt = \infty, \quad i = 1, 2,\dots, n-1. \tag{3}$$

In what follows, the set of all real-valued functions $y(t)$ defined on $[t_y, \infty)$ and such that $L_i y(t)$, $i=0, 1,\dots, n$, exist and are continuous on $[t_y, \infty)$ will be denoted by $\mathscr{D}(L_n)$.

For the inequalities (1) and (2) the following conditions will be assumed without further mention:

(i) $f \in C([t_0, \infty) \times \boldsymbol{R}^m, \boldsymbol{R})$ has the following sign property:
$f(t, x_1, \dots, x_m) > 0$ for $(x_1, \dots, x_m) \in \boldsymbol{R}_+^m$, $t \geqq t_0$,
$f(t, x_1, \dots, x_m) < 0$ for $(x_1, \dots, x_m) \in \boldsymbol{R}_-^m$, $t \geqq t_0$,
where $\boldsymbol{R}_+ = (0, \infty)$ and $\boldsymbol{R}_- = (-\infty, 0)$;

(ii) $g_i \in C([t_0, \infty), \boldsymbol{R})$, $\lim_{t \to \infty} g_i(t) = \infty$, $i = 1, 2, \dots, m$;

(iii) $h \in C([t_0, \infty), \boldsymbol{R})$ and there exists a function $p \in \mathscr{D}(L_n)$ such that $L_n p(t) = h(t)$ and $L_0 p(t)$ is *strongly bounded* on $[t_0, \infty)$ in the sense that for every $T \geqq t_0$ there are T_*, $T^* \geqq T$ such that

$$L_0 p(T_*) = \min_{t \in [T, \infty)} L_0 p(t) \quad \text{and} \quad L_0 p(T^*) = \max_{t \in [T, \infty)} L_0 p(t).$$

As usual, we restrict our considerations only to those solutions $x(t)$ of (1) (or (2)) which exist on some ray $[t_x, \infty)$ and satisfy

$$\sup \{|x(s)| : s \geqq t\} > 0$$

for every $t \in [t_x, \infty)$. Such a solution is called oscillatory if it has arbitrarily large zeros in $[t_x, \infty)$ and it is called nonoscillatory otherwise.

2. Preliminaries

To formulate our results we shall use the following shorthand notation. Let $j_r \in \{1, 2, \dots, n-1\}$, $r = 1, 2, \dots, n-1$, and $t, s \in [t_0, \infty)$. We define $I_0 = 1$ and

$$I_r(t, s; j_1, \dots, j_r) = \int_s^t a_{j_1}^{-1}(\tau) I_{r-1}(\tau, s; j_2, \dots, j_r) d\tau.$$

For the sake of brevity we denote

$$\alpha_k(t, s) = a_0^{-1}(t) I_k(t, s; 1, \dots, k), \qquad \alpha_k(t) = \alpha_k(t, t_0),$$
$$\omega_k(t, s) = a_n^{-1}(t) I_k(t, s; n-1, \dots, n-k), \qquad \omega_k(t) = \omega_k(t, t_0).$$

Moreover, we shall have an occasion to use the following generalized Taylor's formula given in [1]:

$$\begin{aligned} L_i y(t) &= \textstyle\sum_{j=i}^{r} (-1)^{j-i} L_j(s) I_{j-i}(s, t; j, \dots, i+1) + \\ &+ (-1)^{r-i+1} \int_t^s I_{i-1}(\tau, t; r, \dots, i+1) \frac{L_{r+1}(\tau)}{a_{r+1}(\tau)} d\tau, \end{aligned} \tag{4}$$

where $i = 0, 1, \dots, r$; $r = 0, 1, \dots, n-1$; $t, s \in [t_0, \infty)$.

Now we state two well-known Kiguradze's Lemmas which will be needed in proving our results. For the proof see for example [16].

LEMMA 1. *Let $y \in \mathscr{D}(L_n)$ satisfy $y(t) > 0$ and $L_n y(t) < 0$ on $[t_y, \infty)$, $t_y \geqq t_0$.*

Then there exist a $T \geqq t_y$ *and an integer* ℓ, $0 \leqq \ell \leqq n-1$, *such that* $n+\ell$ *is odd and*

$$L_i y(t) > 0 \ \ on \ \ [T, \infty) \ \ for \quad i = 0, 1, \ldots, \ell, \tag{5}$$

$$(-1)^{i-\ell} L_i y(t) > 0 \ \ on \ \ [T, \infty) \ \ for \quad i = \ell, \ell+1, \ldots, n. \tag{6}$$

LEMMA 2. *Let* $y \in \mathscr{D}(L_n)$ *satisfy* $y(t)>0$ *and* $L_n y(t)>0$ *on* $[t_y, \infty)$, $t_y \geqq t_0$. *Then either*

$$L_i y(t) > 0 \ \ on \ \ [T, \infty) \ \ for \quad i = 0, 1, \ldots, n, \tag{7}$$

or there exist an integer ℓ, $0 \leqq \ell \leqq n-2$, *such that* $n+\ell$ *is odd and* (5) *and* (6) *hold on* $[T, \infty)$.

We shall prove further the following lemma which plays an important role in our later considerations.

LEMMA 3. *Suppose that the conditions* (i)–(iii) *hold. If* $x(t)$ *is any nonoscillatory solution of* (1) (*or* (2)) *on an interval* $[t_x, \infty)$, $t_x \geqq t_0$, *then* $L_0 x(t)$ *is bounded away from zero,* i.e. *there exist a* $T \geqq t_x$ *and a positive constant* c *such that* $|L_0 x(t)| \geqq c$ *whenever* $t \geqq T$.

PROOF. We consider only (1). Let $x(t)$ be a nonoscillatory solution of (1) on $[t_x, \infty)$, $t_x \geqq t_0$. Choose t_1 sufficiently large and assume $x(t)>0$ for $t \geqq t_1 \geqq t_x$ (the proof for $x(t)<0$ being similar). Since $g_i(t) \to \infty$ as $t \to \infty$, there is a $t_2 \geqq t_1$ such that $x(g_i(t))>0$ for $t \geqq t_2$ and $i=1, 2,\ldots, m$. Put $u(t)=x(t)-p(t)$. By (1) and condition (i) we have $L_n u(t)<0$ for $t \geqq t_2$ and consequently $L_i u(t)$, $i=0, 1,\ldots, n-1$, have to be eventually of constant sign. In particular, $L_0 u(t)$ is either positive or negative for $t \geqq t_3 \geqq t_2$, where t_3 is sufficiently large.

Assume first that $L_0 x(t)$ is unbounded for large t. Then $L_0 u(t)$ is also unbounded and $L_0 u(t)>0$ for $t \geqq t_3$. From Lemma 1 it follows that $L_1 u(t)>0$ for every large t.

If $L_0 x(t)$ is bounded we use Lemma 1 in the case $L_0 u(t)>0$, resp. Lemma 2 in the case $L_0 u(t)<0$, to conclude that $L_1 u(t)>0$ for $t \geqq t_3$.

Hence in both cases we conclude that the function $L_0 u(t)$ is increasing on $[t_3, \infty)$. Choose $t_* \geqq t_3$ such that $L_0 p(t_*) = \min_{t \in [t_3, \infty)} L_0 p(t)$. Then

$$L_0 x(t) \geqq L_0 x(t_*) + L_0 p(t) - L_0 p(t_*) \geqq L_0 x(t_*) > 0 \quad \text{for} \quad t \geqq t_3$$

and the proof is complete.

3. Main results

On the basis of Lemma 3 we can prove

THEOREM 1. *Suppose that the conditions* (i)–(iii) *and*

$$\limsup_{t\to\infty} a_0(t) < \infty \tag{8}$$

are satisfied and, moreover, for any $c>0$ *there is a* $c_1>0$ *such that for all* $t \geqq t_0$

$$x_i \geqq c,\ i = 1, 2,\dots, m, \text{ implies } f(t, x_1,\dots, x_m) \geqq c_1, \quad \text{and}$$
$$x_i \leqq -c,\ i = 1, 2,\dots, m, \text{ implies } f(t, x_1,\dots, x_m) \leqq -c_1.$$

Then every solution $x(t)$ *of* (1) *is oscillatory.*

PROOF. Assume to the contrary that there exists a nonoscillatory solution $x(t)$ of (1). Let $x(t)$ be positive for $t \geqq t_x \geqq t_0$. It follows from Lemma 3 that there exist a $T \geqq t_x$ and a positive constant c such that $L_0x(t) \geqq c$ for $t \geqq T$. By (ii) we have that there is a $T_1 \geqq T$ such that $L_0x(g_i(t)) \geqq c$ for $i=1, 2,\dots, m$ and $t \geqq T_1$. Hence, putting $u(t)=x(t)-p(t)$ and taking (8) into account, we have from (1)

$$L_nu(t) \leqq -f(t, x(g_1(t)),\dots, x(g_m(t))) \leqq -c_1 < 0, \quad t \geqq T_1,$$

and by Lemma 1 from [16] we get $\lim_{t\to\infty} L_0u(t) = -\infty$. Since $L_0p(t)$ is bounded, $\lim_{t\to\infty} L_0x(t) = \lim_{t\to\infty}(L_0u(t)+L_0p(t)) = -\infty$, a contradiction to the positivity of $x(t)$.

In the case $x(t)<0$ on $[t_x, \infty)$ the proof is similar.

EXAMPLE 1. All assumptions of Theorem 1 hold in the case of the inequality

$$x(t)\{x''(t) + [x^2(t) + x^2(t-\pi/2)]x(t) - \sin t\} \leqq 0, \quad t \geqq \pi/2. \tag{9}$$

Thus all solutions of (9) are oscillatory. One such solution is $x(t)=\sin t$.

EXAMPLE 2. The advanced inequality

$$x(t)\Big\{(tx'(t))' + 2e^{\pi/2}[x^2(t) + e^{\pi}x^2(e^{\pi/2}t)]x(e^{\pi/2}) - \frac{2\sin(\log t)}{t^2}\Big\} \leqq 0, \quad t \geqq 1, \tag{10}$$

has the oscillatory solution $x(t)=\sin(\log t)/t$. Moreover, as it follows from Theorem 1 where we have $p(t)=\cos(\log t)/t$, every solution of (10) is oscillatory.

EXAMPLE 3. Consider the inequality

$$x(t)\{(e^{-t}x'(t))' + e^{-\pi/2}x(t-\pi/2) + e^{-\pi}x(t-\pi) - e^{-2t}(\mathrm{sint} - 3\cos t) + e^{-t}(\cos t + \mathrm{sint})\} \leqq 0 \tag{11}$$

for $t \geqq \pi$. Here $p(t)=(1+e^{-t})\sin t$. Since all conditions of Theorem 1 are satisfied, every solution of (11) is oscillatory. For example, $x(t)=e^{-t}\sin t$ is one such solution.

The next example shows that Theorem 1 is in general false in the case of odd order inequality (2). However, we are able to prove a similar theorem concerning the oscillation of all bounded solutions of (2).

EXAMPLE 4. The third order linear inequality

$$x(t)\{t(t(tx'(t))')' - x(t-\pi) - t(\cos t - 3t\sin t - t^2\cos t)\} \geqq 0, \tag{12}$$

$t \geqq \pi$, satisfies all the conditions of Theorem 1 with $p(t)=\sin t$, but it has $x(t)=t+\sin t$ as an unbounded nonoscillatory solution. On the other hand, the above inequality admits the bounded oscillatory solution $x(t)=\sin t$.

THEOREM 2. *Suppose that the conditions of Theorem 1 are satisfied. Then every bounded solution $x(t)$ of (2) is oscillatory.*

PROOF. Let $x(t)$ be a bounded nonoscillatory solution of the inequality (2). Arguing exactly as in the proof of Theorem 1 we conclude that $\lim_{t\to\infty} L_0x(t)=\infty$ for $x(t)$ eventually positive, resp. $\lim_{t\to\infty} L_0x(t)=-\infty$ for $x(t)$ eventually negative. In view of (8), this contradicts the boundedness of $x(t)$.

EXAMPLE 5. Consider the equation

$$\begin{aligned}(t(tx'(t))')' &- e^{\pi/2}t^2[x^2(t) + e^{\pi}x^2(e^{\pi/2}t)]x(e^{\pi/2}t) = \\ &= \frac{2\sin(\log t)+2\cos(\log t)}{t^2} - \frac{\cos(\log t)}{t}\end{aligned} \tag{13}$$

for $t \geqq 1$. Here the forcing term is the third "quasi-derivative" of the strongly bounded function $p(t)=(1+t^{-1})\sin(\log t)$. Moreover, since the problem of oscillation of the functional differential inequalities (1) and (2) includes the problem of oscillation of the corresponding functional differential equations, we may conclude, by Theorem 2, that all bounded solutions of the above equation are oscillatory. In fact, $x(t)=\sin(\log t)/t$ is one such solution.

Now, let the function $f(t, x_1,\dots, x_m)$ satisfy, in addition to (i), the following condition:

(iv) for any $u \in \mathscr{D}(L_n)$ such that

$$u(t) \geqq c\alpha_{k-1}(t), \text{ resp. } u(t) \leqq -c\alpha_{k-1}(t),$$

for some constant $c>0$, some integer k, $1 \leqq k \leqq n-1$, and $t \geqq t_1 \geqq t_0$, there exists $t_2 \geqq t_1$ such that

$$f(t, u(g_1(t)),\dots, u(g_m(t)))$$
$$\geqq f(t, c\alpha_{k-1}(g_1(t)),\dots, c\alpha_{k-1}(g_m(t))),$$

resp.

$$f(t, u(g_1(t)),\dots, u(g_m(t)))$$
$$\leqq f(t, -c\alpha_{k-1}(g_1(t)),\dots, -c\alpha_{k-1}(g_m(t))),$$

on $[t_2, \infty)$.

THEOREM 3. *Suppose that the conditions* (i)–(iv) *are satisfied. If, moreover,*

$$\int_T^\infty \omega_{n-k-1}(\tau, T) f(\tau, c\alpha_{k-1}(g_1(\tau)),\dots, c\alpha_{k-1}(g_m(\tau)))d\tau = \infty \tag{14}$$

and

$$\int_T^\infty \omega_{n-k-1}(\tau, T) f(\tau, -c\alpha_{k-1}(g_1(\tau)),\dots, -c\alpha_{k-1}(g_m(\tau)))d\tau = -\infty \tag{15}$$

for every $T \geqq t_0$, every positive constant c and every odd integer $k=1, 3,\dots, n-1$, then all solutions of the inequality (1) *are oscillatory.*

PROOF. Assume that there exists a nonoscillatory solution $x(t)$ of (1). Without loss of generality, we may assume that $x(t)$ and $x(g_i(t))$ are positive for $t \geqq t_1 \geqq t_0$ and $i=1, 2,\dots, m$. Put $u(t)=x(t)-p(t)$. In the proof of Lemma 3 we have shown that there is $t_2 \geqq t_1$ such that $L_1u(t)>0$ for $t \geqq t_2$. Thus, by Lemma 1 with $y(t)=L_1u(t)$ and

$$\tilde{L}_{n-1}y(t) = a_n(t)(a_{n-1}(t)(\cdots(a_2(t)y'(t))'\cdots)')'$$

in place of $L_ny(t)$, we conclude that there are an odd integer $\ell \in \{1, 3,\dots, n-1\}$ and a $T \geqq t_2$ such that

$$L_iu(t) = \tilde{L}_{i-1}y(t) > 0 \quad \text{on} \quad [T, \infty) \tag{16}$$

for $i=1, 2,\dots, \ell$, and

$$(-1)^{i-\ell}L_iu(t) = (-1)^{i-\ell}\tilde{L}_{i-1}y(t) > 0 \quad \text{on} \quad [T, \infty) \tag{17}$$

for $i=\ell, \ell+1,\dots, n$.

Using the formula (4) with $y(t)=u(t)$, $i=\ell$, $r=n-1$, and taking (17) into account, we get

$$L_\ell u(t) \geqq (-1)^{n-\ell}\int_t^s I_{n-\ell-1}(\tau, t; n-1,\dots, \ell+1)\frac{L_nu(\tau)}{a_n(\tau)}d\tau$$
$$= -\int_t^s I_{n-\ell-1}(\tau, t; n-1,\dots, \ell+1)\frac{L_nu(\tau)}{a_n(\tau)}d\tau$$

on $[T, \infty)$. Thus, for $s\to\infty$

$$L_\ell u(t) \geqq -\int_t^\infty I_{n-\ell-1}(\tau, t; n-1,\dots, \ell+1)\frac{L_n u(\tau)}{a_n(\tau)} s\tau, \quad t\geqq T,$$

and from (1)

$$L_\ell u(t) \geqq \int_t^\infty \omega_{n-\ell-1}(\tau, t) f(\tau, x(g_1(\tau)),\dots, x(g_m(\tau)))d\tau, \quad t\geqq T. \tag{18}$$

On the other hand, it is not difficult to verify that

$$I_r(t, s; j_1,\dots, j_r) = (-1)^r I_r(s, t; j_r,\dots, j_1)$$

for $1\leqq r\leqq n-1$ and, therefore, we can rewrite (4) as

$$L_i y(t) = \textstyle\sum_{j=i}^r L_j y(s) I_{j-i}(t, s; i+1,\dots, j) + \\ + \displaystyle\int_s^t I_{r-i}(t, \tau; i+1,\dots, r)\frac{L_{r+1}y(\tau)}{a_{r+1}(\tau)}d\tau, \tag{19}$$

$i=0, 1,\dots, r$; $r=0, 1,\dots, n-1$.

If $\ell>1$, then using the above formula with $y(t)=u(t)$, $i=0$, $r=\ell-2$, $s=T$, and taking (16) into account, we have

$$L_0 u(t) \geqq L_0 u(T) + L_{\ell-1}u(T) I_{\ell-1}(t, T; 1,\dots, \ell-1) \quad \text{for} \quad t\geqq T. \tag{20}$$

Since $L_0 p(t)$ is strongly bounded there is a $T_*\geqq T$ such that $L_0 p(T_*)= \min_{t\in[T,\infty)} L_0 p(t)$ and it follows from (20) with T_* in place of T that

$$\begin{aligned} L_0 x(t) &\geqq L_0 p(t) - L_0 p(T_*) + L_0 x(T_*) + L_{\ell-1}u(T_*) I_{\ell-1}(t, T_*; 1,\dots, \ell-1) \\ &\geqq L_0 x(T_*) + L_{\ell-1}u(T_*) I_{\ell-1}(t, T_*; 1,\dots, \ell-1) \\ &\geqq L_{\ell-1}u(T_*) I_{\ell-1}(t, T_*; 1,\dots, \ell-1) \end{aligned}$$

for $t\geqq T_*$. Thus there exist a $c>0$ and a $T_1\geqq T_*$ such that

$$x(t) \geqq c\alpha_{\ell-1}(t) \tag{21}$$

for $t\geqq T_1$ and $\ell>1$.

From Lemma 3 it follows that (21) holds also for $\ell=1$.

By (iv) and (18) we have now for sufficiently large t

$$L_\ell u(t) \geqq \int_t^\infty \omega_{n-\ell-1}(s, t) f(s, c\alpha_{\ell-1}(g_1(s)),\dots, c\alpha_{\ell-1}(g_m(s)))ds,$$

a contradiction to (14).

Similarly we can prove

Theorem 4. *Suppose that the conditions* (i)–(iv) *are satisfied. If, moreover,* (14) *and* (15) *hold for every* $T \geqq t_0$, *every positive constant c and every odd integer* $k=1, 3,\dots, n-2$, *then all solutions* $x(t)$ *of* (2) *such that*

$$x(t) = O(\alpha_{n-1}(t)) \quad as \quad t \to \infty$$

are oscillatory.

Example 6. For an illustration of Theorem 3 consider the equation

$$(22) \qquad (t(t^{-1}(t(t^{-1}x(t))')')')' + 3t^{-3}x(e^{-\pi}t) = \\ = -4t^{-4}[3\sin(\log t)+5\cos(\log t)] - 3t^{-3}\sin(\log t), \quad t \geqq 1.$$

It is not difficult to verify that all the conditions of Theorem 3 are satisfied with $p(t)=(1+t)\sin(\log t)$ for which $L_0p(t)$ is strongly bounded, and so all solutions of (22) are oscillatory. One such solution is $x(t)=\sin(\log t)$.

We now give an example which illustrates that the conclusion of Theorem 3 is in general false if $L_0p(t)$ is assumed only to be bounded. Similar examples can be found also for our other results.

Example 7. The inequality

$$(23) \qquad x(t)\{(t(t^{-1}x(t))')' + 2t^{-2}x(t) \\ - 2t^{-2}[\sin(\log t)+\cos(\log t)+3]\} \leqq 0, \quad t \geqq 1,$$

has the nonoscillatory solution $x(t)=2+\sin(\log t)$. Here all the hypotheses of Theorem 3 are satisfied except that $L_0p(t)=t^{-1}[\sin[\log t)+\cos(\log t)+6]$ is not strongly bounded.

Our next result concerns the oscillation of all bounded solutions of (1) (or (2)).

Theorem 5. *Let the conditions* (i)–(iii) *and* (8) *be satisfied and let the function f have the following property*:

(v) *for any* $c>0$ *there is a* $c_1>0$ *such that for all* $t \geqq t_0$

$$x_i \geqq c,\ 1 \leqq i \leqq m, \textit{ implies } f(t, x_1,\dots, x_m) \geqq f(t, c_1,\dots, c_1)$$

and

$$x_i \leqq -c,\ 1 \leqq i \leqq m, \textit{ implies } f(t, x_1,\dots, x_m) \leqq f(t, -c_1,\dots, c_1).$$

If, moreover,

$$(24) \qquad \int_T^\infty \omega_{n-1}(\tau, T)f(\tau, c,\dots, c)d\tau = \infty$$

and

$$\int_T^\infty \omega_{n-1}(\tau, T)f(\tau, -c,\dots, -c)d\tau = -\infty \tag{25}$$

for every $T \geqq t_0$ *and every positive constant* c, *then all bounded solutions of the inequality* (1) (*or* (2)) *are oscillatory.*

PROOF. We consider only the inequality (1).

Assume the existence of a bounded nonoscillatory solution $x(t)$ of (1). Let this solution be positive for $t \geqq t_1 \geqq t_0$. Introducing the function $u(t) = x(t) - p(t)$ and proceeding as in the proof of Lemma 3, we get $L_n u(t) < 0$ for $t \geqq t_2 \geqq t_1$, where t_2 is sufficiently large. Since $L_0 u(t)$ is bounded, we have by Lemma 1 in the case $L_0 u(t) > 0$, resp. by Lemma 2 in the case $L_0 u(t) < 0$, that there exists a $t_3 \geqq t_2$ such that

$$(-1)^{i-1} L_i u(t) > 0 \quad \text{for} \quad t \geqq t_3 \quad \text{and} \quad i = 1, 2,\dots, n. \tag{26}$$

Moreover, it follows from Lemma 3 that there are a $T \geqq t_3$ and a constant $c > 0$ such that

$$x(g_i(t)) \geqq c \quad \text{for} \quad t \geqq T \quad \text{and} \quad i = 1, 2,\dots, m.$$

Now, an application of formula (4) with $i = 1$ and $r = n-1$ to $u(t)$ and taking (26) into consideration give

$$L_1 u(t) \geqq -\int_t^s I_{n-2}(\tau, t;\, n-1,\dots, 2)\, \frac{L_n u(\tau)}{a_n(\tau)}\, d\tau \tag{27}$$

for $s \geqq t \geqq T$. Dividing (27) by $a_1(t)$ and integrating from T to t, we obtain after some manipulations

$$L_0 u(t) \geqq L_0 u(T) - \int_T^t I_{n-1}(\tau, T;\, n-1,\dots, 1)\, \frac{L_n u(\tau)}{a_n(\tau)}\, d\tau, \quad t \geqq T,$$

which by (1) and (v) yields

$$L_0 u(t) \geqq L_0 u(T) + \int_T^t \omega_{n-1}(\tau, T) f(\tau, c_1,\dots, c_1) d\tau$$

for $t \geqq T$ and some constant $c_1 > 0$. Finally, if we let $t \to \infty$ in the last relation, we get a contradiction to the boundedness of $L_0 u(t)$.

A similar argument holds for $x(t)$ eventually negative, and this completes the proof.

Following the results of Grace and Lalli [6] we can similarly establish

THEOREM 6. *Let the conditions* (i)–(iii), (8) *and* (v) *be satisfied. If, moreover,*

(28) $$\limsup_{t\to\infty} \frac{1}{\alpha_2(t)} \int_T^t \omega_{n-1}(\tau, T) f(\tau, c, \ldots, c) d\tau > 0$$

and

(29) $$\liminf_{t\to\infty} \frac{1}{\alpha_2(t)} \int_T^t \omega_{n-1}(\tau, T) f(\tau, -c, \ldots, -c) d\tau < 0$$

for every $T \geqq t_0$ *and every positive constant* c, *then every solution* $x(t)$ *of* (1) (*or* (2)) *such that* $L_0 x(t)/\alpha_2(t) \to 0$ *as* $t \to \infty$ *is oscillatory.*

In order to prove this theorem it sufficies to show that the inequalities (26) remain valid for the positive solution $x(t)$ of (1) (or (2)) such that $L_0 x(t)/\alpha_2(t) \to 0$ as $t \to \infty$. But this is possible to do in an analogous way as in the proof of Lemma in [6]. The rest of the proof follows along the lines of that of Theorem 5, and so we omit it.

REMARK. As mentioned in the Introduction, the class of strongly bounded functions contains the following particular classes of continuous functions which have frequently appeared in the literature concerning the oscillation of forced differential equations and inequalities:

(I) the class of functions φ: $[t_0, \infty) \to \boldsymbol{R}$ which are oscillatory and such that $\lim_{t\to\infty} \varphi(t) = 0$,

(II) the class of functions φ: $[t_0, \infty) \to \boldsymbol{R}$ such that there exist sequences $\{t'_n\}_{n=1}^{\infty}$, $\{t''_n\}_{n=1}^{\infty}$ and constants q_1, q_2 such that $\lim_{n\to\infty} t'_n = \lim_{n\to\infty} t''_n = \infty$, $\varphi(t'_n) = q_1$, $\varphi(t''_n) = q_2$, and $q_1 \leqq \varphi(t) \leqq q_2$ for $t \geqq t_0$.

Obviously, the function $L_0 p(t)$ in Example 2 is of the type (I), while $L_0 p(t)$ in Examples 1 and 4 are of the type (II). On the other hand, there exist strongly bounded functions which need not satisfy (I) or (II). In fact, the forcings in Examples 3, 5 and 6 represent such functions.

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