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The first eigenvalue of the Laplacian for the compact quotient of a certain Riemannian symmetric space

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§ 1. Introduction

Let (M, g) be a Riemmanian symmetric space of noncompact type. Then M is isometric to a coset space G/K where G is a non-compact semisimple Lie group with finite center and K is a maximal compact subgroup of G. Put $o=\{K\}\in M$. Normalize g in such a way that it is induced by the Killing form of the Lie algebra of G. Let Γ be a discrete subgroup of G acting fixed point freely on M whose quotient manifold $M_\Gamma=\Gamma\backslash M$ is compact. Let Δ_Γ be the Laplacian (cf. [1]) acting on C^∞ functions on M_Γ for the Riemannian metric induced by g. The compactness of M_Γ implies that the spectrum of Δ_Γ forms a discrete subset of the set of non-negative real numbers. Let $\lambda_1(\Gamma)$ denote the the first positive eigenvalue of Δ_Γ. Consider all such cocompact discrete subgroups Γ of G. Then we know the following inequality for several (M, g)'s,

$$\limsup_{\mathrm{vol}(M_\Gamma)\to\infty} \lambda_1(\Gamma) \leq |\rho|^2,$$

where the positive constant $|\rho|^2$ depends only on M (cf. §2). When (M, g) is the unit disc with the Poincaré metric, H. Huber showed this inequality in [6]. H. Urakawa in [7] generalized it to the case when (M, g) is a Riemannian symmetric space of noncompact type of rank one.

The purpose of the present article is to show this inequality when (M, g) is the Riemannian symmetric space of noncompact type such that G is complex.

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§ 2. Preliminaries

Let $M=G/K$ be a Riemannian symmetric space of noncompact type where G is a semisimple Lie group with finite center and K is a maximal compact subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively. Let B denote the Killing form of $\mathfrak{g}$ and let $\mathfrak{p}$ denote the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. Then $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ is the Cartan decomposition and $\mathfrak{p}$ is identified with the tangent space T_0M. We assume that the Riemannian metric g on M is induced by the Killing form B restricted to $\mathfrak{p}$. Let Δ be the corresponding Laplacian on M. Fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$. If α is a linear form on $\mathfrak{a}$ and $\alpha \neq 0$, let $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X$ for all $H \in \mathfrak{a}\}$. α is called a restricted root if $\mathfrak{g}_\alpha \neq 0$. Let Σ denote the set of all restricted roots of $(\mathfrak{g}, \mathfrak{a})$. Let $\mathfrak{a}'$ denote the open set of $\mathfrak{a}$ on which all roots in Σ are $\neq 0$. Fix a Weyl chamber $\mathfrak{a}^+$ in $\mathfrak{a}$, i.e., a connected component of $\mathfrak{a}'$. $\alpha \in \Sigma$ is called positive (denoted $\alpha > 0$) if its values on $\mathfrak{a}^+$ are positive. Let Σ^+ denote the set of positive elements of Σ. For $\alpha \in \Sigma^+$, put $m_\alpha = \dim \mathfrak{g}_\alpha$, which is called the multiplicity of α. We know that $m_\alpha = 2$ for all α if G is complex. Let the linear form ρ on $\mathfrak{a}$ be defined by $2\rho = \sum_{\alpha>0} m_\alpha \alpha$. Put $\mathfrak{n} = \sum_{\alpha>0} \mathfrak{g}_\alpha$ and let N be the corresponding analytic subgroup of G. The Iwasawa decomposition says that each $g \in G$ can be uniquely written as $g = \kappa(g) \exp H(g) n(g)$ where $\kappa(g) \in K$, $H(g) \in \mathfrak{a}$, $n(g) \in N$. For any complex linear form λ on $\mathfrak{a}$, the zonal spherical function corresponding to λ is a function on G defined by

$$\varphi_\lambda(x) = \int_K e^{(i\lambda - \rho)H(xk)} dk, \quad x \in G,$$

where dk is the normalized Haar measure on K. φ_λ satisfies (1) $\varphi_\lambda(e) = 1$, (2) $\varphi_\lambda(kxk') = \varphi_\lambda(x)$, $k, k' \in K$, $x \in G$, and hence it is regarded as a function on $M = G/K$, (3) φ_λ is a joint eigen-function of every operators $D \in \boldsymbol{D}(G/K)$ where $\boldsymbol{D}(G/K)$ denotes the algebra of G-invariant differential operators on $M = G/K$. In particular we have $\Delta\varphi_\lambda = (|\lambda|^2 + |\rho|^2)\varphi_\lambda$. Because of the Cartan decomposition $G = K\,\mathrm{Cl}\,(A^+)K$ where Cl denotes the closure, φ_λ is completely determined by its values on A^+o. If $D \in \boldsymbol{D}(G/K)$ and if $f \in C^\infty(G/K)$ is left invariant under K, then the map $\bar{f} \to (Df)^-$ (bar denoting restriction to A^+o) can be realized by the differential operator $\delta(D)$ on A^+o which is called the radial part of D, i.e., $(Df)^- = \delta(D)\bar{f}$. The radial part of the Laplacian is computed as

$$\delta(\Delta) = \Delta_A - \sum_{\alpha>0} m_\alpha (\coth \alpha) H_\alpha .$$

Here Δ_A is the Laplacian on A, i.e., $\Delta_A = -\sum_i H_i^2$ where $(H_i)_{i=1}^l$ is an orthonomal basis of $\mathfrak{a}$ and vectors H_α, H_i are viewed as first order differential operators on A^+o (cf. [5] Ch II. Proposition 3.9). We normalize the Haar measure dg on G and dk on K by $\int_K dk = 1$, $\int_G f(g)dg = \int_{G/K} \int_K f(gk)dk d\dot{g}$ for $f \in C_c(G/K)$, where $d\dot{g}$ is the induced measure on G/K. Then we have for a suitable positive constant c,

$$\int_{G/K} f(go)d\dot{g} = c \int_{\mathfrak{a}} \int_K f(k \exp Ho) dk D(H) dH, \quad f \in C_c(G/K),$$

where dH is the Euclidean measure on $\mathfrak{a}$ induced by the Killing form and $D(H) = \prod_{\alpha>0} |\mathrm{sh}\, \alpha(H)|^{m_\alpha}$.

From now on the assumption that G is complex is imposed. In this case the zonal spherical functions have a simpler expression:

$$\varphi_\lambda(a) = c(\lambda)\frac{A_{i\lambda}(a)}{A_\rho(a)}, \quad a \in A,$$

where $c(\lambda)=\pi(\rho)/\pi(i\lambda)$, $\pi(\lambda)=\prod_{\alpha>0}\langle\alpha, \lambda\rangle$ and $A_{i\lambda}(a)=\sum_s(\det s)e^{is\lambda(H)}$ for $a=\exp H$ ($H\in\mathfrak{a}$) the summation extending over the Weyl group W of $(\mathfrak{g}, \mathfrak{a})$. We know that $\rho=\sum_{\alpha>0}\alpha$ and $A_\rho(a)=\prod_{\alpha>0}(e^{\alpha(H)}-e^{-\alpha(H)})=\prod_{\alpha>0}2\operatorname{sh}\alpha(H)$. Put $\lambda=t\rho$ ($t>0$, a real number) in the above. Then we get that $\varphi_{t\rho}(a)=\prod_{\alpha>0}(\sin t\alpha(H))/(t\operatorname{sh}\alpha(H))$. Put $\mathfrak{a}_t=\{H\in\mathfrak{a}\mid |t\alpha(H)|<\pi \text{ for all } \alpha\in\Sigma^+\}$, $A_t=\exp\mathfrak{a}_t$ and $M_t=KA_to$. Then A_t (resp. M_t) is a nodal set of $\varphi_{t\rho}$, i.e., a connected component of the set $\{x\in Ao \text{ (resp. } M)\mid \varphi_{t\rho}(x)>0\}$ containing o. Notice that $\mathfrak{a}_t$ is a relatively compact, convex, open set in $\mathfrak{a}$. We consider the functions $\psi_{t,\varepsilon}$ ($0<\varepsilon<1$) on M defined by

$$\psi_{t,\varepsilon}(x) = \begin{cases} \varphi_{t\rho}(x)^{1+\varepsilon} & \text{if } x\in M_t. \\ 0 & \text{if } x\notin M_t \end{cases}$$

The continuous function $\psi_{t,\varepsilon}$ on M has the following properties.

LEMMA 1. (1) *$\psi_{t,\varepsilon}$ is a K-invariant function belonging to $C^1(M)$,* supp $\psi_{t,\varepsilon}$ $\subset \mathrm{Cl}(M_t)$ *and $\psi_{t,\varepsilon}$ is C^∞ on $M\smallsetminus\partial M_t$ (∂ denoting the boundary).* (2) *$\Delta\psi_{t,\varepsilon}$ is defined on $M\smallsetminus\partial M_t$ and belongs to $L^1(M)$.* (3) *$\Delta\psi_{t,\varepsilon}\leq(1+\varepsilon)(1+t^2)|\rho|^2\psi_{t,\varepsilon}$ on $M\smallsetminus\partial M_t$.*

PROOF. (1) is clear. For (2) the assertion $\Delta\psi_{t,\varepsilon}\in L^1(M)$ will be proved in the proof of the next Lemma. As for (3), first we see that $\Delta\psi_{t,\varepsilon}=\psi_{t,\varepsilon}=0$ on $M\smallsetminus\mathrm{Cl}(M_t)$. We calculate $\Delta\psi_{t,\varepsilon}$ on A^+o by its K-invariance. On $(A_t\cap A^+)o$,

$$\begin{aligned}
(\Delta\psi_{t,\varepsilon})^- &= \delta(\Delta)\bar{\psi}_{t,\varepsilon} = (\Delta_A-\textstyle\sum_{\alpha>0}2(\coth\alpha)H_\alpha)\bar{\psi}_{t,\varepsilon} \\
&= (1+\varepsilon)[\bar{\varphi}_{t\rho}^\varepsilon\Delta_A\bar{\varphi}_{t\rho}-\varepsilon\bar{\varphi}_{t\rho}^{\varepsilon-1}\textstyle\sum_i(H_i\bar{\varphi}_{t\rho})^2] - (1+\varepsilon)\bar{\varphi}_{t\rho}^\varepsilon\sum_{\alpha>0}2(\coth\alpha)H_\alpha\bar{\varphi}_t \\
&= (1+\varepsilon)\bar{\varphi}_{t\rho}^\varepsilon\delta(\Delta)\bar{\varphi}_{t\rho} - \varepsilon(1+\varepsilon)\bar{\varphi}_{t\rho}^{\varepsilon-1}\textstyle\sum_i(H_i\bar{\varphi}_{t\rho})^2 \\
&= (1+\varepsilon)(1+t^2)|\rho|^2\bar{\psi}_{t,\varepsilon} - \varepsilon(1+\varepsilon)\bar{\varphi}_{t\rho}^{\varepsilon-1}\textstyle\sum_i(H_i\bar{\varphi}_{t\rho})^2 \\
&\leq (1+\varepsilon)(1+t^2)|\rho|^2\bar{\psi}_{t,\varepsilon},
\end{aligned}$$

since $\delta(\Delta)\bar{\varphi}_{t\rho}=(1+t^2)|\rho|^2\bar{\varphi}_{t\rho}$ and $\bar{\varphi}_{t\rho}>0$ on A_t.

According to Lemma 1, we can construct an approximate sequence $\{\psi_{t,\varepsilon}^{(m)}\}_{m=1}^\infty$ of $\psi_{t,\varepsilon}$ having the following properties.

LEMMA 2. *There exists a sequence $\{\psi_{t,\varepsilon}^{(m)}\}_{m=1}^\infty$ of C^∞-functions on M such that* (4) supp $\bar{\psi}_{t,\varepsilon}^{(m)}\subset\{X\in\mathfrak{a}\mid dist(X, \mathfrak{a}_t)\leq 1\}$, (5) *$\psi_{t,\varepsilon}^{(m)}$ converges to $\psi_{t,\varepsilon}$ uniformly on M as $m\to\infty$ and* (6)

$$\int_M |\Delta\psi_{t,\varepsilon}^{(m)} - \Delta\psi_{t,\varepsilon}|d\dot{g} \longrightarrow 0 \quad as \quad m \longrightarrow \infty.$$

PROOF. We will mollify $\psi_{t,\varepsilon}$ on $\mathfrak{a}$. We identify $\mathfrak{a}$ and A by the exponential map and write $\bar{\psi}_{t,\varepsilon}(H)$ for $\bar{\psi}_{t,\varepsilon}(\exp H)$. Take a function $\phi \in C_c^\infty(\boldsymbol{R})$ such that $\operatorname{supp}\phi \subset [-1, 1]$ and $c_l\int_0^\infty \phi(x)x^{l-1}dx = 1$ where $l = \dim A$ and $c_l = (l-1)$-volume of the unit sphere $\boldsymbol{S}^{l-1}$ in $\boldsymbol{R}^l$. For each $0 < \delta \leq 1$ put $\rho_\delta(X) = (1/\delta)^l\phi(|X/\delta|)$ for $X \in \mathfrak{a}$. Then ρ_δ is radially symmetric, $\operatorname{supp}\rho_\delta \subset \{X \in \mathfrak{a} \mid |X| \leq \delta\}$ and $\int_{\mathfrak{a}} \rho_\delta(X)dX = 1$. Convoluting with $\bar{\psi}_{t,\varepsilon}$ on $\mathfrak{a}$, we obtain the function

$$(\bar{\psi}_{t,\varepsilon}*\rho_\delta)(H) = \int_{\mathfrak{a}} \bar{\psi}_{t,\varepsilon}(H-X)\rho_\delta(X)dX, \quad H \in \mathfrak{a}.$$

It is easy to see that $\bar{\psi}_{t,\varepsilon}*\rho_\delta$ is invariant under W and hence $\bar{\psi}_{t,\varepsilon}*\rho_\delta$ extends to a K-invariant C^∞-function $\psi_{t,\varepsilon}^\delta$ on $M = G/K$. The function $\psi_{t,\varepsilon}^{(m)}$ of the lemma is now defined by $\psi_{t,\varepsilon}^{(m)} = \psi_{t,\varepsilon}^{1/m}$. We have to show the properties (4), (5), (6). (4) and (5) are clear from the definition. As for (6) we put $\psi = \psi_{t,\varepsilon}$ for convenience. Here we prove the assertion that $\delta(\Delta)\bar{\psi} \in L^1(\mathfrak{a}, dH)$, from which Lemma 1, (2) follows since $\int_M |\Delta\psi|d\dot{g} = c\int_{\mathfrak{a}} |\delta(\Delta)\bar{\psi}|D(H)dH \leq c(\sup_{\mathfrak{a}_t} D(H))\int_{\mathfrak{a}_t} |\delta(\Delta)\bar{\psi}|\,dH < \infty$. We previously observed that for $0 < \varepsilon < 1$,

$$\delta(\Delta)\bar{\psi} = (1+\varepsilon)(1+t^2)|\rho|^2\bar{\psi} - \varepsilon(1+\varepsilon)\bar{\varphi}_{t\rho}^{\varepsilon-1}\textstyle\sum_i (H_i\bar{\varphi}_{t\rho})^2 \quad \text{on} \quad \mathfrak{a}_t.$$

We have to estimate $\bar{\varphi}_{t\rho}^{\varepsilon-1}(H\bar{\varphi}_{t\rho})^2$ (H = some H_i) near the boundary $\partial\mathfrak{a}_t$. We see that $s_\alpha = H((\sin t\alpha)/(t\operatorname{sh}\alpha))$ is C^∞ on $\mathfrak{a}$ and $H\bar{\varphi}_{t\rho} = \bar{\varphi}_{t\rho}\sum_{\alpha>0}((t\operatorname{sh}\alpha)/(\sin t\alpha))s_\alpha$. Hence we get

$$\begin{aligned}\bar{\varphi}_{t\rho}^{\varepsilon-1}(H\bar{\varphi}_{t\rho})^2 &= \bar{\varphi}_{t\rho}^{\varepsilon+1}\textstyle\sum_{\alpha,\beta}((t\operatorname{sh}\alpha)/(\sin t\alpha))((t\operatorname{sh}\beta)/(\sin t\beta))s_\alpha s_\beta \\ &= \textstyle\sum_{\alpha\neq\beta}\bar{\varphi}_{t\rho}^{\varepsilon}(\prod_{\gamma\neq\alpha,\beta}(\sin t\gamma)/(t\operatorname{sh}\gamma))s_\alpha s_\beta \\ &\quad + \textstyle\sum_{\alpha>0}(\prod_{\gamma\neq\alpha}(\sin t\gamma)(t\operatorname{sh}\gamma))^{\varepsilon+1}((\sin t\alpha)/(t\operatorname{sh}\alpha))^{\varepsilon-1}s_\alpha^2. \qquad (*)\end{aligned}$$

The first term is C^∞ on $\mathfrak{a}$ and for the second term there exists a positive constant M_α such that

$$|(\textstyle\prod_{\gamma\neq\alpha}(\sin t\gamma)/(t\operatorname{sh}\gamma))^{\varepsilon+1}((\sin t\alpha)/(t\operatorname{sh}\alpha))^{\varepsilon-1}s_\alpha^2| \leq M_\alpha|\sin t\alpha|^{\varepsilon-1} \quad \text{on} \quad \mathfrak{a}_t.$$

Near the boundary $t\alpha(X) = \pi$, for example, we have $|\sin t\alpha(X)|^{\varepsilon-1} = |\sin(\pi - t\alpha(X)|^{\varepsilon-1} = O(|\pi - t\alpha(X)|^{\varepsilon-1})$. We can take a coordinate system $(x_1, \dots, x_l)$ near $t\alpha(X) = \pi$ with $x_1 = \pi - t\alpha(X)$ and $dH = dx_1dx_2\cdots dx_l$. On a small relatively compact neighborhood U of a boundary point of $t\alpha(X) = \pi$, we get

$$\begin{aligned}&\int_U |(\textstyle\prod_{\gamma\neq\alpha}(\sin t\gamma)/(t\operatorname{sh}\gamma))^{\varepsilon+1}((\sin t\alpha)/(t\operatorname{sh}\alpha))^{\varepsilon-1}s_\alpha^2|dH \\ &\leq M_\alpha\int\cdots\int_U |\sin x_1|^{\varepsilon-1}dx_1\cdots dx_l \leq M_1\int_0^c x_1^{\varepsilon-1}dx_1 < \infty.\end{aligned}$$

This shows that the second term of $(*)$ belongs to $L^1_{loc}(\mathfrak{a}, dH)$. Since $\delta(\Delta)\bar{\psi}$ has compact support $\mathfrak{a}_t$, we obtain our assertion. It is easy to check that $\delta(\Delta)\cdot(\bar{\psi}*\rho_\delta)=(\delta(\Delta)\bar{\psi})*\rho_\delta$. We have

$$\int_M |\Delta\psi^\delta-\Delta\psi|d\dot{g} = c\int_{\mathfrak{a}} |(\delta(\Delta)\bar{\psi})*\rho_\delta-\delta(\Delta)\bar{\psi}|D(H)dH \leq M_2\int_{\mathfrak{a}} |g*\rho_\delta-g|dH,$$

where we put $g=\delta(\Delta)\bar{\psi}$ and $M_2=c\sup\{D(H)\,|\,dist(H, \mathfrak{a}_t)\leq 1\}$, and we have

$$|g*\rho_\delta(H)-g(H)| \leq \int_{|X|\leq\delta} |g(H-X)-g(H)|\rho_\delta(X)dX.$$

Put $\tau(g, \delta)=\sup_{|X|\leq\delta}\int_{\mathfrak{a}} |g(H-X)-g(H)|dH$. Then it is well known that for $g\in L^1(\boldsymbol{R}^l)$, $\tau(g, \delta)\to 0$ as $\delta\to +0$. Hence we obtain by the Fubini theorem that

$$\begin{aligned}\int_{\mathfrak{a}} |g*\rho_\delta-g|dH &\leq \int_{|X|\leq\delta}\int_{\mathfrak{a}} |g(H-X)-g(H)|\rho_\delta(X)dHdX\\ &\leq \tau(g, \delta)\int\rho_\delta(X)dX = \tau(g, \delta)\longrightarrow 0 \quad\text{as}\quad \delta\longrightarrow +0.\end{aligned}$$

This completes the proof of Lemma 2.

§3. Main Theorem

Let Γ be a cocompact discrete subgroup of G acting fixed point freely on $M=G/K$. Then $M_\Gamma=\Gamma\backslash M$ is a compact manifold with the Riemannian metric induced from g on M. The corresponding Laplacian is denoted by Δ_Γ. The volume element on M_Γ induced by $d\dot{g}$ is denoted by $d\omega$. Let F be the fundamental domain in M for Γ-action, i.e. $F=\{x\in M\,|\,d(x, o)\leq d(x, \gamma o)$ for all $\gamma\in\Gamma\}$ where d is the distance function on (M, g). It is known (see [2] for example) that $M=\bigcup_{\gamma\in\Gamma}\gamma F$ and $\gamma F\cap F$ has measure zero for all $\gamma\in\Gamma$, $\gamma\neq 1$.

We can define Γ-invariant functions $f_{t,\varepsilon}$ and $f^{(m)}_{t,\varepsilon}$ on M, which are regarded as functions on M_Γ, from $\psi_{t,\varepsilon}$ and $\psi^{(m)}_{t,\varepsilon}$ since they have compact supports. Namely put

$$f_{t,\varepsilon}(x) = \sum_{\gamma\in\Gamma}\psi_{t,\varepsilon}(\gamma x),\quad f^{(m)}_{t,\varepsilon}(x) = \sum_{\gamma\in\Gamma}\psi^{(m)}_{t,\varepsilon}(\gamma x),\quad x\in M.$$

These functions satisfy the following:

LEMMA 3. (1) *The function $f_{t,\varepsilon}$ belongs to $C^1(M_\Gamma)$ and is C^∞ on $M_\Gamma\smallsetminus\pi(\partial M_t)$ where $\pi\colon M\to M_\Gamma$ is the natural projection.* (2) *$\Delta_\Gamma f_{t,\varepsilon}\in L^1(M_\Gamma)$ and $\Delta_\Gamma f_{t,\varepsilon}\leq(1+\varepsilon)(1+t^2)|\rho|^2 f_{t,\varepsilon}$ on $M_\Gamma\smallsetminus\pi(\partial M_t)$.* (3) *$f^{(m)}_{t,\varepsilon}\in C^\infty(M_\Gamma)$ converges uniformly to $f_{t,\varepsilon}$ and*

$$\int_{M_\Gamma} |\Delta_\Gamma f^{(m)}_{t,\varepsilon}-\Delta_\Gamma f_{t,\varepsilon}|d\omega\longrightarrow 0\quad as\quad m\longrightarrow\infty.$$

Moreover we have

(4) $$\int_{M_\Gamma} |f_{t,\varepsilon}^{(m)} \Delta_\Gamma f_{t,\varepsilon}^{(m)} - f_{t,\varepsilon} \Delta_\Gamma f_{t,\varepsilon}| d\omega \longrightarrow 0 \quad as \quad m \longrightarrow \infty.$$

PROOF. (1) and (2) follow from Lemma 1, (1), (2) and (3). For (3) notice that supports of $\psi_{t,\varepsilon}$ and $\psi_{t,\varepsilon}^{(m)}$ are contained in a compact set $M_t' = K \exp\{X \in \mathfrak{a} \mid \mathrm{dist}(X, \mathfrak{a}_t) \leq 1\} o$. Hence if $\Gamma_t = \{\gamma \in \Gamma \mid \gamma F \cap M_t' \neq \emptyset\}$, the order $|\Gamma_t|$ is finite. We have at $x \in F$,

$$\begin{aligned} |f_{t,\varepsilon}^{(m)}(x) - f_{t,\varepsilon}(x)| &\leq \textstyle\sum_{\gamma \in \Lambda_t} |\psi_{t,\varepsilon}^{(m)}(\gamma x) - \psi_{t,\varepsilon}(\gamma x)| \\ &\leq |\Gamma_t| \sup_M |\psi_{t,\varepsilon}^{(m)} - \psi_{t,\varepsilon}| \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty \end{aligned}$$

by Lemma 2, (5). Notice that $\Delta_\Gamma f_{t,\varepsilon}(x) = \sum_{\gamma \in \Gamma} \Delta \psi_{t,\varepsilon}(\gamma x)$ at $x \in F \smallsetminus \Gamma(\partial M_t)$. Hence we have

$$\begin{aligned} \int_{M_\Gamma} |\Delta_\Gamma f_{t,\varepsilon}^{(m)} - \Delta_\Gamma f_{t,\varepsilon}| d\omega &= \int_F |\Delta f_{t,\varepsilon}^{(m)} - \Delta f_{t,\varepsilon}| dx \\ \leq \int_F \textstyle\sum_{\gamma \in \Gamma} |\Delta \psi_{t,\varepsilon}^{(m)}(\gamma x) - \Delta \psi_{t,\varepsilon}(\gamma x)| dx &= \int_M |\Delta \psi_{t,\varepsilon}^{(m)} - \Delta \psi_{t,\varepsilon}| dx \longrightarrow 0 \end{aligned}$$

as $m \to \infty$ by Lemma 2, (6). The inequalities

$$\begin{aligned} &\int_{M_\Gamma} |f_{t,\varepsilon}^{(m)} \Delta_\Gamma f_{t,\varepsilon} - f_{t,\varepsilon} \Delta_\Gamma f_{t,\varepsilon}| d\omega \\ &\leq \int_{M_\Gamma} (|f_{t,\varepsilon}^{(m)}| \, |\Delta_\Gamma f_{t,\varepsilon}^{(m)} - \Delta_\Gamma f_{t,\varepsilon}| + |f_{t,\varepsilon}^{(m)} - f_{t,\varepsilon}| \, |\Delta_\Gamma f_{t,\varepsilon}|) d\omega \\ &\leq \sup_{M_\Gamma} |f_{t,\varepsilon}^{(m)}| \int_{M_\Gamma} |\Delta_\Gamma f_{t,\varepsilon}^{(m)} - \Delta_\Gamma f_{t\,\varepsilon}| d\omega + \sup_{M_\Gamma} |f_{t,\varepsilon}^{(m)} - f_{t,\varepsilon}| \int_{M_\Gamma} |\Delta_\Gamma f_{t,\varepsilon}| d\omega \end{aligned}$$

together with (2) and (3) imply (4).

Notice that

(5) $$\int_{M_\Gamma} f_{t,\varepsilon} d\omega = \int_M \psi_{t,\varepsilon} d\dot{g} = c \int_{\mathfrak{a}_t} \varphi_{t\rho}(\exp H)^{1+\varepsilon} D(H) dH,$$

(6) $$\int_{M_\Gamma} f_{t,\varepsilon}^2 d\omega \geq \int_M \psi_{t,\varepsilon}^2 d\dot{g} = c \int_{\mathfrak{a}_t} \varphi_{t\rho}(\exp H)^{2(1+\varepsilon)} D(H) dH.$$

Then we obtain the following proposition.

PROPOSITION 4. $\lambda_1(\Gamma)\{1 - K_t \operatorname{vol}(M_\Gamma)^{-1}\} \leq (1+t^2)|\rho|^2$, *where* K_t *is a positive constant depending only on* t, *i.e.,*

$$K_t = c\left(\int_{\mathfrak{a}_t} \varphi_{t\rho}(\exp H) D(H) dH\right)^2 \left(\int_{\mathfrak{a}_t} \varphi_{t\rho}(\exp H)^2 D(H) dH\right)^{-1}.$$

PROOF. The minimum principle for the first eigenvalue $\lambda_1(\Gamma)$ of Δ_Γ says (cf. [1], p. 186) that

$$\lambda_1(\Gamma) = \inf\left\{\left(\int_{M_\Gamma} \eta\Delta_\Gamma\eta d\omega\right)\Big/\left(\int_{M_\Gamma} \eta^2 d\omega\right)\Big|\int_{M_\Gamma} \eta d\omega = 0,\ \eta\in C^\infty(M_\Gamma)\right\}.$$

Therefore we get the inequality

$$\lambda_1(\Gamma)\int_{M_\Gamma} \eta^2 d\omega \le \int_{M_\Gamma} \eta\Delta_\Gamma\eta d\omega \quad \text{if} \quad \int_{M_\Gamma} \eta d\omega = 0.$$

Apply this for $\eta = f_{t,\varepsilon}^{(m)} - \alpha$ where $\alpha = \mathrm{vol}\,(M_\Gamma)^{-1}\int_{M_\Gamma} f_{t,\varepsilon}^{(m)} d\omega$. Then we get by Green's formula

$$\lambda_1(\Gamma)\left\{\int_{M_\Gamma} (f_{t,\varepsilon}^{(m)})^2 d\omega - \mathrm{vol}\,(M_\Gamma)^{-1}\left(\int_{M_\Gamma} f_{t,\varepsilon}^{(m)} d\omega\right)^2\right\} \le \int_{M_\Gamma} f_{t,\varepsilon}^{(m)}\Delta_\Gamma f_{t,\varepsilon}^{(m)} d\omega.$$

Letting $m\to\infty$, we obtain by Lemma 3, (3) and (4) that

$$\lambda_1(\Gamma)\left\{\int_{M_\Gamma} f_{t,\varepsilon}^2 d\omega - \mathrm{vol}\,(M_\Gamma)^{-1}\left(\int_{M_\Gamma} f_{t,\varepsilon} d\omega\right)^2\right\} \le \int_{M_\Gamma} f_{t,\varepsilon}\Delta_\Gamma f_{t,\varepsilon} d\omega.$$

By Lemma 3, (2), we have

$$\int_{M_\Gamma} f_{t,\varepsilon}\Delta_\Gamma f_{t,\varepsilon} d\omega \le (1+\varepsilon)(1+t^2)|\rho|^2\int_{M_\Gamma} f_{t,\varepsilon}^2 d\omega,$$

since $\pi(\partial M_t)$ has measure zero, and hence

$$\lambda_1(\Gamma)\left\{\int_{M_\Gamma} f_{t,\varepsilon}^2 d\omega - \mathrm{vol}\,(M_\Gamma)^{-1}\left(\int_{M_\Gamma} f_{t,\varepsilon} d\omega\right)^2\right\} \le (1+\varepsilon)(1+t^2)|\rho|^2\int_{M_\Gamma} f_{t,\varepsilon}^2 d\omega.$$

Divide this inequality by $\int_{M_\Gamma} f_{t,\varepsilon}^2\, d\omega$ and use (5) and (6). Then we obtain

$$\lambda_1(\Gamma)\{1 - \mathrm{vol}\,(M_\Gamma)^{-1}K_{t,\varepsilon}\} \le (1+\varepsilon)(1+t^2)|\rho|^2,$$

where we put $K_{t,\varepsilon} = c\left(\int_{\mathfrak{a}_t} \varphi_{t\rho}^{1+\varepsilon} DdH\right)^2\left(\int_{\mathfrak{a}_t} \varphi_{t\rho}^{2(1+\varepsilon)} DdH\right)^{-1}$. Letting $\varepsilon\to+0$, we obtain the inequality in the proposition.

THEOREM. *Let $M = G/K$ be a Riemannian symmetric space of noncompact type such that G is a complex semisimple Lie group. Consider all discrete cocompact subgroups Γ of G acting fixed point freely on M. Then we have*

$$\limsup_{\mathrm{vol}(M_\Gamma)\to\infty} \lambda_1(\Gamma) \le |\rho|^2,$$

where $|\rho|^2$ is the positive constant defined in §2.

PROOF. For a discrete subgroup Γ with sufficient large $\mathrm{vol}(M_\Gamma)$ (such Γ always exists; see [2], Theorem 2.1) such that $\mathrm{vol}(M_\Gamma) > K_t$, we have by Proposition 4,

$$\lambda_1(\Gamma) \leq (1+t^2)|\rho|^2\{1-\mathrm{vol}(M_\Gamma)^{-1}K_t\}^{-1}.$$

Hence $\limsup_{\mathrm{vol}(M_\Gamma)\to\infty} \lambda_1(\Gamma) \leq (1+t^2)|\rho|^2$ for every $t>0$. Letting $t\to+0$, we finally obtain the inequality of the theorem.

References

[1] M. Berger, P. Gaudachon and E. Mazet, Le spectre d'une variété riemannienne, Lecture Note in Math, **194**, Springer-Verlag, New York, 1971.

[2] D. L. DeGeorge and N. R. Wallach, Limit formulas for multiplicities in $L^2(\Gamma\backslash G)$, Ann. of Math., **107** (1978), 133–150.

[3] R. Gangolli, Asymptotic behavior of spectra of compact quatients of certain symmetric spaces, Acta Math., **12** (1968), 151–192.

[4] Harish-Chandra, Spherical functions on a semisimple Lie group, I, Amer. J. Math., **80** (1958), 241–310.

[5] S. Helgason, Group and Geometric Analysis. Integral Geometry, Invariant Differential Operators and Spherical Functions, Academic Press, New York, 1984.

[6] H. Huber, Über den ersten Eigenwert des Laplace-Operators auf kompacten Riemannschen Flächen, Comment. Math. Helv., **49** (1974), 251–259.

[7] H. Urakawa, On the least positive eigenvalue of the Laplacian for the compact quotient of a certain Riemannian symmetric space, Nagoya Math. J., **78** (1980), 137–152.

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