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Block one-step methods for starting multistep methods

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1. Introduction

Consider the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0, \tag{1.1}$$

where the function $f(x, y)$ is assumed to be sufficiently smooth. Let $y(x)$ be the solution of this problem, let

$$x_t = x_0 + th \qquad (t>0,\ h>0) \tag{1.2}$$

and denote by y_t an approximation to $y(x_t)$, where h is a stepsize. Multistep methods for solving (1.1) numerically require starting values. For instance, a linear k-step method needs y_t $(t=1, 2,\dots, k-1)$ (see [2]), and a two-step method with one off-step node x_v $(0<v<1)$ requires y_v and y_1 (see [6]).

Rosser [3] and Shintani [4, 5] have proposed block one-step methods of the form

$$y_t = y_0 + h \textstyle\sum_{i=1}^{q} p_{it}k_i \tag{1.3}$$

that provide y_t for $t=1, 2,\dots, N$, where

$$k_1 = f(x_0, y_0), \tag{1.4}$$

$$k_i = f(x_0 + a_i h,\ y_0 + h \textstyle\sum_{j=1}^{i-1} b_{ij}k_j) \qquad (i=2, 3,\dots, q), \tag{1.5}$$

$$\textstyle\sum_{j=1}^{i-1} b_{ij} = a_i, \quad a_i \neq 0 \qquad (i=2, 3,\dots, q), \tag{1.6}$$

p_{kt} $(k=1, 2,\dots, q)$, a_i and b_{ij} $(j=1, 2,\dots, i-1;\ i=2, 3,\dots, q)$ are constants. Rosser has shown that for $q=N(N+3)/2$ there exists a method (1.3) of order $N+1$ for $t=1, 2,\dots, N$ and that for $t=N=2p$ the order can be raised to $N+2$ with one more evaluation of f. Shintani [5] has proved that for $q=4$, 6 there exists a method (1.3) which is of order 3, 4 for $t=1$ and is of order 4, 5 for $t=2$ respectively. But these methods cannot be used to start multistep methods with off-step nodes. Gear [1] has considered methods of the form (1.3) that can provide y_t for any t's and has shown that there exists a method (1.3) of order 3 for $q=4$ but none for $q=3$, that a method of order 4 exists for $q=6$ and that q must not be less than 9 for constructing a method of order 5.

Let a be a specified value of t. Then in this paper it is shown that for $q=3, 4, 6, 9$ there exists a method (1.3) which is of order 2, 3, 4, 5 for $t\neq a$ respectively and is of order 3, 4, 5, 6 for $t=a$ respectively.

2. Preliminaries

We write p_{it} simply as p_i for $i=1, 2,\dots, q$, but no confusion will arise. Let

(2.1) $c_i = \sum_{j=2}^{i-1} a_j b_{ij}, \quad d_i = \sum_{j=2}^{i-1} a_j^2 b_{ij}, \quad e_i = \sum_{j=2}^{i-1} a_j^3 b_{ij} \qquad (i=3, 4,\dots, q),$

(2.2) $l_i = \sum_{j=3}^{i-1} c_j b_{ij}, \quad m_i = \sum_{j=3}^{i-1} d_j b_{ij}, \quad g_i = \sum_{j=3}^{i-1} a_j c_j b_{ij} \qquad (i=4, 5,\dots, q).$

Let D be the differential operator defined by

$$D = \frac{\partial}{\partial x} + k_1 \frac{\partial}{\partial y} \tag{2.3}$$

and put

(2.4) $D^j f(x_0, y_0) = T^j, \quad D^j f_y(x_0, y_0) = S^j \qquad (j=1, 2,\dots),$

$(Df)^2(x_0, y_0) = P, \quad (Df_y)^2(x_0, y_0) = Q, \quad Df_{yy}(x_0, y_0) = R,$

$f_y(x_0, y_0) = f_y, \quad f_{yy}(x_0, y_0) = f_{yy}.$

Then y_t can be expanded into power series in h as follows:

(2.5) $y_t = y_0 + hA_1k_1 + h^2A_2T + (h^3/2!)(A_3T^2 + 2A_4f_yT) + (h^4/3!)(B_1T^3 + 6B_2TS + 3B_3f_yT^2 + 6B_4f_y^2T) + (h^5/4!)(C_1T^4 + 12C_2TS^2 + 12C_3T^2S + 12C_4f_{yy}P + 4C_5f_yT^3 + 12C_6f_y^2T^2 + 24C_7f_y^3T + 24C_8f_yTS) + (h^6/5!)(D_1T^5 + 20D_2TS^3 + 30D_3T^2S^2 + 20D_4T^3S + 60D_5f_{yy}TT^2 + 60D_6PR + 120D_7TQ + 60D_8f_yf_{yy}P + 60D_9f_yTS^2 + 60D_{10}f_yT^2S + 120D_{11}f_y^2TS + 5D_{12}f_yT^4 + 20D_{13}f_y^2T^3 + 60D_{14}f_y^3T^2 + 120D_{15}f_y^4T) + O(h^7),$

where

(2.6) $A_1 = \sum_{i=1}^q p_i, \quad A_2 = \sum_{i=2}^q a_i p_i,$

(2.7) $A_3 = \sum_{i=2}^q a_i^2 p_i, \quad B_1 = \sum_{i=2}^q a_i^3 p_i, \quad C_1 = \sum_{i=2}^q a_i^4 p_i, \quad D_1 = \sum_{i=2}^q a_i^5 p_i,$

(2.8) $A_4 = \sum_{i=3}^q c_i p_i, \quad B_2 = \sum_{i=3}^q a_i c_i p_i, \quad B_3 = \sum_{i=3}^q d_i p_i, \quad C_2 = \sum_{i=3}^q a_i^2 c_i p_i,$

$C_3 = \sum_{i=3}^q a_i d_i p_i, \quad C_4 = \sum_{i=3}^q c_i^2 p_i, \quad C_5 = \sum_{i=3}^q e_i p_i, \quad D_2 = \sum_{i=3}^q a_i^3 c_i p_i,$

$$D_3 = \sum_{i=3}^q a_i^2 d_i p_i, \quad D_4 = \sum_{i=3}^q a_i e_i p_i, \quad D_5 = \sum_{i=3}^q c_i d_i p_i,$$

$$D_6 = \sum_{i=3}^q a_i c_i^2 p_i,$$

(2.9) $$B_4 = \sum_{i=4}^q l_i p_i, \quad C_6 = \sum_{i=4}^q m_i p_i, \quad C_7 = \sum_{i=5}^q (\sum_{j=4}^{i-1} l_j b_{ij}) p_i,$$

$$C_8 = \sum_{i=4}^q (a_i l_i + g_i) p_i, \quad D_7 = \sum_{i=4}^q a_i g_i p_i,$$

(2.10) $$D_8 = \sum_{i=4}^q (2c_i l_i + \sum_{j=3}^{i-1} c_j^2 b_{ij}) p_i, \quad D_9 = \sum_{i=4}^q (a_i^2 l_i + \sum_{j=3}^{i-1} a_j^2 c_j b_{ij}) p_i,$$

$$D_{10} = \sum_{i=4}^q (a_i m_i + \sum_{j=3}^{i-1} a_j d_j b_{ij}) p_i,$$

$$D_{11} = \sum_{i=5}^q [\sum_{j=4}^{i-1} (a_i l_j + a_j l_j + g_j) b_{ij}] p_i,$$

$$D_{12} = \sum_{i=3}^q (\sum_{j=2}^{i-1} a_j^4 b_{ij}) p_i, \quad D_{13} = \sum_{i=4}^q (\sum_{j=3}^{i-1} e_j b_{ij}) p_i,$$

$$D_{14} = \sum_{i=5}^q (\sum_{j=4}^{i-1} m_j b_{ij}) p_i, \quad D_{15} = \sum_{i=6}^q [\sum_{j=5}^{i-1} (\sum_{k=4}^{j-1} l_k b_{jk}) b_{ij}] p_i.$$

Put

(2.11) $$A_{1t} = A_1 - t, \quad A_{2t} = A_2 - t^2/2, \quad A_{3t} = A_3 - t^3/3, \quad A_{4t} = A_4 - t^3/6,$$

(2.12) $$B_{it} = B_i - t^4/(4u_i) \quad (i=1, 2, 3, 4), \quad C_{jt} = C_j - t^5/(5v_j) \quad (j=1, 2,\dots, 8),$$

$$D_{kt} = D_k - t^6/(6w_k) \qquad (k=1, 2,\dots, 15),$$

where

(2.13) $$u_1 = 1, \quad u_2 = 2, \quad u_3 = 3, \quad u_4 = 6, \quad v_i = i \quad (i=1, 2, 3, 4), \quad v_5 = 4,$$

$$v_6 = 12, \quad v_7 = 24, \quad v_8 = 24/7,$$

(2.14) $$w_i = i \quad (i=1, 2, 3, 4), \quad w_5 = 6, \quad w_6 = 4, \quad w_7 = 8, \quad w_8 = 60/13,$$

$$w_9 = 15/4, \quad w_{10} = 20/3, \quad w_{11} = 10, \quad w_{12} = 5, \quad w_{13} = 20, \quad w_{14} = 60,$$

$$w_{15} = 120.$$

Then we have

(2.15) $$y_t - y(x_t) = hA_{1t}k_1 + h^2 A_{2t} T + (h^3/2)(A_{3t} T^2 + 2A_{4t} f_y T) + \dots.$$

For simplicity assume that a_i $(i=2, 3,\dots, q)$ are all distinct and put

(2.16) $$L_{ij} = a_i \prod_{k=2}^j (a_i - a_k), \quad M_{ij} = a_i \prod_{k=3}^j (a_i - a_k) \qquad (i>j).$$

If we impose the condition

(2.17) $$p_2 = 0, \quad c_i = a_i^2/2, \quad d_i = a_i^3/3 \qquad (i=3, 4,\dots, q),$$

then we have

(2.18) $$2A_{4t} = A_{3t}, \quad 2B_{2t} = 3B_{3t} = B_{1t}, \quad 2C_{2t} = 3C_{3t} = 4C_{4t} = C_{1t},$$

$$2D_{2t} = 3D_{3t} = 6D_{5t} = 4D_{6t} = D_{1t},$$

(2.19) $$3a_2 = 2a_3,$$

(2.20) $$a_3^2 b_{i3} + 3\sum_{j=4}^{i-1} a_j(a_j - a_2)b_{ij} = a_i^2(a_i - a_3) \qquad (i=4, 5,\dots, q).$$

Put

(2.21) $$X_1 = a_2 + a_3, \quad Y_1 = a_2a_3, \quad U_1 = a_4 + X_1, \quad V_1 = a_4X_1 + Y_1,$$
$$W_1 = a_4Y_1, \quad X = a_3 + a_4, \quad Y = a_3a_4, \quad U = a_5 + X, \quad V = a_5X + Y,$$
$$W = a_5Y,$$

(2.22) $$Q_1(t) = 3t^2 - 4X_1t + 6Y_1, \quad Q_2(t) = 12t^3 - 15U_1t^2 + 20V_1t - 30W_1,$$
$$R_1(t) = 3t^2 - 4Xt + 6Y, \quad R_2(t) = 3t^2 - 5Xt + 10Y,$$
$$R_3(t) = 12t^3 - 15Ut^2 + 20Vt - 30W,$$

(2.23) $$P_{ik} = \sum_{j=k+1}^{i-1} M_{jk}b_{ij} \ (i \geqq k+2), \quad Q_{ik} = \sum_{j=k+2}^{i-1} P_{jk}b_{ij} \ (i \geqq k+3),$$

(2.24) $$N_k = \sum_{j=k}^{q} M_{jk-1}p_j \ (k \geqq 4), \quad S_k = \sum_{j=k}^{q} P_{jk-2}p_j \ (k \geqq 6),$$
$$T_k = \sum_{j=k}^{q} Q_{jk-3}p_j \ (k \geqq 6),$$

(2.25) $$P_{i3} = \sum_{j=4}^{i-1} M_{ij}E_j \ (i \geqq 5), \quad P_{i4} = \sum_{j=5}^{i-1} M_{ij}F_j \ (i \geqq 6),$$
$$P_{i5} = \sum_{j=6}^{i-1} M_{ij}G_j \ (i \geqq 7), \quad P_{i6} = \sum_{j=7}^{i-1} M_{ij}H_j \ (i \geqq 8),$$
$$P_{i7} = \sum_{j=8}^{i-1} M_{ij}J_j \ (i \geqq 9),$$

(2.26) $$U_2 = a_6 + U, \quad V_2 = a_6U + V, \quad W_2 = a_6V + W, \quad X_2 = a_6W,$$
$$U_3 = a_7 + U_2, \quad V_3 = a_7U_2 + V_2, \quad W_3 = a_7V_2 + W_2,$$
$$X_3 = a_7W_2 + X_2, \quad Y_3 = a_7X_2,$$

(2.27) $$r = a^7/7 - U_3a^6/6 + V_3a^5/5 - W_3a^4/4 + X_3a^3/3 - Y_3a^2/2.$$

3. Construction of methods

3.1. Case $q=3$

The choice $A_{it}=0$ $(i=1, 2, 3)$ yields

(3.1) $$\sum_{i=1}^{3} p_i = t, \quad 2\sum_{i=2}^{3} a_ip_i = t^2, \quad 6L_{32}p_3 = t^2(2t - 3a_2),$$

(3.2) $$6L_{32}A_{4t} = t^2[(2t - 3a_2)c_3 - tL_{32}],$$

(3.3) $$12B_{1t} = -t^2Q_1(t), \quad 12B_{2t} = 12a_3A_{4t} + t^3(4a_3 - 3t)/2,$$
$$6B_{3t} = 6a_2A_{4t} + t^3(2a_3 - t)/2, \quad 24B_{4t} = -t^4.$$

Setting $A_{4a}=0$, we have

(3.4) $$(2a-3a_2)c_3 = aL_{32}, \quad 2(2a-3a_2)A_{4t} = t^2a_2(t-a).$$

Hence for any t, a_2, a_3 and a such that $2a\neq 3a_2$ other constants are determined uniquely.

For the choice $a_2=a/2$ and $a_3=a$ we have

(3.5) $$b_{31} = -a, \quad b_{32} = 2a, \quad 2A_{4t} = t^2(t-a), \quad 4B_{1t} = -t^2(t-a)^2,$$
$$24B_{2t} = t^2(16at-3t^2-12a^2), \quad 12B_{3t} = t^2(a-t)(t-3a),$$
$$24B_{4t} = -t^4.$$

3.2. Case $q=4$

Put

(3.6) $$c_3 = L_{32}K_3, \quad c_4 = L_{42}K_3 + L_{43}K_4.$$

Then the condition $A_{it}=0$ ($i=1, 2, 3, 4$) leads to

(3.7) $$\textstyle\sum_{i=1}^4 p_i = t, \quad 2\sum_{i=2}^4 a_ip_i = t^2, \quad 6\sum_{i=3}^4 L_{i2}p_i = t^2(2t-3a_2),$$
$$6L_{43}K_4p_4 = t^3 - t^2(2t-3a_2)K_3,$$

(3.8) $$12B_{1t} = -t^2Q_1(t) + 12L_{43}p_4, \quad 12B_{3t} = t^3(2a_2-t) + 12L_{32}b_{43}p_4,$$
$$24B_{2t} = t^3(4a_3-3t) + 24[K_3+(a_4-a_3)K_4]L_{43}p_4,$$
$$24B_{4t} = -t^4 + 24c_3b_{43}p_4.$$

If $K_4\neq 0$, then p_i ($i=1, 2, 3, 4$) are determined from (3.7).

Choosing $B_{ia}=0$ ($i=1, 2, 3, 4$), we have

(3.9) $$a_4 = a, \quad 2(a-2a_2)K_3 = a, \quad (2a_2-a)Q_1(a)K_4 = a_2a,$$
$$Q_1(a)L_{32}b_{43} = a(a-2a_2)L_{43}, \quad 2a_2(a-2a_2)b_{32} = aL_{32},$$

(3.10) $$4aB_{1t} = (t-a)t^2(Q_1(t)+2Y_1-at), \quad 8B_{2t} = (t-a)t^2(t+4a_3-3a),$$
$$12B_{3t} = (t-a)t^2(3-3a+6a_2), \quad 24B_{4t} = (t-a)t^2(t-3a),$$

(3.11) $$60C_{1t} = -t^2Q_2(t)/2 + 60U_1B_{1t},$$
$$60C_{2t} = -t^3(12t^2-15Xt+20Y)/2 + 60XB_{2t},$$
$$40C_{3t} = -t^3(8t^2-15a_2t-10a_4t+20a_2a_4)/3 + 40a_2B_{2t} + 40a_4B_{3t},$$
$$60C_{4t} = t^3(10c_3-3t^2) + 60c_4(c_4-c_3)p_4,$$
$$60C_{5t} = -t^3(3t^2-5X_1t+10Y_1) + 60X_1B_{3t},$$

$$120C_{6t} = t^4(5a_2-2t) + 120a_2B_{4t}, \quad 120C_{7t} = -t^5,$$
$$120C_{8t} = t^4(5X-7t) + 120XB_{4t}.$$

Hence for any t, a_2, a_3 and a such that $(2a_2-a)Q_1(a)\neq 0$ other constants are determined uniquely.

For the choice $a_2=a/4$ and $a_3=7a/10$ we have

$$(3.12)\quad b_{31} = -14a/25, \quad b_{32} = 63a/50, \quad b_{41} = 83a/35, \quad b_{42} = -14a/5,$$
$$b_{43} = 10a/7,$$

$$(3.13)\quad C_{1a} = C_{5a} = 0, \quad 240C_{2a} = -a^5, \quad 160C_{3a} = a^5, \quad 480C_{4a} = a^5,$$
$$160C_{6a} = -a^5, \quad 120C_{7a} = -a^5, \quad 80C_{8a} = a^5.$$

3.3. Case $q=6$

We impose the condition (2.17). Setting $A_{it}=B_{it}=0$ $(i=1, 2, 3, 4)$, we have

$$(3.14)\quad p_1 + \textstyle\sum_{i=3}^{6} p_i = t, \quad 2\sum_{i=3}^{6} a_ip_i = t^2, \quad 6N_4 = t^2(2t-3a_3),$$
$$12N_5 = t^2R_1(t), \quad 12M_{65}E_5p_6 = t^3(t-2a_3) - t^2R_1(t)E_4,$$

$$(3.15)\quad 60C_{1t} = -t^2R_3(t) + 60M_{65}p_6, \quad 60C_{5t} = 180C_{6t} = -t^3R_2(t) + 60P_{64}p_6,$$
$$24C_{7t} = -t^5/5 + a_4t^4 - 2Yt^3 + 12(P_{64}-2Q_{63})p_6,$$
$$24C_{8t} = -7t^5/5 + t^3(3t-4a_3)X + t^2R_1(t)a_5 + t^3(t-2a_3)(a_4-2a_3-2a_5)$$
$$+ 12[M_{65}+P_{64}-2(a_6-a_5)P_{63}]p_6.$$

Thus if $E_5\neq 0$, then p_i $(i=1, 3, 4, 5, 6)$ can be determined from (3.14).

The condition $C_{ia}=0$ $(i=1, 2,\dots, 8)$ leads to

$$(3.16)\quad 2R_2(a)E_4 = a(2a-5a_3), \quad 2R_2(a)R_3(a)E_5 = -5aR_4(a),$$
$$R_3(a)M_{54}b_{65} = aR_2(a)M_{65}, \quad (a-a_6)R_4(a) = 0,$$

where

$$R_4(t) = (2a_4-a_3)t^2 - 8Yt + 10a_3Y.$$

Thus E_4, E_5 and b_{65} are determined for any given a_i $(i=3, 4, 5, 6)$ and a such that $R_2(a)R_3(a)\neq 0$. By the condition $E_5\neq 0$ we have $R_4(a)\neq 0$, so that $a_6=a$.

For the choice

$$(3.17)\quad a_3 = a/4, \quad a_4 = a/2, \quad a_5 = 3a/4$$

we have $D_{1a}=0$,

(3.18) $a_2 = a/6, \quad b_{31} = a/16, \quad b_{32} = 3a/16, \quad b_{41} = a/4, \quad b_{42} = -3a/4,$
$b_{43} = a, \quad b_{51} = 57a/16, \quad b_{52} = 9a/16, \quad b_{53} = -9a/2, \quad b_{54} = 9a/8,$
$b_{61} = 71a/14, \quad b_{62} = 579a/28, \quad b_{63} = -114a/7, \quad b_{64} = 15a/28,$
$b_{65} = 8a/7.$

3.4. Case $q=9$

We impose the condition (2.17). Choosing $A_{it}=B_{it}=0$ $(i=1, 2, 3, 4)$ and $C_{jt}=0$ $(j=1, 2, \ldots, 8)$, we have

(3.19) $p_1 + \sum_{i=3}^{9} p_i = t, \quad 2\sum_{i=3}^{9} a_i p_i = t^2, \quad 6N_4 = t^2(2t-3a_3),$
$N_5 = r_2(t),$

(3.20) $N_6 = r_3(t),$

(3.21) $\sum_{i=6}^{9}(\sum_{j=5}^{i-1} M_{ij}E_j)p_i = r_4(t) - E_4 r_2(t), \quad S_6 = r_5(t),$

(3.22) $T_6 = r_6(t),$

(3.23) $\sum_{i=6}^{9}(a_i - a_5)P_{i3}p_i = r_7(t) - a_5 r_4(t),$

(3.24) $D_{1t} = N_7 - r_8(t), \quad 2D_{7t} = U_7 - r_9(t) = D_{4t},$
$4D_{8t} = S_7 - 4V_7 + 2(D_{1t} + t^6/6 - a_3 t^5/5) + s_1(t) + 4s_2(t) - 13t^6/90,$
$2D_{9t} = S_7 - 2V_7 + D_{1t} + t^6/6 - a_3 t^5/5 + 2s_2(t) - 4t^6/45 + s_1(t),$
$3D_{10t} = S_7 + U_7 + a_3(t^5/5 - a_3 t^4/4) + s_1(t) + (a_4 - 2a_3)r_7(t)$
$+ a_6 r_5(t) - 3t^6/40,$
$2D_{11t} = -2\sum_{i=7}^{9}[(a_i - a_6)Q_{i3} + \sum_{j=6}^{i-1}(a_j - a_5)P_{j3}b_{ij}]p_i + S_7$
$+ U_7 + T_7 + s_3(t) + s_0(t) + a_6 r_5(t) - t^6/30 + a_4 r_7(t),$
$9D_{12t} = 9S_7 + a_3^2[t^4/3 - 4r_1(t) + 8r_4(t)] + 9s_1(t) - 3t^6/10,$
$9D_{13t} = 9T_7 + 2s_1(t) + 3s_3(t) + 6a_4 r_6(t) - 2a_4^2 r_4(t)$
$- 2(a_4 + a_5)r_5(t) - 3t^6/40,$
$3D_{14t} = T_7 + s_3(t) - t^6/120,$
$2D_{15t} = -2\sum_{i=7}^{9}(\sum_{j=6}^{i-1} Q_{j3}b_{ij})p_i + T_7 + a_4 r_6(t) - t^6/360,$

where

(3.25) $U_7 = \sum_{i=7}^{9}(a_i - a_6)P_{i4}p_i, \quad V_7 = \sum_{i=7}^{9}(a_i - a_5)(a_i - a_6)P_{i3}p_i,$

(3.26) $12r_1(t) = t^3(3t - 4a_3), \quad 12r_2(t) = t^2 R_1(t), \quad 60r_3(t) = t^2 R_3(t),$

$$12r_4(t) = t^3(t-2a_3), \quad 60r_5(t) = t^3R_2(t), \quad 120r_6(t) = t^4(2t-5a_3),$$
$$120r_7(t) = t^2(8t-15a_3),$$
$$60r_8(t) = t^2[10t^4-12U_2t^3+15V_2t^2-20W_2t+30X_2],$$
$$120r_9(t) = t^4(5t^2-8Xt+15Y) - 120a_6r_5(t),$$

(3.27) $$s_0(t) = a_4^2r_4(t) + (a_4+a_5)r_5(t) - 2(a_5+a_6)r_6(t),$$
$$s_1(t) = a_3^2r_1(t) + (a_4-a_3)(a_4+2a_3)r_4(t) + Ur_5(t),$$
$$s_2(t) = a_5a_6r_4(t) - (a_5+a_6)r_7(t),$$
$$s_3(t) = (a_4-2a_3)r_6(t) + a_3r_5(t) + Yr_4(t).$$

Thus p_i $(i=1, 3, 4, 5, 6)$ can be determined from (3.19) and (3.20).

Let

$$w_1r_6(t) + w_2r_5(t) + w_3[r_4(t)-E_4r_2(t)] + w_4r_3(t) \equiv 0$$

so that (3.22) is expressed as a linear combination of (3.20) and (3.21). Then if

(3.28) $$a_4 \neq 2a_3,$$

the choice $w_4=a_3/2$ yields

(3.29) $$w_1 = 3(a_3-a_4), \quad w_2 = a_4 - 3a_3, \quad w_3 = a_4(a_4-2a_3),$$
$$2a_4(2a_3-a_4)E_4 = a_3a_5,$$

(3.30) $$w_1Q_{i3} + w_2P_{i4} + w_3\sum_{j=5}^{i-1}M_{ij}E_j + w_4M_{i5} = 0 \qquad (i=6, 7, 8, 9).$$

Equating the coefficients of M_{ij} $(j=5, 6,\dots, i-1)$ to zero in (3.30), we have

(3.31) $$w_0F_5 + w_3E_5 + w_4 = 0, \quad w_0F_6 + w_1E_5G_6 + w_3E_6 = 0,$$
$$w_0F_7 + w_1(E_5G_7+E_6H_7) + w_3E_7 = 0,$$
$$w_0F_8 + w_1(E_5G_8+E_6H_8+E_7J_8) + w_3E_8 = 0,$$

where $w_0=w_1E_4+w_2$. If

(3.32) $$w_0w_1E_5 \neq 0,$$

then F_5 and G_k $(k=6, 7, 8)$ can be determined for any given w_3, w_4, E_k, F_k $(k=6, 7, 8)$, H_7, H_8 and J_8.

Eliminating p_6 from (3.21) and (3.23), we have

(3.33) $$\sum_{k=6}^{8} E_kN_{k+1} = s_4(t), \quad \sum_{k=6}^{8} F_kN_{k+1} = s_5(t),$$
$$\sum_{k=6}^{8} [E_{k-1}+(a_{k+1}-a_7)E_k]N_{k+1} = s_7(t) - (a_7-a_5)s_4(t),$$

where

$$s_4(t) = r_4(t) - E_4 r_2(t) - E_5 r_3(t), \quad s_5(t) = r_5(t) - F_5 r_3(t),$$
$$s_7(t) = r_7(t) - a_5 r_4(t) - [E_4 + (a_6 - a_5)E_5] r_3(t). \tag{3.34}$$

Put

$$E_{i+4} = q_i E_{i+5}, \quad F_{i+5} = h_i E_{i+5} \qquad (i=1, 2, 3), \tag{3.35}$$

$$z_1 = q_2 - q_1 + a_8 - a_7, \quad z_2 = q_3 - q_2 + a_9 - a_8. \tag{3.36}$$

Then if

$$q_2 q_3 E_8[(h_2 - h_1)z_2 - (h_3 - h_2)z_1] \neq 0, \tag{3.37}$$

from (3.33) p_j ($j=7, 8, 9$) can be determined for any t.

Now we consider the case $t=a$, so that p_i denotes p_{ia} for $i=1, 2, \ldots, q$. The choice $D_{ia}=0$ ($i=1, 2, \ldots, 15$) yields

$$N_7 = r_8(a), \tag{3.38}$$

$$U_7 = r_9(a), \quad S_7 = r_{10}, \quad V_7 = r_{11}, \quad T_7 = r_{12}, \quad \textstyle\sum_{i=7}^{9} \sum_{j=6}^{i-1} Q_{j3} b_{ij} p_i = r_{13},$$
$$\textstyle\sum_{i=7}^{9} [(a_i - a_6)Q_{i3} + \sum_{j=6}^{i-1} (a_j - a_5) P_{j3} b_{ij}] p_i = r_{14} \quad , \tag{3.39}$$

where

$$r_{10} = a^6/30 - s_1(a), \quad r_{11} = a^6/18 - a_3 a^5/10 + s_2(a),$$
$$r_{12} = a^6/120 - s_3(a), \quad 2r_{13} = a^6/180 - s_3(a) + a_4 r_6(a),$$
$$2r_{14} = a^6/20 - Xa^5/15 + Ya^4/8 - s_1(a) + a_4 r_7(a) + s_0(a). \tag{3.40}$$

From (3.33) and (3.39) we have by (3.31) and (3.38)

$$E_7 N_8 + E_8 N_9 = v_4, \quad F_7 N_8 + F_8 N_9 = v_5,$$
$$[E_6 - (a_9 - a_8)E_7] N_8 + E_7 N_9 = v_7,$$
$$[F_6 - (a_9 - a_8)F_7] N_8 + F_7 N_9 = v_9 + v_{15} h_1,$$
$$[E_5 - (a_9 - a_7)E_6] N_8 + E_6 N_9 = v_{11},$$
$$F_6 S_8 + F_7 S_9 = v_{12}, \quad E_6 S_8 + E_7 S_9 = v_{10},$$
$$E_5 S_8 + [E_6 + (a_8 - a_7)E_7] S_9 = v_{14}, \tag{3.41}$$

where

$$v_0 = s_7(a) - [E_5 + (a_7 - a_5)E_6] r_8(a), \quad v_4 = s_4(a) - E_6 r_8(a),$$
$$v_5 = s_5(a) - F_6 r_8(a), \quad v_6 = v_0 - (a_8 - a_5) v_4, \quad v_7 = v_0 - (a_9 - a_5) v_4, \tag{3.42}$$

$$v_9 = r_9(a) - (a_9 - a_6)s_5(a) - r_8(a)F_5, \quad v_{12} = r_{12} - F_5 r_{10},$$
$$v_{10} = r_6(a) - E_4 r_5(a) - E_5 r_{10}, \quad v_{15} = (a_9 - a_7) r_8(a) E_6,$$
$$v_{11} = r_{11} - [E_4 + (a_7 - a_5)E_5 + (a_7 - a_5)(a_7 - a_6)E_6] r_8(a) - (a_8 - a_6)v_0 - (a_9 - a_5)v_6,$$
$$v_{14} = r_{15} - [r_{10} - (a_7 - a_5) r_5(a)] E_4 + (a_7 - a_6) r_{10} E_5,$$
$$120 r_{15} = 4a^6/3 - (3a_3 + 2a_7)a^5 + 5a_3 a_7 a^4.$$

Choosing $a_9 = a$, by (3.35) we have from (3.41)

(3.43) $$L_1 q_3 = L_2,$$

(3.44) $$L_3 h_1 + L_4 h_2 = z_1 v_{12}, \quad L_6 h_3 = (v_7 - q_3 v_4) h_2 + z_2 v_5,$$

(3.45) $$z_1 E_6 N_8 = - L_5, \quad z_2 E_8 N_9 = L_6,$$

(3.46) $$L_5 H_7 + [(z_2 - q_3) v_{11} + q_2 (q_2 - a_9 + a_7) v_7] H_8 = L_4 - z_1 v_{10},$$
$$q_3 z_1 L_6 J_8 = z_2 L_4,$$

where

(3.47) $$L_1 = L_5 + q_1 z_1 v_4, \quad L_2 = q_2 z_1 v_7 + (q_2 - a_9 + a_8) L_5,$$
$$L_3 = (z_1 + q_1) v_{10} - v_{14}, \quad L_4 = v_{14} - q_1 v_{10}, \quad L_5 = v_{11} - q_2 v_7,$$
$$L_6 = v_6 - q_2 v_4.$$

For any $q_1 \neq 0$, $q_2 \neq 0$, $E_5 \neq 0$, h_1 and H_8 such that

$$L_i \neq 0 \ (i = 1, 2, 4, 5, 6), \quad z_j \neq 0 \ (j = 1, 2),$$

the values of q_3, h_2 and h_3 are determined from (3.43) and (3.44) successively and we have $q_3 \neq 0$, so that $E_i \neq 0$ $(i = 6, 7, 8)$; p_{8a} and p_{9a} are obtained from (3.45); H_7 and J_8 are determined from (3.46).

For the choice $a_j = (j-1)a/8$ $(j = 3, 4, \ldots, 9)$, $N_8 = r$, $q_1 = 2.15a$, $a^2 E_6 = -1.27$, $h_1 = -0.02a$, $aH_8 = 1.08$ we have

$$a_2 = a/6, \quad \tilde{b}_{21} = 1/6, \quad \tilde{b}_{31} = 1/16, \quad \tilde{b}_{32} = 3/16, \quad \tilde{b}_{41} = 3/32, \quad \tilde{b}_{42} = 0,$$
$$\tilde{b}_{43} = 9/32, \quad \tilde{b}_{51} = -1/36, \quad \tilde{b}_{52} = 3/4, \quad \tilde{b}_{53} = -2/3, \quad \tilde{b}_{54} = 4/9,$$
$$\tilde{b}_{61} = -0.1951934524, \quad \tilde{b}_{62} = 1.841337891, \quad \tilde{b}_{63} = -2.035284598,$$
$$\tilde{b}_{64} = 0.8786086310, \quad \tilde{b}_{65} = 0.1355315290, \quad \tilde{b}_{71} = -0.05786087521,$$
$$\tilde{b}_{72} = 1.124085938, \quad \tilde{b}_{73} = -1.003943034, \quad \tilde{b}_{74} = -0.03009339701,$$
$$\tilde{b}_{75} = 0.7396718335, \quad \tilde{b}_{76} = -0.02186046512, \quad \tilde{b}_{81} = 1.100504144,$$
$$\tilde{b}_{82} = -5.070306927, \quad \tilde{b}_{83} = 5.571648504, \quad \tilde{b}_{84} = -0.2884653995,$$

$\tilde{b}_{85} = -2.265681048$, $\tilde{b}_{86} = 2.356071743$, $\tilde{b}_{87} = -0.5287710161$,
$\tilde{b}_{91} = 3.928841861$, $\tilde{b}_{92} = -19.59914764$, $\tilde{b}_{93} = 19.22514986$,
$\tilde{b}_{94} = 2.296193618$. $\tilde{b}_{95} = -9.056014744$, $\tilde{b}_{96} = 3.444657007$,
$\tilde{b}_{97} = 1.528150063$, $\tilde{b}_{98} = -0.7678300215$, $\tilde{p}_1 = -0.01388244549$, $\tilde{p}_2 = 0$,
$\tilde{p}_3 = 2.786592076$, $\tilde{p}_4 = -9.453246618$, $\tilde{p}_5 = 17.35023425$,
$\tilde{p}_6 = -17.82289539$, $\tilde{p}_7 = 11.15624084$, $\tilde{p}_8 = -3.586992329$,
$\tilde{p}_9 = 0.5839496094$, $(h_2-h_1)z_2 - (h_3-h_2)z_1 = 1.62853a^2$,
where $p_{ia} = a\tilde{p}_i$ and $b_{ij} = a\tilde{b}_{ij}$ $(j=1, 2,\dots, i-1;\ i=1, 2,\dots, 9)$.

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